Longwave stability of three-layer plane Poiseuille flow with an inner thin layer. Application to the coextrusion of polymers

Stéphane Scotto and Patrice Laure
Institut Non Linéaire de Nice (UMR CNRS 6618), Université de Nice
1361 Rte des Lucioles, Sophia-Antipolis, 06560 Valbonne, France
Email: {scotto,laure}@inln.cnrs.fr

Abstract . This paper deals with the longwave stability of three-layer plane Poiseuille flow of Oldroyd-B fluids. The thickness of the inner layer is assumed to be very thin with respect to the outer layers. The stability is given by two eigenvalues: one corresponds to this obtained for the two-layer flow composed by the two outer layers mode whereas the other one expresses the effect of the thin layer. In this way, simple conditions on the rheologic properties of the inner layer are given so that the stability was only led by the eigenvalue associated to the two larger layers.

Keywords: multifluid, interface, stability, longwave, Oldroyd-B, Poiseuille

Stabilité aux grandes ondes de l'écoulement de Poiseuille constitué de deux couches séparées par une couche fine. Application à la coextrusion de polymères

Résumé . On étudie la stabilité aux longues ondes de l'écoulement Poiseuille plan pour trois couches de fluides suivant le modèle constitutif Oldroyd-B. On suppose que la couche centrale est fine par rapport aux deux couches extérieures. On montre que la stabilité de l'écoulement est donnée par deux valeurs propres : l'une correspondant à celle provenant des deux couches extérieures tandis que l'autre traduit l'influence de la couche mince. De cette manière, on donne des règles simples permettant de choisir les constantes rhéologiques du fluide central de manière à ce que la stabilité soit donnée par la valeur propre associée aux deux couches extérieures.

Mots clés : multicouche, interface, stabilité, longues ondes, Oldroyd-B, Poiseuille

La coextrusion consiste à faire écouler dans un canal plusieurs polymères fondus ayant des caractéristiques différentes, afin d'obtenir un produit stratifié multicouche réunissant les propriétés de chacun des polymères constituants. La nécessité d'un "collage" parfait entre ces couches impose l'introduction d'une fine couche intermédiaire ayant un rôle de liant. Toutefois, lors de ce procédé, des instabilités localisées à l'interface entre les différents produits se manifestent sous la forme de vagues et altèrent les propriétés du produit.

Cela a conduit à de nombreux articles sur la stabilité de l'écoulement de Poiseuille constitué de deux (Su et Khomami, 1992 ; Chen, 1991 ; Joseph et Renardy, 1992 ; Laure *et al.*, 1997) ou trois couches (Scotto et Laure, 1998). Cependant, le cas industriel le plus simple correspondant à deux polymères collés par un liant n'a jamais été étudié et constitue le sujet de cette note.

Les notations reprennent celles utilisées dans (Scotto et Laure, 1998) pour le cas trois-couche. On considère trois fluides qui suivent la loi constitutive Oldroyd-B. L'écoulement stationnaire de Poiseuille correspond à des interfaces planes situées à $y = d_1$ et d_2. La hauteur du canal est normalisée à un et l'épaisseur de la couche centrale δ est supposée très petite par rapport aux épaisseurs des deux couches extérieures ($\delta \ll d_1$ et $\delta \ll 1-d_2$). Les paramètres du problème sont donc le nombre de Reynolds Re, les viscosités η_k, les élasticités We_k et les composantes polymériques α_k des trois fluides.

On regarde la stabilité temporelle de l'écoulement de Poiseuille par rapport aux perturbations périodiques dans la direction de l'écoulement. Elles sont de la forme (1), où q est le nombre d'onde et σ le taux d'amplification.

Dans nos articles précédents (Laure *et al.*, 1997 ; Scotto et Laure, 1998), on a montré que l'étude aux longues ondes (c'est à dire l'étude de la stabilité pour des perturbations ayant un nombre d'onde q qui tend vers 0), permettait sous certaines conditions de donner des indications sur la stabilité par rapport à des perturbations ayant un nombre d'onde modéré. Il faut pour cela que le nombre de Reynolds soit très faible, que la période des vagues observées soit supérieure ou égale à l'épaisseur du canal (les deux premières conditions sont habituellement vérifiées lors des expériences de coextrusion) et que le profil de Poiseuille soit convexe.

L'étude aux longues ondes est basée sur un développement des valeurs propres σ de la forme (2) et il permet de découpler le calcul de l'écoulement de celui des coefficients c_0 et c_1 intervenant dans (2).

De plus, on effectue un développement limité de ces coefficients c_0 et c_1 par rapport à l'épaisseur δ de la couche centrale. Le calcul a été effectué pour des fluides newtoniens et viscoélastiques à l'aide de Maple (voir le détail des calculs dans (Scotto, 1998)).

Si les fluides ont des viscosités différentes, le premier coefficient c_0 est réel et la stabilité de l'écoulement est dirigée par l'ordre le plus bas des coefficients c_1 des deux valeurs propres (mode "C" et "L") correspondant aux deux interfaces. Le mode "C" (3) et (7) est exactement le mode obtenu par Laure *et al.* (1997) lors de l'étude aux longues ondes de l'écoulement bi-couche composé des fluides extérieurs. Le mode "L" (4) et (8) traduit l'influence de la couche mince au centre sur la stabilité de l'écoulement. Finalement, on montre :

- La stabilité asymptotique de l'écoulement de fluides newtoniens est assurée si la couche extérieure la moins visqueuse est aussi la plus fine et que les viscosités des fluides varient de façon monotone avec la position relative des fluides. Une telle configuration correspond à un profil des vitesses convexe (Charru et Fabre, 1994).
- La stabilité de l'écoulement trois-couche de fluides Oldroyd-B est reliée à celle de l'écoulement bi-couche, si la couche centrale a une elasticité plus petite que les deux couches extérieures et une viscosité comprise entre celles des deux autres couches.

1 Introduction

The stability of multilayer Poiseuille flow has been recently studied by numerous authors due to the growing industrial importance of multilayer products in polymer processing. One of porposes was to give practical tools in order to prevent wavy interfaces which occur in the coextrusion process.

In this paper, we focus our attention to the three-layer plane Poiseuille flow when the inner layer is very thin. This configuration is related to the industrial process in which two polymers having different properties are sticked together by a tie layer. Unless this large amount of literature rather devoted on the stability of the two-layer or concentric flows, the role of an additional inner layer on the stability of the flow was not clearly analysed. As shown in (Scotto et Laure, 1998), the general three-layer case is rather straigforward to handle. That is why we would exhibit the conditions which allow to neglect this inner thin layer in the stability analysis. In this way, the stability analysis becomes simpler and one can directly apply on the two outer layers the very known theoretical results of the stability of the two-layer Poiseuille flow (Su et Khomami, 1992 ; Chen, 1991 ; Joseph et Renardy, 1992 ; Laure *et al.*, 1997).

2 Notations

We used the same notations as those introduced in (Scotto et Laure, 1998). The geometry consists of three distinct superposed Oldroyd-B fluids flowing through a parallel plate die. Therefore, we have to deal with two interfaces located at $y = d_1$ and d_2. Each layer k ($k \in \{1, 2, 3\}$) follows the momentum conservation equation, the incompressibility and Oldroyd-B model for the stress tensor. At the two interfaces, the standard immiscibility conditions and the continuity of the jump of the stress tensor are used.

The space variables are scaled by the total thickness d of the channel. Therefore, the sizes of the layer 1 and 3 are respectively d_1 and $1 - d_2$. In the sequel, we look at the particular case for which $\delta = d_2 - d_1 \ll d_1$ and $\delta \ll 1 - d_2$. The other parameters are the Reynolds number Re, the three Weissenberg numbers We_k, the polymeric components α_k, the viscosities η_k. Two other parameters are also used in order to make comparisons with two-layer flow composed by the layers 1 and 3; namely, the viscosities ratio $m = \eta_3/\eta_1$ and thicknesses ratio $\epsilon = (1 - d_1)/d_1$ (Laure *et al.*, 1997).

The Poiseuille solution corresponds to flat interfaces and parabolic velocity profile whose the coefficients depend on viscosities and thicknesses ratio. As usual, we look at the temporal stability analysis of this profile with respect to two dimensional periodic perturbations having the following form

$$f(x,y,t) \;=\; \widetilde{f}(y)\,\exp[iqx+\sigma t] \tag{1}$$

where f represents any variable, q the wavenumber in the flow direction and σ the growth rate.

3 Longwave analysis

In our previous work (Scotto et Laure, 1998) on Oldroyd-B fluids, we have shown that the stability of the three-layer Poiseuille profile is given by two kind of eigenvalues: two "longwave" eigenvalues such that $\sigma(q=0)=0$, and, six "shortwave" eigenvalues satisfaying $\sigma(q=0)=-1/We_k$. The latter eigenvalue can give rised to unstable mode at large wavenumber if the Weissenberg numbers are large. Nevertheless, wavy interfaces observed in the experiments correspond rather to moderate or small wavenumber (that also means that a spatial period of the disturbance is equal or greater than the total thickness of the die). Moreover, we have also found that the real part of the "longwave" eigenvalue remains negative if the Poiseuille profile is both convex and stable with respect to longwave disturbances.

Therefore, we restrict our computations to longwave analysis which seems to give the most revelant informations for the coextrusion process. Thanks to this remark, we only have to deal with two "longwave" eigenvalues which have the following expansion around $q=0$,

$$\sigma = -i\,q\,(c_0 \;+\; i\,q\,c_1 \;+\; ...) \tag{2}$$

The longwave analysis consists in the expansion of the other variables with respect to the wavenumber q, the substitution of these expansions into the whole system and the collecting of terms of like order. In this way, the velocity field is solution of a sequence of "o.d.e", whereas the coefficients c_0 and c_1 in (2) are given by the two kinematic conditions.

Moreover, we also expand the eigenvalues and the variable in function of the small parameter δ (δ is the thickness of the inner layer) and the lowest order terms of expansion of c_0 and c_1 are analysed. All these computations are made with Maple and all the details can be found in (Scotto, 1998).

If the fluids have different viscosities, the coefficients c_0 are real and the lowest order terms of these coefficients depend only on the two parameters m and ϵ. Therefore the stability is given by the sign of the two coefficients c_1. In the sequel, the two coefficients c_1 are labelled by "C" and "L". That means that the first term of c_{1_C} depends only on the two parameters m and ϵ whereas the first term of the rheologic constants of the tie layer interneves only in the lowest order term of c_{1_L}.

4 Newtonian fluids

First, we discuss the simpler case corresponding to three newtonian fluids. One gets the following expansions with respect to the small parameter δ:

$$c_{1_C} = i\,Re\,J_1(\epsilon, m) + \mathcal{O}(\delta) \quad (3)$$

$$c_{1_L} = i\,Re\,J_{1_L}(\eta_3, \eta_2, \eta_1, d_1)\delta^2 + \mathcal{O}(\delta^3) \quad (4)$$

The function J_1 depends only of the two parameter m and ϵ and is similar to this written in (Laure *et al.*, 1997) (there is a constant factor between the two expressions due to a different scaling). Therefore, we recover the stability criteria of the two-layer Poiseuille flow composed by the fluid 1 and 3 (Yiantsios et Higgins, 1988),

$$sign(J_1) = sign\left((m-1)\,(m-\epsilon^2)\right) \quad (5)$$

that means that the less viscous fluid has also to be in the thinner layer.

Moreover, we found that the sign of the function J_{1_L} is given by the following simple rule:

$$sign(J_{1_L}) = sign\left((m-1)\,(m-\epsilon^2)(\eta_3-\eta_2)(\eta_2-\eta_1)\right) \quad (6)$$

Then, it follows that if the inner layer has a viscosity between the two outer viscosities, we can neglect this layer in the discussion and only take into account the two main outer layers. Note that if the configuration composed by the outer fluids is stable ($J_1 < 0$) and the viscosity η_2 is between η_1 and η_3 then the Poiseuille velocity profile is convex (Charru et Fabre, 1994 ; Scotto et Laure, 1998).

5 Coextrusion of Oldroyd-B fluids

In the coextrusion conditions, we can assume that the Reynolds number is very small ($Re \sim 0$). So, the stability with respect to longwave stability analysis is given

$$c_{1_C} = i\,\alpha_1 We_1\,[J_4(\epsilon, m) + (M_\lambda - 1)J_5(\epsilon, m)] + \mathcal{O}(\delta) \quad (7)$$

$$c_{1_L} = i\,(\epsilon-1)\,[\alpha_1 We_1(\eta_2-\eta_3) + \alpha_2 We_2(\eta_3-\eta_1) + \alpha_3 We_3(\eta_1-\eta_2)]\,\frac{F(\epsilon, m)}{\eta_2}\,\delta + \mathcal{O}(\delta^2) \quad (8)$$

where $M_\lambda = \alpha_3 We_3/\alpha_1 We_1$ represents the elastic stratification between the two outer layers. The two functions J_4 and J_5 depend only on the parameters m and ϵ, and the function $J_\lambda = J_4 + (M_\lambda - 1)J_5(\epsilon, m)$ is studied numerically in (Laure *et al.*, 1997).

For the eigenvalue c_{1_L}, the inertial effects (represented by the Reynolds number) are less important than the elastic influence as the first contribution of the Reynolds number occurs at order δ^2 (see equation 4). We found that the function $F(\epsilon, m)$ is always positive (Scotto, 1998), then the sign of c_{1_L} depends on the function

$$J_{2_L} = (\epsilon-1)\,[\alpha_1 We_1(\eta_2-\eta_3) + \alpha_2 We_2(\eta_3-\eta_1) + \alpha_3 We_3(\eta_1-\eta_2)]$$

As noted in (Scotto et Laure, 1998), the Poiseuille velocity profile has to be convex in order to insure the stability with respect to disturbance having a moderate wavenumber. Therefore, if $\epsilon < 1$, this latter condition yields to $\eta_3 < \eta_2 < \eta_1$ and finally one gets that $J_{2_L} < 0$ if $\alpha_2 \, We_2 < Min(\alpha_1 \, We_1, \alpha_3 \, We_3)$. The same kind of result is obtained for $\epsilon > 1$. Finally, if the inner viscosity is between the two outer viscosities and the inner elasticity is smaller than the two outer elasticities, the stability is given by the rheological properties of the two outer layers (namely the two functions J_1 and J_λ has to be both negative).

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