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Finite Volume discretization of the Black and Scholes model

The Black and Scholes model computes the price of the option $P(S,t)$ as function of the price of the asset S and of time $t \in (0,T)$. To fix ideas we will consider an European call option with the following parameters :

- $T > 0$: the maturity time at which the owner of the option can buy the asset
- $K > 0$: the strike defined as the price at which the call option gives the right to buy the asset
- σ : the volatility of the asset price model
- r : the free risk interest rate of the market

The Black and Scholes model is defined by the following linear unstationary convection diffusion reaction equation :

$$\left\{\begin{array}{lr} \partial_t P(S,t) + \frac{\sigma^2}{2} S^2 \partial_{S^2} P(S,t) + rS\partial_S P(S,t) - rP(S,t) = 0 & \text{on } (0,S_{max}) \times (0,T), \\ P(S,T) = \max(0, S-K), & \text{on } (0,S_{max}) \\ P(S_{max},t) = f(S_{max},t), & \text{on } (0,T), \end{array}\right.$$

where $S_{max} > K$ is given and $f(S_{max},t)$ is the right side boundary condition determined by financing reasoning. In our case it will be defined by

$$f(S,t) = S - Ke^{-r(T-t)}.$$

Note that no boundary condition is needed at the left side $S = 0$ since the equation reduces to $\partial_t P(S,t) = rP(S,t)$ which imposes

$$P(0,t) = e^{r(t-T)} P(0,T) = e^{r(t-T)} \max(0,-K) = 0.$$

To write the finite volume discretization of the Black and Scholes model, one must first reformulate the PDE in conservative form :

$$\partial_t P(S,t) + \partial_S \Big(\frac{\sigma^2}{2} S^2 \partial_S P(S,t) + (r-\sigma^2) S P(S,t) \Big) - (2r - \sigma^2) P(S,t) = 0.$$

For conveniency we also make the change of of variable $\tau = T - t$ and of function $V(S,\tau) = P(S, T-\tau)$ to obtain our (BS) model :

$$\text{(BS)} \quad \left\{\begin{array}{l} \partial_\tau V(S,\tau) + \partial_S \Big(-\frac{\sigma^2}{2} S^2 \partial_S V(S,\tau) - (r-\sigma^2) S V(S,\tau) \Big) + (2r-\sigma^2) V(S,\tau) = 0 \text{ on } (0,S_{max}) \times (0,T), \\ V(S,0) = \max(0, S-K) \text{ on } (0,S_{max}) \\ V(S_{max},\tau) = S - Ke^{-r\tau} \text{ on } (0,T). \end{array}\right.$$

We consider the following subdivision of the interval $(0, S_{max})$ with $N+1$ points :

$$S_{1/2} = 0 < S_{3/2} < \cdots < S_{i-1/2} < S_{i+1/2} < \cdots < S_{N-1/2} < S_{N+1/2} = S_{max}.$$

Keeping the notations of the course, the finite volume discretization of the interval $(0,1)$ consists of the set of N cells $\kappa_i = (S_{i-1/2}, S_{i+1/2})$ for $i = 1, \cdots, N$, and of the cell centers $S_i = \frac{S_{i-1/2}+S_{i+1/2}}{2}$

for $i = 1, \cdots, N$. We also set $S_0 = 0$ and $S_{N+1} = S_{max}$, $h_{i+1/2} = |S_{i+1} - S_i|$ for $i = 0, \cdots, N$, and $h_i = |S_{i+1/2} - S_{i-1/2}|$ for $i = 1, \cdots, N$.

The time discretization is denoted by $\tau^0 = 0 < \tau^1 < \tau^2 \cdots \tau^n < \tau^{n+1} < \cdots < \tau^M = T$, and we set $\Delta\tau^n = \tau^n - \tau^{n-1} > 0$ for all $n \geq 1$.

(1) Write the finite volume discretization of the (BS) model on the previous space time discretization of $(0, S_{max}) \times (0, T)$ using a θ scheme for the time integration and an upwind scheme for the convection term
(2) Implement the code in scilab
(3) Test the code for the following values of the parameters
- $T = 0.25$
- $K = 100$
- $\sigma = 0.3$
- $r = 0.2$
- $S_{max} = 200$