GEODESICS IN MARGULIS SPACETIMES

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Dedicated to the memory of Dan Rudolph

ABSTRACT. Let M^3 be a Margulis spacetime whose associated complete hyperbolic surface Σ^2 has compact convex core. Generalizing the correspondence between closed geodesics on M^3 and closed geodesics on Σ^2, we establish an orbit equivalence between recurrent spacelike geodesics on M^3 and recurrent geodesics on Σ^2. In contrast, no timelike geodesic recurs in neither forward nor backwards time.

CONTENTS

INTRODUCTION

A *Margulis spacetime* is a complete flat affine 3-manifold M^3 with free nonabelian fundamental group Γ. It necessarily carries a unique parallel Lorentz metric. Parallelism classes of timelike geodesics form a noncompact complete hyperbolic surface Σ^2. This complete hyperbolic surface is naturally associated to the flat 3-manifold M^3 and we regard M^3 as an *affine deformation* of Σ^2. This note relates the dynamics of the geodesic flow of the flat affine manifold M^3 to the dynamics of the geodesic flow on the hyperbolic surface Σ^2.

We restrict to the case that Σ^2 has a compact convex core (that is, Σ^2 has finite type and no cusps). Equivalently, the Fuchsian group Γ_0 corresponding to $\pi_1(\Sigma^2)$ is *convex cocompact.* In particular Γ_0 is finitely generated and contains no parabolic elements. Under this assumption,

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every free homotopy class of an essential closed curve in Σ^2 contains a unique closed geodesic. Since Σ^2 and M^3 are homotopy-equivalent, free homotopy classes of essential closed curves in M correspond to free homotopy classes of essential closed curves in Σ^2. Every essential closed curve in M^3 is likewise homotopic to a unique closed geodesic in M^3.

In her thesis [4, 7], Charette studied the next case of dynamical behavior: geodesics spiralling around closed geodesics both in forward and backward time. She proved bispiralling geodesics in M^3 exist, and correspond to bispiralling geodesics in Σ^2.

This paper extends the above correspondence between geodesics on Σ^2 and M^3 to recurrent geodesics.

A geodesic (either in Σ^2 or in M^3) is *recurrent* if and only if it (together with its velocity vector) is recurrent in *both* directions. These correspond to recurrent points for the corresponding geodesic flows as in Katok-Hasselblatt [15], §3.3. Under our hypotheses on Σ^2, a geodesic on Σ^2 is recurrent if and only if the corresponding orbit of the geodesic flow is precompact.

Theorem 1. *Let M^3 be a Margulis spacetime whose associated complete hyperbolic surface Σ has compact convex core.*

- *The recurrent part of the geodesic flow for Σ^2 is* topologically orbit-equivalent *to the recurrent spacelike part of the geodesic flow of M^3.*
- *The set of recurrent spacelike geodesics in a Margulis spacetime is the closure of the set of periodic geodesics.*
- *No timelike geodesic recurs.*

A semiconjugacy between these flows was observed by D. Fried [11].

This note is the sequel to [13], which characterizes properness of affine deformations by positivity of a marked Lorentzian length spectrum, the *generalized Margulis invariant*. A crucial step in the proof that properness implies positivity is the construction of sections of the associated flat affine bundle, called *neutralized sections.* A further modification of neutralized sections produces an orbit equivalence between recurrent geodesics in Σ and recurrent geodesics in M.

It follows that the set of recurrent spacelike geodesics is a Smale hyperbolic set in $\mathsf{T}M$.

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1. Geodesics on affine manifolds

An *affinely flat manifold* is a smooth manifold with a distinguished atlas of local coordinate systems whose charts map to an affine space $\mathbb{E}$ such that the coordinate changes are restrictions of affine automorphisms of $\mathbb{E}$. Denote the group of affine automorphisms of $\mathbb{E}$ by $\mathsf{Aff}(\mathbb{E})$. This structure is equivalent to a flat torsionfree affine connection. The affine coordinate atlas globalizes to a *developing map*

$$\tilde{M} \xrightarrow{\mathsf{dev}} \mathbb{E}$$

where $\tilde{M} \to M$ denotes a universal covering space of M. The coordinate changes globalize to an affine holonomy homomorphism

$$\pi_1(M) \xrightarrow{\rho} \mathsf{Aff}(\mathbb{E})$$

where $\pi_1(M)$ denotes the group of deck transformations of $\tilde{M} \to M$. The developing map is equivariant respecting ρ.

Denote the vector space of translations $\mathbb{E} \to \mathbb{E}$ by $\mathbb{V}$. The action of $\mathbb{V}$ by translations on $\mathbb{E}$ defines a trivialization of the tangent bundle $\mathsf{T}M \cong M \times \mathbb{V}$. In these local coordinate charts, a geodesic is a path

$$p \longmapsto p + t\mathsf{v}$$

where $p \in \mathbb{E}$ and $\mathsf{v} \in \mathbb{V}$ is a vector. In terms of the trivialization the geodesic flow is:

$$\begin{aligned} \mathbb{E} \times \mathbb{V} &\xrightarrow{\tilde{\psi}_t} \mathbb{E} \times \mathbb{V} \\ (p, \mathsf{v}) &\longmapsto (p + t\mathsf{v}, \mathsf{v}) \end{aligned}$$

for $t \in \mathbb{R}$. Clearly this $\mathbb{R}$-action commutes with $\mathsf{Aff}(\mathbb{E})$.

Geodesic completeness implies that dev is a diffeomorphism. Thus the universal covering $\tilde{M}$ is affinely isomorphic to affine space $\mathbb{E}$ and $M \cong \mathbb{E}/\Gamma$, where $\Gamma := \rho\big(\pi_1(M)\big)$ is a discrete group of affine transformations acting properly and freely on $\mathbb{E}$.

2. Flat Lorentz 3-manifolds

Let $\mathsf{Aff}(\mathbb{E}) \xrightarrow{\mathsf{L}} \mathsf{GL}(\mathbb{V})$ denote the homomorphism given by *linear part,* that is, $\mathsf{L}(\gamma) = A$ where

$$p \xrightarrow{\gamma} \mathsf{L}(\gamma)(p) + u(\gamma).$$

Any $\mathsf{L}(\Gamma)$-invariant nondegenerate inner product $\langle , \rangle$ on $\mathbb{V}$ defines a Γ-invariant flat pseudo-Riemannian structure on $\mathbb{E}$ which descends to $M = \mathbb{E}/\Gamma$. In particular affine manifolds with $\mathsf{L}(\Gamma) \subset \mathsf{O}(n-1,1)$ are precisely the *flat Lorentzian manifolds,* and the underlying affine structures their Levi-Civita connections.

For this reason we henceforth fix the invariant Lorentzian inner product on $\mathbb{V}$, and hence the (parallel) flat Lorentzian structure on $\mathbb{E}$. The group $\mathsf{Isom}(\mathbb{E})$ of Lorentzian isometries is the semidirect product of the group $\mathbb{V}$ of translations of $\mathbb{E}$ with the orthogonal group $\mathsf{SO}(n-1,1)$ of linear isometries. Linear part $\mathsf{Isom}(\mathbb{E}) \xrightarrow{\mathsf{L}} \mathsf{SO}(n-1,1)$ defines the projection homomorphism for the semidirect product. For $l \in \mathbb{R}$, define

$$\mathsf{S}_l \;:=\; \{\mathsf{v} \in \mathbb{V} \mid \langle \mathsf{v}, \mathsf{v} \rangle \;=\; l\}.$$

When $l > 0$, S_l is a Riemannian submanifold of constant curvature $-l^{-2}$ and when $l < 0$, it is a Lorentzian submanifold of constant curvature l^{-2}. In particular S_{-1} is a disjoint union of two isometrically embedded copies of hyperbolic $n-1$-space H^{n-1} and S_1 is *de Sitter space,* a model space of Lorentzian curvature $+1$.

The subset $\mathsf{T}_l(M)$ consisting of tangent vectors v such that $\langle v, v \rangle = l$ is invariant under the geodesic flow. Indeed, using parallel translation, these bundles trivialize over the universal covering $\mathbb{E}$:

$$\mathsf{T}_l(\mathbb{E}) \xrightarrow{\cong} \mathbb{E} \times \mathsf{S}_l$$

By Abels-Margulis-Soifer [2, 3], if a discrete group of Lorentz isometries acts properly on Minkowski space $\mathbb{E}$, and $\mathsf{L}(\Gamma)$ is Zariski dense in $\mathsf{SO}(n-1,1)$, then $n = 3$. Consequently every complete flat Lorentz manifold is a flat Euclidean affine fibration over a complete flat Lorentz 3-manifold. Thus we henceforth restrict to $n = 3$.

Let M^3 be a complete affinely flat 3-manifold. By Fried-Goldman [12], either Γ is solvable or $\mathsf{L} \circ h$ embeds Γ as a discrete subgroup in (a conjugate of) the orthogonal group

$$\mathsf{SO}(2,1) \subset \mathsf{GL}(3,\mathbb{R}).$$

The cases when Γ is solvable are easily classified (see [12]) and we assume we are in the latter case. In that case, M^3 is a complete flat Lorentz 3-manifold.

In the early 1980's Margulis, answering a question of Milnor [20]), constructed the first examples [17, 18], which are now called *Margulis spacetimes.* Explicit geometric constructions of these manifolds have been given by Drumm [8, 9] and his coauthors [5, 6, 10].

Since the hyperbolic plane H^2 is the symmetric space of $\mathsf{SO}(2,1)$, Γ acts properly and discretely on H^2. Since M^3 is aspherical, its fundamental group $\pi_1(M^3) \cong \Gamma$ is torsionfree, so Γ acts freely as well. Therefore the quotient $\mathsf{H}^2/\mathsf{L}(\Gamma)$ is a complete hyperbolic surface Σ^2. Furthermore Σ is noncompact (Mess [19], see also Goldman-Margulis [14] and Labourie [16] for alternate proofs), and every noncompact complete hyperbolic surface occurs for a Margulis spacetime (Drumm [8])

The points of Σ^2 correspond to parallelism classes of timelike geodesics on M^3 as follows. A timelike geodesic is given by an ψ-orbit in

$$\mathsf{T}_{-1}M \;\cong\; \big(\mathbb{E}\times\mathsf{H}^2\big)/\Gamma$$

Since $\Gamma \xrightarrow{\mathsf{L}} \mathsf{SO}(2,1)$ is a discrete embedding [12], $\mathsf{SO}(2,1)$ acting properly on H^2 implies that Γ acts properly on H^2. Cartesian projection $\mathbb{E}\times\mathsf{H}^2 \to \mathsf{H}^2$ defines a projection $\mathsf{T}^{-1}M \longrightarrow \mathsf{H}^2/\mathsf{L}(\Gamma) = \Sigma$ which is invariant under the restriction of the geodesic flow ψ to $\mathsf{T}^{-1}M$; and defines an $\mathbb{E}$-bundle. Its fiber over a fixed future-pointing unit-timelike vector v is the union of geodesics parallel to v. In particular properness of the $\mathsf{L}(\Gamma)$-action on H^2 implies nonrecurrence of timelike geodesics, the last statement in Theorem 1.

More generally, any $\mathsf{L}(\Gamma)$-invariant subset $\Omega \subset \mathbb{V}$ defines a subset $\mathsf{T}_\Omega(M) \subset \mathsf{T}M$ invariant under the geodesic flow. If Ω is an open set upon which $\mathsf{L}(\Gamma)$ acts properly, then the geodesic flow defines a proper $\mathbb{R}$-action on $\mathsf{T}_\Omega(M)$. In particular every geodesic whose velocity lies in Ω is properly immersed and is neither positively nor negatively recurrent.

3. From geodesics in Σ^2 to geodesics in M^3

While timelike directions correspond to points of Σ^2, spacelike directions correspond to geodesics in H^2. The recurrent geodesics in Σ intimately relate to the recurrent spacelike geodesics on M^3.

Denote the set of oriented spacelike geodesics in $\mathbb{E}$ by $\mathscr{S}$. It identifies with the orbit space of the geodesic flow $\tilde{\psi}$ on $\mathsf{T}_{+1}\mathbb{E} \cong \mathbb{E}\times\mathsf{S}_{+1}$. The natural map $\mathscr{S} \xrightarrow{\Upsilon} \mathsf{S}_{+1}$ which associates to a spacelike vector its direction is equivariant respecting $\mathsf{Isom}(\mathbb{E}) \xrightarrow{\mathsf{L}} \mathsf{SO}(2,1)$.

The identity component of $\mathsf{SO}(2,1)$ acts simply transitively on the unit tangent bundle UH^2, and therefore we identify $\mathsf{SO}(2,1)^0$ with UH^2 by choosing a basepoint u_0 in UH^2. Unit-spacelike vectors in S_{+1} correspond to oriented geodesics in H^2. Explicitly, if $v \in \mathsf{S}_{+1}$, then there is a one-parameter subgroup $a(t) \in \mathsf{SO}(2,1)$, having v as a fixed vector, and such that $\det(v, v^-, v^+) > 0$, where v^+ is an expanding eigenvector of $a(t)$ (for $t > 0$) and v^- is the contracting eigenvector. Choose a basepoint $v_0 \in \mathsf{S}_{+1}$ corresponding to the orbit of u_0 under the geodesic flow on $\mathsf{U}\Sigma$. Geodesics in H^2 relate to spacelike directions by an equivariant mapping

$$\begin{aligned} \mathsf{UH}^2 &\xrightarrow{\tilde{\nu}} \mathsf{S}_{+1} \\ g(u_0) &\longmapsto g(v_0) \end{aligned}$$

The unit tangent bundle $\mathsf{U}\Sigma$ of Σ identifies with the quotient

$$\mathsf{L}(\Gamma)\backslash\mathsf{UH}^2 \;\cong\; \mathsf{L}(\Gamma)\backslash\mathsf{SO}(2,1)^0,$$

where the geodesic flow ψ corresponds the right-action of $a(-t)$ (see, for example, [13],§1.2).

Observe that a geodesic in Σ^2 is recurrent if and only if the endpoints of any of its lifts to $\tilde{\Sigma} \approx \mathsf{H}^2$ lie in the limit set Λ of $\mathsf{L}(\Gamma)$. If the convex core of Σ^2 is compact, then the union $\mathsf{U}_{\mathsf{rec}}\Sigma$ of recurrent ϕ-orbits is compact.

Lemma 2. *There exists an orbit-preserving map*

$$\mathsf{U}_{\mathsf{rec}}\Sigma \xrightarrow{\widehat{\mathbf{N}}} \mathsf{T}_{+1}(M)$$

mapping ϕ-orbits injectively to recurrent ψ-orbits.

Proof. The associated flat affine bundle E over $\mathsf{U}\Sigma$ associated to the affine deformation Γ is defined as follows. The affine representation of Γ defines a diagonal action of Γ on $\widetilde{\mathsf{U}\Sigma} \times \mathbb{E}$

Its total space is the quotient of the product $\widetilde{\mathsf{U}\Sigma} \times \mathbb{E}$ by the diagonal action of $\pi_1(\Sigma)$

$$\pi_1(\mathsf{U}\Sigma) \longrightarrow \pi_1(\Sigma) \longrightarrow \mathsf{Isom}(\mathbb{E}).$$

Similarly the flat vector bundle V over $\mathsf{U}\Sigma$ is the quotient of $\widetilde{\mathsf{U}\Sigma} \times \mathbb{V}$ by the diagonal action

$$\pi_1(\mathsf{U}\Sigma) \longrightarrow \pi_1(\Sigma) \longrightarrow \mathsf{Isom}(\mathbb{E}) \xrightarrow{\mathsf{L}} \mathsf{SO}(2,1).$$

According to [13], the *neutral section* of V is a $\mathsf{SO}(2,1)$-invariant section which is parallel with respect to the geodesic flow on $\mathsf{U}\Sigma$. and arises from the graph of the $\mathsf{SO}(2,1)$-equivariant map

$$\mathsf{U}\tilde{\Sigma} \cong \mathsf{UH}^2 \longrightarrow \mathbb{V}$$

with image S_{+1}, the space of unit-spacelike vectors in $\mathbb{V}$.

Here is the main construction of [13]. To every section σ of E continuously differentiable along ϕ, associate the function

$$F_\sigma \;:=\; \langle \nabla_\phi \sigma, \nu \rangle$$

on $\mathsf{U}\Sigma$. (Here the covariant derivative of a section of E along a vector field ϕ in the base is a section of the associated vector bundle V.) Different choices of section σ yield cohomologous functions F_σ. (Recall that two functions f_1, f_2 are *cohomologous,* written $f_1 \sim f_2$, if

$$f_1 - f_2 \;=\; \phi\, g$$

for a function g which is differentiable with respect to the vector field ϕ ([15],§2.2).

Restrict the affine bundle E to $\mathsf{U}_{\mathsf{rec}}\Sigma$. Lemma 8.4 of [13] guarantees the existence of a *neutralized section*, that is, a section N of $\mathsf{E}|_{\mathsf{U}_{\mathsf{rec}}\Sigma}$ satisfying

$$\nabla_X N = f\nu,$$

for some function f.

Although the following lemma is well known, we could not find a proof in the literature. For completeness, we supply a proof in the appendix.

Lemma 3. *Let X be a compact space equipped with a flow ϕ. Let $f \in C(X)$, such that for all ϕ-invariant measures μ on X,*

$$\int f \; \mathrm{d}\mu \; > \; 0.$$

Then f is cohomologous to a positive function.

Since Γ acts properly, Proposition 8.1 of [13] implies that $\int F_\sigma d\mu \neq 0$ for all ϕ-invariant probability measures μ on $\mathsf{U}_{\mathsf{rec}}\Sigma$. Since the set of invariant measures is connected, $\int F_\sigma d\mu$ is either positive for all ϕ-invariant probability measures μ on $\mathsf{U}_{\mathsf{rec}}\Sigma$ or negative for all ϕ-invariant probability measures μ on $\mathsf{U}_{\mathsf{rec}}\Sigma$. Conjugating by $-I$ if necessary we may assume that $\int F_\sigma d\mu > 0$. Lemma 3 implies $F_\sigma + L_\phi g > 0$ for some function g. Write

$$\widehat{N} \; = \; N + \; g\nu.$$

$\widehat{N}$ remains neutralized, and $\nabla_X \widehat{N}$ vanishes nowhere.

Let $\widetilde{\mathsf{U}_{\mathsf{rec}}\Sigma}$ be the preimage of $\mathsf{U}_{\mathsf{rec}}\Sigma$ in UH^2. Then $\widehat{N}$ determines a Γ-equivariant map

$$\widetilde{\mathsf{U}_{\mathsf{rec}}\Sigma} \xrightarrow{\widetilde{N}} \mathbb{E}.$$

Each $\tilde{\phi}$-orbit injectively maps to a spacelike geodesic. The map

$$\begin{aligned} \mathsf{U}_{\mathsf{rec}}\Sigma \xrightarrow{\widehat{\mathbf{N}}} \big(\mathbb{E} \times \mathsf{S}_{+1}\big)/\Gamma \; &= \; \mathsf{T}_{+1}(M) \\ \widehat{\mathbf{N}}(x) := \; &\big[(\widehat{N}(x), \nu(x))\big]. \end{aligned}$$

is the desired orbit equivalence $\mathsf{U}_{\mathsf{rec}}\Sigma \longrightarrow \mathsf{T}_{+1}(M)$. $\square$

Lemma 4. *Any spacelike recurrent geodesic parallel to a geodesic γ in the image of $\widehat{\mathbf{N}}$ coincides with γ.*

Proof. Let $t \xmapsto{g} \phi_t(v)$ be an orbit in $\mathsf{U}_{\mathsf{rec}}\Sigma$. A geodesic ξ parallel to $\widehat{\mathbf{N}}(g)$ determines a parallel section u of $\mathbb{V}$ along g. Since g recurs, the resulting parallel section is a bounded invariant parallel section along the closure of g. By the Anosov property, such a section is along ν, and therefore, up to reparametrization, $\gamma = \widehat{\mathbf{N}}(g)$. $\square$

Proposition 5. $\widehat{\mathbf{N}}$ *is injective and its image is the set of recurrent spacelike geodesics.*

Proof. An orbit of the geodesic flow ϕ recurs if and only if the corresponding Γ-orbit in the space $\mathscr{S}$ of spacelike geodesics in $\mathbb{E}$ recurs. Similarly a ϕ-orbit in $\mathsf{T}_{+1}(M)$ recurs if and only if the corresponding $\mathsf{L}(\Gamma)$-orbit in S_{+1} recurs. The map $\mathscr{S} \xrightarrow{\Upsilon} \mathsf{S}_{+1}$ recording the direction of a spacelike geodesic is L-equivariant. If the Γ-orbit of $g \in \mathscr{S}$ corresponds to a recurrent spacelike geodesic in M, then the $\mathsf{L}(\Gamma)$-orbit of $\Upsilon(g)$ corresponds to a recurrent ϕ-orbit in $\mathsf{U}\Sigma$.

$\widehat{\mathbf{N}}$ is injective along orbits of the geodesic flow. Thus it suffices to prove that the restriction of Υ to the subset of Γ-recurrent geodesics in $\mathscr{S}$ is injective. Since the fibers of Υ are parallelism classes of spacelike geodesics, Lemma 4 implies injectivity of $\widehat{\mathbf{N}}$.

Finally let g be a ψ-recurrent point in $\mathsf{T}_{+1}(M)$, corresponding to a spacelike recurrent geodesic γ in M. It corresponds to a recurrent Γ-orbit Γg in $\mathscr{S}$. Then $\Upsilon(\Gamma g)$ is a recurrent $\mathsf{L}(\Gamma)$-orbit in S_{+1}, and corresponds to a recurrent ϕ-orbit in $\mathsf{U}\Sigma$. The image of this ϕ-orbit under $\widehat{\mathbf{N}}$ is a spacelike recurrent geodesic in $\mathsf{T}_{+1}(M)$ parallel to γ. Now apply Lemma 4 again to conclude that g lies in the image of $\mathbf{h}N$. $\square$

The proof of Theorem 1 is complete.

4. Appendix: Cohomology and positive functions

Let X be a smooth manifold equipped with a smooth flow ϕ. A function $g \in C(X)$ is *continuously differentiable along* ϕ if for each $x \in X$, the function

$$t \mapsto g\big(\phi_t(x)\big)$$

is a continuously differentiable map $\mathbb{R} \longrightarrow X$. Denote the subspace of $C(X)$ consisting of functions continuously differentiable along ϕ by $C_\phi(X)$. For $g \in C_\phi(X)$, denote its directional derivative by:

$$\phi(g) \;:=\; \frac{d}{dt}\bigg|_{t=0} g \circ \phi_t.$$

The proof of Lemma 3 will be based on two lemmas.

Lemma 6. *Let $f \in C_\phi(X)$. For any $T > 0$, define*

$$f_T(x) \;:=\; \frac{1}{T}\int_0^T f\big(\phi_s(x)\big)\, \mathrm{d}s.$$

Then $f \sim f_T$.

Proof. We must show that there exists a function $g \in C_\phi(X)$ such that:

$$f_T - f \;=\; \phi g.$$

By the fundamental theorem of calculus,

$$f \circ \phi_t \;=\; f + \int_0^t (\phi f \circ \phi_s)\; \mathrm{d}s.$$

Writing

$$g = \frac{1}{T}\int_0^T \int_0^t (f \circ \phi_s)\; \mathrm{d}s\; \mathrm{d}t,$$

then

$$\begin{aligned} f_T - f &= \frac{1}{T}\int_0^T (f \circ \phi_t - f)\; \mathrm{d}t \\ &= \frac{1}{T}\int_0^T \int_0^t \phi(f \circ \phi_s)\; \mathrm{d}s\; \mathrm{d}t \\ &= \phi g. \end{aligned}$$

as desired. □

Lemma 7. *Assume that for all ϕ-invariant measures μ,*

$$\int f \mathrm{d}\mu > 0.$$

Then $f_T > 0$ for some $T > 0$.

Proof. Otherwise sequences $\{T_m\}_{m \in \widehat{\mathbb{N}}}$ of positive real numbers and sequences $\{x_m\}_{m \in \widehat{\mathbb{N}}}$ of points in M exist such that

$$f_{T_n}(x_n) \leq 0.$$

Using the flow ϕ_t, push forward the (normalized) Lebesgue measure

$$\frac{1}{T_m}\mu_{[0,T_m]}$$

on the interval $[0, T_m]$ to X, to obtain a sequence of probability measures μ_n on X such that

$$\int f\; \mathrm{d}\mu_n \;\leq\; 0.$$

As in [13], §7, a subsequence weakly converges to an ϕ-invariant measure μ for which

$$\int f\; \mathrm{d}\mu \leq 0,$$

contradicting our hypotheses. □

Proof of Lemma 3. By Lemma 6, $f \sim f_T$ for any $T > 0$, and Lemma 7 implies that $f_T > 0$ for some T. $\square$

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