Correction 5 : Eléments finis mixtes

On veut résoudre le problème de Stokes

$$\nabla.\sigma + f = 0$$

avec $\sigma = 2\mu(\nabla u + (\nabla u)^T) - pE$ et $\nabla.u = 0$. Sous forme développée

$$\sigma = 2\mu \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial x} \end{pmatrix}$$

On écrit l'équation sous forme développée :

$$\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right) - \frac{\partial p}{\partial x} + f_x = 0$$

$$\frac{\partial}{\partial x}\left(\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right) + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}\right) - \frac{\partial p}{\partial y} + f_y = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

On a la formulation variationnelle si $\phi = (w_1, w_2, q)$ est la fonction test

$$\begin{aligned}
\int_\Omega \left[2\mu\frac{\partial u}{\partial x}\frac{\partial w_1}{\partial x} + \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\frac{\partial w_1}{\partial y} - p\ \frac{\partial w_1}{\partial x} - f_x\ w_1\right]\ dx\ dy - \int_\Gamma \tau_x w_1\ ds &= 0 \\
\int_\Omega \left[\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\frac{\partial w_2}{\partial x} + 2\mu\frac{\partial v}{\partial y}\frac{\partial w_2}{\partial y} - p\ \frac{\partial w_2}{\partial x} - f_y\ w_2\right]\ dx\ dy - \int_\Gamma \tau_y w_2\ ds &= 0 \\
\int_\Omega \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] q\ dx\ dy &= 0
\end{aligned}$$

avec

$$\begin{aligned}
\tau_x &= \left(2\mu\frac{\partial u}{\partial x} - p\right)\ n_x + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\ n_y \\
\tau_y &= \left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\ n_x + 2\mu\frac{\partial v}{\partial x} - p\right)\ n_y
\end{aligned}$$

Si on considère un carré avec les conditions aux limites

$$u(x,0) = v(x,0) = v(x,1) = 0 \quad \text{et} \quad u(x,1) = 1$$

$$u(0,y) = v(0,y) = u(1,y) = v(1,y) = 0$$

τ_x et τ_y n'interviennent pas car on prend des fonctions tests qui s'annulent sur les cotés.

Sur chaque élement, on utilise des fonctions d'interpolation differentes pour la vitesse ($N_j^u(x,y)$) et la pression ($N_j^p(x,y)$)

$$u = \sum_{j=1}^{n} u_j N_j^u \quad ; \quad v = \sum_{j=1}^{n} v_j N_j^u \quad ; \quad p = \sum_{j=1}^{m} p_j N_j^p$$

Sur l'élement, on obtient le système local

$$\begin{bmatrix} K^{11} & K^{12} & K^{13} \\ K^{21} & K^{22} & K^{23} \\ K^{31} & K^{32} & 0 \end{bmatrix} \begin{bmatrix} \{u\} \\ \{v\} \\ \{p\} \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \\ 0 \end{bmatrix}$$

avec

$$K_{ij}^{11} = \int_\Omega \mu \left(2 \frac{\partial N_j^u}{\partial x} \frac{\partial N_i^u}{\partial x} + \frac{\partial N_j^u}{\partial y} \frac{\partial N_i^u}{\partial y} \right) \; d_x \; d_y \;\; ; \;\; K_{ij}^{12} = \int_\Omega \mu \frac{\partial N_j^u}{\partial x} \frac{\partial N_i^u}{\partial y} \, d_x \, d_y \;\; ; \;\; K_{ij}^{13} = - \int_\Omega N_j^p \, \frac{\partial N_i^u}{\partial x} \, d_x \, d_y$$

$$K_{ij}^{21} = \frac{\partial N_j^u}{\partial y} \frac{\partial N_i^u}{\partial y} \;\; ; \;\; K_{ij}^{21} = \int_\Omega \mu \left(\frac{\partial N_j^u}{\partial x} \frac{\partial N_i^u}{\partial x} + 2 \frac{\partial N_j^u}{\partial y} \frac{\partial N_i^u}{\partial y} \right) \; d_x \; d_y \;\; ; \;\; K_{ij}^{23} = - \int_\Omega N_j^p \, \frac{\partial N_i^u}{\partial y} \; d_x \; d_y$$

$$K_{ij}^{31} = \int_\Omega \frac{\partial N_j^u}{\partial x} N_i^p \; d_x \; d_y \;\; ; \;\; K_{ij}^{32} = \int_\Omega \frac{\partial N_j^u}{\partial y} N_i^p \; d_x \; d_y$$