Master MathMod,
University Nice Sophia Antipolis

Exercize : finite volume discretization of the 1D Laplacian with Dirichlet and Neumann boundary conditions.

We consider the following problem

$$(P) \begin{cases} -u''(x) = f(x) \quad \text{on } (0,1), \\ u(0) = u_D, \\ -u'(1) = g, \end{cases}$$

which has a unique solution in $H^1(0,1)$ for all $u_D \in \mathbb{R}$, $g \in \mathbb{R}$, $f \in L^2(0,1)$.

We consider the following subdivision of the interval $(0,1)$ with $N+1$ points :

$$x_{1/2} = 0 < x_{3/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N-1/2} < x_{N+1/2} = 1.$$

Keeping the notations of the course, the finite volume discretization of the interval $(0,1)$ consists of the set of N cells $\kappa_i = (x_{i-1/2}, x_{i+1/2})$ for $i = 1, \cdots, N$, and of the cell centers $x_i = \frac{x_{i-1/2}+x_{i+1/2}}{2}$ for $i = 1, \cdots, N$. We also set $x_0 = 0$ and $x_{N+1} = 1$, $h_{i+1/2} = |x_{i+1} - x_i|$ for $i = 0, \cdots, N$, and $h_i = |x_{i+1/2} - x_{i-1/2}|$ for $i = 1, \cdots, N$. Finally, we set $h = \max_{i=1,\cdots,N} h_i$.

(1) Let us consider the N discrete unknowns u_i approximating $u(x_i)$ for $i = 1, \cdots, N$. Write the discrete fluxes $f_{i+1/2}$ approximating $-u'(x_{i+1/2})$, $i = 0, \cdots, N$, and taking into account the boundary conditions for $i = 0$ and $i = N$.
Write the finite volume discretization of (P) consisting of N discrete conservation equations on the cells κ_i, $i = 1, \cdots, N$ using the previous fluxes.
Let $u_i^0 = u_i - u_D$, $i = 1, \cdots, N$. Write the previous finite volume scheme for the unknowns $u_i^0, i = 1, \cdots, N$

(2) Let V_h be the vector space of cellwise constant functions. We denote by v_i the value of v_h on the cell κ_i for $i = 1, \cdots, N$. For all $v_h, w_h \in V_h \times V_h$, we define the discrete scalar product and discrete norms

$$\|v_h\|_{0,h} = \|v_h\|_{L^2(0,1)} = \Big(\sum_{i=1}^{N} h_i v_i^2\Big)^{1/2},$$

$$< v_h, w_h >_{1,h} = \frac{(0 - v_1)(0 - w_1)}{h_{1/2}} + \sum_{i=1}^{N-1} \frac{(v_i - v_{i+1})(w_i - w_{i+1})}{h_{i+1/2}}.$$

and

$$\|v_h\|_{1,h} = \Big(< v_h, v_h >_{1,h}\Big)^{1/2}.$$

Prove the following discrete Poincaré inequality : for all $v_h \in V_h$

$$\|v_h\|_{0,h} \le \|v_h\|_{1,h}.$$

Prove the following trace inequality : for all $v_h \in V_h$

$$|v_N| \le \|v_h\|_{1,h}.$$

(3) Prove that the finite volume discretization of (P) is equivalent to the following discrete variational formulation : find $u_h^0 \in V_h$ such that

$$(FVP) \qquad < u_h^0, v_h >_{1,h} = \int_0^1 f(x) v_h(x) dx - g\ v_N,$$

for all $v_h \in V_h$. Using the previous Poincaré and trace inequalities, deduce that the discrete solution $u_h^0 \in V_h$ of (FVP) satisfies the following a priori estimate

$$\|u_h^0\|_{1,h} \leq |g| + \|f\|_{L^2(0,1)}.$$

What can be deduced for the finite volume scheme solution ?

(4) For all $v \in C^2[0,1]$, let us define the fluxes residuals

$$r_{i+1/2}(v) = \frac{v(x_i) - v(x_{i+1})}{h_{i+1/2}} + v'(x_{i+1/2}),$$

for all $i = 0, \cdots, N-1$. Show that there exists a constant $C(v)$ independent on h and such that

$$\max_{i=0,\cdots,N-1} |r_{i+1/2}(v)| \leq C(v) h.$$

Assuming that the solution u of (P) is in $C^2[0,1]$, prove that

$$\|e_h\|_{1,h} \leq C(u) h,$$

with $e_h(x) = u(x_i) - u_i$ on κ_i, $i = 1, \cdots, N$.