Course on Finite Volume discretization of PDEs,
University Nice Sophia Antipolis

**Exercize** : finite volume discretization of the 1D Laplacian with Dirichlet and Neumann boundary conditions.

We consider the following problem

$$(P)\left\{\begin{array}{r}-u''(x)=f(x)\quad\text{on }(0,1),\\ u(0)=u_D,\\ -u'(1)=g,\end{array}\right.$$

which has a unique solution in $H^1(0,1)$ for all $u_D \in \mathbb{R}$, $g \in \mathbb{R}$, $f \in L^2(0,1)$.

We consider the following subdivision of the interval $(0,1)$ with $N+1$ points :

$$x_{1/2}=0<x_{3/2}<\cdots<x_{i-1/2}<x_{i+1/2}<\cdots<x_{N-1/2}<x_{N+1/2}=1.$$

Keeping the notations of the course, the finite volume discretization of the interval $(0,1)$ consists of the set of $N$ cells $\kappa_i=(x_{i-1/2},x_{i+1/2})$ for $i=1,\cdots,N$, and of the cell centers $x_i=\frac{x_{i-1/2}+x_{i+1/2}}{2}$ for $i=1,\cdots,N$. We also set $x_0=0$ and $x_{N+1}=1$, $h_{i+1/2}=|x_{i+1}-x_i|$ for $i=0,\cdots,N$, and $h_i=|x_{i+1/2}-x_{i-1/2}|$ for $i=1,\cdots,N$. Finally, we set $h=\max_{i=1,\cdots,N} h_i$.

(1) Let us consider the $N$ discrete unknowns $u_i$ approximating $u(x_i)$ for $i=1,\cdots,N$. Write the discrete fluxes $f_{i+1/2}$ approximating $-u'(x_{i+1/2})$, $i=0,\cdots,N$, and taking into account the boundary conditions for $i=0$ and $i=N$.
Write the finite volume discretization of (P) consisting of $N$ discrete conservation equations on the cells $\kappa_i$, $i=1,\cdots,N$ using the previous fluxes.
Let $u_i^0=u_i-u_D$, $i=1,\cdots,N$. Write the previous finite volume scheme for the unknowns $u_i^0, i=1,\cdots,N$

**Corrigé** : The numerical fluxes are defined by

$$\begin{aligned}F_{1/2}(u_h)&=\frac{u_D-u_1}{h_{1/2}},\\ F_{i+1/2}(u_h)&=\frac{u_i-u_{i+1}}{h_{i+1/2}},\ i=1,\cdots,N-1,\\ F_{N+1/2}(u_h)&=g.\end{aligned}$$

and the finite volume conservation equations in each cell are

$$F_{i+1/2}(u_h)-F_{i-1/2}(u_h)=h_if_i,\ i=1,\cdots,N,$$

with $f_i=\frac{1}{h_i}\int_{\kappa_i}f(x)dx$.

We remark that

$$F_{1/2}(u_h^0) == \frac{0-u_1^0}{h_{1/2}},$$
$$F_{i+1/2}(u_h^0) = \frac{u_i^0-u_{i+1}^0}{h_{i+1/2}},\ i=1,\cdots,N-1,$$
$$F_{N+1/2}(u_h^0) = g.$$

and that

$$F_{i+1/2}(u_h^0) - F_{i-1/2}(u_h^0) = h_i f_i,\ i=1,\cdots,N,$$

(2) Let $V_h$ be the vector space of cellwise constant functions. We denote by $v_i$ the value of $v_h$ on the cell $\kappa_i$ for $i=1,\cdots,N$. For all $v_h, w_h \in V_h \times V_h$, we define the discrete scalar product and discrete norms

$$\|v_h\|_{0,h} = \|v_h\|_{L^2(0,1)} = \Big(\sum_{i=1}^{N} h_i v_i^2\Big)^{1/2},$$

$$< v_h, w_h >_{1,h} = \frac{(0-v_1)(0-w_1)}{h_{1/2}} + \sum_{i=1}^{N-1} \frac{(v_i-v_{i+1})(w_i-w_{i+1})}{h_{i+1/2}}.$$

and

$$\|v_h\|_{1,h} = \Big(< v_h, v_h >_{1,h}\Big)^{1/2}.$$

Prove the following discrete Poincaré inequality : for all $v_h \in V_h$

$$\|v_h\|_{0,h} \le \|v_h\|_{1,h}.$$

Prove the following trace inequality : for all $v_h \in V_h$

$$|v_N| \le \|v_h\|_{1,h}.$$

**Corrigé :** Let us set for $v_h \in V_h$, $v_0 = 0$. Then

$$|v_i| = |\sum_{j=0}^{i-1} v_{j+1} - v_j| \le \sum_{j=0}^{N-1} |v_{j+1}-v_j| \le \|v_h\|_{1,h}\Big(\sum_{j=0}^{N-1} h_{j+1/2}\Big)^{1/2} \le \|v_h\|_{1,h}.$$

It results that

$$\|v_h\|_{0,h}^2 \le \|v_h\|_{1,h}^2 (\sum_{i=1}^{N} h_i) \le \|v_h\|_{1,h}^2,$$

hence

$$\|v_h\|_{0,h} \le \|v_h\|_{1,h}.$$

(3) Prove that the finite volume discretization of (P) is equivalent to the following discrete variational formulation : find $u_h^0 \in V_h$ such that

$$(FVP) \quad < u_h^0, v_h >_{1,h} = \int_0^1 f(x) v_h(x) dx - g\ v_N,$$

for all $v_h \in V_h$. Using the previous Poincaré and trace inequalities, deduce that the discrete solution $u_h^0 \in V_h$ of (FVP) satisfies the following a priori estimate

$$\|u_h^0\|_{1,h} \leq |g| + \|f\|_{L^2(0,1)}.$$

What can be deduced for the finite volume scheme solution ?

**Corrigé** : Let us set for conveniency $v_0 = 0$. Multiplying the conservation equation in cell $i$ by $v_i$ and summing over $i = 1, \cdots, N$ we obtain that

$$\sum_{i=1}^{N} v_i F_{i+1/2}(u_h^0) - \sum_{i=1}^{N} v_i F_{i-1/2}(u_h^0) = \sum_{i=1}^{N} v_i h_i f_i.$$

Let us note that

$$\sum_{i=1}^{N} v_i h_i f_i = \int_0^1 f(x) v_h(x) dx,$$

and that

$$\sum_{i=1}^{N} v_i F_{i-1/2}(u_h^0) = \sum_{i=0}^{N-1} v_{i+1} F_{i+1/2}(u_h^0)$$

Using that $v_0$ is set to 0 we have that

$$\sum_{i=1}^{N} v_i F_{i+1/2}(u_h^0) = \sum_{i=0}^{N-1} v_i F_{i+1/2}(u_h^0) + g v_N.$$

It results that

$$< u_h^0, v_h >_{1,h} = \int_0^1 f(x) v_h(x) dx - g\ v_N.$$

(4) For all $v \in C^2[0,1]$, let us define the fluxes residuals

$$r_{i+1/2}(v) = \frac{v(x_i) - v(x_{i+1})}{h_{i+1/2}} + v'(x_{i+1/2}),$$

for all $i = 0, \cdots, N-1$. Show that there exists a constant $C(v)$ independent on $h$ and such that

$$\max_{i=0,\cdots,N-1} |r_{i+1/2}(v)| \leq C(v) h.$$

Assuming that the solution $u$ of (P) is in $C^2[0,1]$, prove that

$$\|e_h\|_{1,h} \leq C(u) h,$$

with $e_h(x) = u(x_i) - u_i$ on $\kappa_i$, $i = 1, \cdots, N$.

**Corrigé** : Note that

$$-r_{i+1/2}(v) = \frac{1}{h_{i+1/2}} \int_{x_i}^{x_{i+1}} (v'(x) - v'(x_{i+1/2})) dx.$$

Using by Taylor expansion that for all $x \in (x_i, x_{i+1})$ one has

$$|v'(x) - v'(x_{i+1/2})| \leq \sup_{y \in (x_i, x_{i+1})} |v''(y)| |x - x_{i+1/2}| \leq \sup_{y \in (0,1)} |v''(y)| h_{i+1/2},$$

we deduce that

$$|r_{i+1/2}(v)| \le C(v)h,$$

with $C(v) = \sup_{y\in(0,1)} |v''(y)|$.
Let us derive an equation for the error $e_h$. Integrating the equation $-u''(x) = f(x)$ over the cell $\kappa_i$ we obtain that

$$-u'(x_{i+1/2}) - (-u'(x_{i-1/2})) = h_i f_i.$$

From the definition of the flux consistency errors $r_{i+1/2}(u)$ we obtain for $i = 1, \cdots, N-1$ that

$$\frac{u(x_i) - u(x_{i+1})}{h_{i+1/2}} - \frac{u(x_{i-1}) - u(x_i)}{h_{i-1/2}} = h_i f_i + r_{i+1/2}(u) - r_{i-1/2}(u).$$

For $i = N$ we obtain by setting $r_{N+1/2}(u) = 0$ that

$$g - \frac{u(x_{N-1}) - u(x_N)}{h_{N-1/2}} = h_N f_N + r_{N+1/2}(u) - r_{N-1/2}(u).$$

Substrating the finite volume equations for $u_h$ from these equations we get, setting $e_0 = 0$ that

$$F_{i+1/2}(e_h) - F_{i-1/2}(e_h) = r_{i+1/2}(u) - r_{i-1/2}(u),$$

for all $i = 1, \cdots, N-1$ and

$$0 - F_{N-1/2}(e_h) = r_{N+1/2}(u) - r_{N-1/2}(u).$$

Multiplying each equation by $e_i$ and summing over $i = 1, \cdots, N$ and using that $e_0 = 0$ and that $r_{N+1/2} = 0$, we obtain that

$$\|e_h\|_{1,h}^2 = \sum_{i=1}^{N} e_i(r_{i+1/2}(u) - r_{i-1/2}(u)) = \sum_{i=0}^{N-1} (e_i - e_{i+1}) r_{i+1/2}(u).$$

By Cauchy Scharwz inequality and from the flux consistency error estimate we get that

$$\|e_h\|_{1,h}^2 \le \|e_h\|_{1,h} \Big( \sum_{i=0}^{N-1} (r_{i+1/2}(u))^2 h_{i+1/2} \Big)^{1/2} \le C(u)h\|e_h\|_{1,h}.$$

It results that

$$\|e_h\|_{1,h} \le C(u)h.$$