An Introduction to Heterogeneous Domain Decomposition, and a New Approach based on Factorization

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Heterogeneous domain decomposition methods, a terminology that was introduced in [22], are domain decomposition methods to solve problems which are using different models in different parts of the domain. There are two reasons for using different models for different regions: the first one is that there are problems where the physics is really different in different regions, and hence different models need to be used, for example in fluid-structure interaction, see [21, 5] and references therein. A very important area of application in this case is also the simulation of the cardiovascular system [10, 12], with the interaction between the blood and the arterial wall.

A second main area of heterogeneous domain decomposition methods is when homogeneous objects are simulated, but the partial differential equation modeling the physical phenomenon is too expensive to solve over the entire domain, and a simpler, less expensive model would suffice in most of the domain to reach the desired accuracy; air flow around an airplane is a typical example, where viscous effects are important close to the airplane, but can be neglected further away, see the early publication [6], and also [23, 4] and the references therein. An automatic approach for neglecting the diffusion in parts of the domain is the χ-formulation [2], and there are also techniques based on virtual control originating in [6], see [1] for the case with overlap, and [19] for the case without, and also [7, 8] for virtual control with variational coupling conditions. Transmission conditions for this situation have been developed in the seminal paper [17], but with the first situation described above in mind, i.e. there is indeed a viscous and an inviscid physical domain, and the coupling conditions are obtained by a limiting process as the viscosity goes to zero, see also [18], and [3] for an innovative correction layer. An important situation that fits also into this second area is the coupling of equations across dimensions, for example the blood flow in the artery can be modeled by a one dimensional model, but in the heart, it needs to be three dimensional, see for example [11].

After an introduction to heterogeneous domain decomposition methods, I will focus on the second case, i.e. where the global physical problem is indeed the same across the entire domain, but we want to use a heterogeneous domain decomposition method to be able to use a cheaper model in parts of the domain where the full model is not needed. Using as a model problem the coupling of an advection diffusion reaction equation to a pure advection reaction equation, I will first define a rigorous mathematical criterion which permits to measure the quality of a heterogeneous domain decomposition method [15]. I will then give a comparison of several heterogeneous domain decomposition methods from the literature, and also introduce a new approach based on operator factorization [20], whose underlying idea goes back the the PhD thesis of Dubach [9], see also [14]. The results I will show for the steady case have appeared in [13], and for the unsteady case, see [16].

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