# - Numerical simulation of a stratigraphic model - two lithologies

Two lithologies stratigraphic model

The aim of this work is to simulate the formation of clastic sedimentary basins at large space and time scales. In this case we have two lithologies, $i = 1, 2$. Let us denote

- $\Omega = [0, L_x]$: horizontal extension of the basin

- $(0, t_f)$ is the time scale with $t_f > 0$

- $h(x, t)$ is the sediments layer thickness with $h(x, 0) = h_{\text{init}}(x)$

- $H_{\text{m}}(t)$ is the sea level

- $b(x, t) = H_{\text{m}}(t) - h(x, t)$ is called the bathymetry

- z is the vertical axis

- $c_i(x, z, t)$ is the composition of the sediments in the lithology $i = 1, 2$

- $c_i^s(x, t)$ is the surface concentration

These notations are illustrated on the figure below :
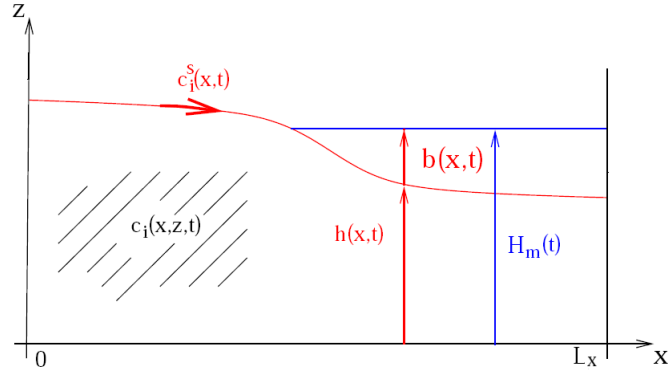
The basin sediment is described by its concentrations $0 \leqslant c_i \leqslant 1$ in the lithology $i = 1, 2$ defined in the domain $\mathcal{B} = \{(x, z, t) \mid (x, t) \in \Omega \times (0, t_f), z \in (-\infty, h(x, t))\}$. Since the compaction is not considered, there is no evolution in time of the concentrations c_i inside the basin. Thus the sediments transported at the surface of the basin, i.e deposited in case of **sedimentation**, are described by their surface concentrations $0 \leqslant c_i^s \leqslant 1$. This leads to the following first set of linear hyperbolic equations for $i = 1, 2$

$$\begin{cases} \partial_t c_i(x, z, t) = 0 \text{ on } \mathcal{B} \\ c_i(x, h(x, t), t) = c_i^s(x, t) \text{ if } \partial_t h(x, t) > 0 \text{ (sedimentation)} \\ c_i(x, z, 0) = c_{i,\text{init}}(x) \text{ for } z \in (-\infty, h_{\text{init}}(x)) \end{cases} \tag{1}$$

In case of an erosion, the surface concentrations are no longer connected to the concentrations of the sediments c_i at the top of the basin. In that case, they represent the concentrations of

the sediments passing through the surface as they appear in front of the fluxes included in the mass conservation of the sediments :

$$\partial_t h_i(x,t) + \text{div}\ (c_i^s(x,t)\ k_i\ \nabla\psi(b(x,t))) = 0 \text{ on } \Omega \times (0,t_f) \text{ for } i = 1,2 \tag{2}$$

where

$$h_i(x,t) = \int_{-\infty}^{h(x,t)} c_i(x,z,t)\mathrm{d}z$$

and k_i is a constant defined for each lithology and $\psi(b) = \int_0^b k(u)\mathrm{d}u$ with $k > 0$ is the diffusion coefficient of the sediments measuring their capability to be transported by gravity. This coefficient is modeled by a nonlinear function of the bathymetry as follow

$$k(b) = \left\{ \begin{array}{l} k^m \text{ if } b \geqslant 0 \\ k^c \text{ otherwise} \end{array} \right.$$

The model is closed by conservativity of the concentrations and by boundary conditions

$$\left\{ \begin{array}{l} c_1 + c_2 = 1 \text{ and } c_1^s + c_2^s = 1 \\[2mm] c_i^s(x,t)\ k_i\ \nabla\psi(b(x,t)) \cdot \boldsymbol{n} = g_i \text{ on } \{x = 0\} \times (0,t_f), i = 1,2 \text{ (input boundary)} \\[2mm] \nabla\psi(b(x,t)) \cdot \boldsymbol{n} = 0 \text{ on } \{x = L_x\} \times (0,t_f), \text{ (output boundary)} \end{array} \right. \tag{3}$$

Numerical scheme

The equations are discretized using an Euler implicit time integration and the upwind Two Points Flux Approximation finite volume scheme. We obtained on each cell K and each time iteration $n > 0$

$$\left\{ \begin{array}{l} |K|\ \dfrac{\Delta h_{i,K}^n}{\Delta t^n}\ + \sum_{\sigma \in \mathcal{F}_K \cap \mathcal{F}_{\text{int}}=K|L} k_i\ c_{i,\sigma}^{s,n}\ T_\sigma\ (\psi(b_L^n) - \psi(b_K^n)) + \sum_{\sigma \in \mathcal{F}_K \cap \{x=0\}} g_{\sigma,i} = 0, \text{ for } i = 1,2 \\[5mm] \text{with } T_{\sigma=K|L} = \dfrac{|\sigma|}{\text{dist}(x_K, x_L)} \\[5mm] \text{and the upwinding } c_{i,\sigma}^{s,n} = \left\{ \begin{array}{ll} c_{i,K}^{s,n} & \text{if } T_\sigma(\psi(b_L^n) - \psi(b_K^n)) \geqslant 0 \\ c_{i,L}^{s,n} & \text{otherwise} \end{array} \right. \\[5mm] \text{and with } \Delta h_{i,K}^n = \int_{-\infty}^{h_K^n} c_{i,K}^n(z)\mathrm{d}z - \int_{-\infty}^{h_K^{n-1}} c_{i,K}^{n-1}(z)\mathrm{d}z \\[5mm] b_K^n = H_\text{m}(t^n) - h_K^n \\[3mm] c_{1,k}^{s,n} + c_{2,k}^{s,n} = 1 \text{ and } c_{1,K}^n(z) + c_{1,K}^n(z) = 1, \forall z \\[3mm] h_K^0 = h_\text{init}(x_K) \end{array} \right.$$

$$\tag{4}$$

The two main discrete unknowns of the Newton algorithm are h_K^n and $c_{1,K}^{s,n}$, which are the piecewise constant approximations of $h(x,t)$ and $c_1^s(x,t)$ in K at time t^n. We have clearly $c_{2,K}^{s,n} = 1 - c_{1,K}^{s,n}$.

Last but not least, we have to calculate $c_{1,K}^n(z)$, the approximation of $c_1(x, z, t)$ in the cell K at time t^n. The approximate concentration $c_{1,K}^n$ is the solution at time t^n of the conservation equation

$$\begin{cases} \partial_t c_{1,K}(z, t) = 0 \text{ for all } t^{n-1} < t < t^n, z < h_K(t) \\[2mm] c_{1,K}(h_K(t), t) = c_{1,K}^{s,n} \text{ if } h_K^n > h_K^{n-1} \text{ (sedimentation)} \\[2mm] c_{1,K}(z, t^{n-1}) = c_{1,K}^{n-1}(z) \text{ for all } z < h_K^{n-1} \end{cases} \qquad (5)$$

with

$$h_K(t) = h_K^{n-1} + \frac{t - t^{n-1}}{t^n - t^{n-1}}(h_K^n - h_K^{n-1})$$

This system is solved exactly on the column moving mesh. The solution is straightforward considering separately the sedimentation $(h_K^n > h_K^{n-1})$ and the erosion $(h_K^n < h_K^{n-1})$ cases. It leads to the update formula below illustrated by the following figure :
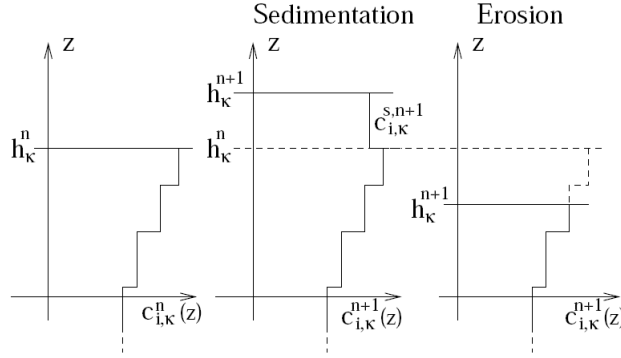
if $h_K^n > h_K^{n-1}$ (sedimentation) we have

$$\begin{cases} c_{1,K}^n(z) = c_{1,K}^{n-1}(z) \text{ for all } z < h_K^{n-1} \\[2mm] c_{1,K}^n(z) = c_{1,K}^{s,n} \text{ for all } h_K^{n-1} < z < h_K^n \\[2mm] \Delta h_{i,K}^n = c_{1,K}^{s,n}(h_K^n - h_K^{n-1}) \end{cases}$$

otherwise if $h_K^n \leqslant h_K^{n-1}$ (erosion) we have

$$\begin{cases} c_{1,K}^n(z) = c_{1,K}^{n-1}(z) \text{ for all } z < h_K^n \\[2mm] \Delta h_{i,K}^n = \int_{h_K^{n-1}}^{h_K^n} c_{1,K}^{n-1}(z)\mathrm{d}z \leqslant 0 \end{cases}$$

★ You have to write the Scilab `residual` and `Jacobian` functions.

The concentration $c_{1,K}^{n-1}(z)$ is piecewise constant on the intervals

$$]-\infty, hcol(K,1)], \;]hcol(K,1), hcol(K,2)], \cdots]hcol(K, ncol(K) - 1), hcol(K, ncol(K))],$$

where $ncol(K)$ is the number of intervals. The value of the concentration $c_{1,K}^n(z)$ in each interval is stored in $ccol(K, l)$ for $l = 1, \cdots, ncol(K)$.

The arrays $ncol$, $hcol$ and $ccol$ will be updated at each time step to take into account erosion and sedimentation in each cell K.