Numerical simulation of two phase porous media flow models with application to oil recovery

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Outline

- Discretization of single phase flows
  - Two Point Flux Finite Volume Approximation of Darcy Fluxes
    - Conservativity
    - Consistency
    - Stability
  - Exercise: single phase incompressible Darcy flow in 1D (using Scilab)

Outline

- Discretization of hyperbolic scalar conservation laws
  - Euler explicit time integration
  - Two point monotone fluxes (upwind scheme)
  - Maximum principle and stability (CFL) condition on the time step

Outline

- Discretization of two phase immiscible incompressible Darcy flows
  – Elliptic pressure equation
  – Hyperbolic saturation equation
  – IMPES (Implicit in pressure, Explicit in Saturation) discretization
  – CFL condition

  – Exercise: Impes discretization of water oil two phase flow in 1D (using Scilab)

Outline

- Fully implicit discretization
  - Phase by phase upwind scheme
  - Implicit Euler time integration
  - Newton algorithm

  - Exercise using Scilab

Outline

- Discretization of Black Oil Models
  - Formulation of the model
  - Fully implicit discretization
  - Newton algorithm taking into account the gas phase appearance and disappearance

  - Exercise using scilab

Finite Volume Discretization of single phase Darcy flows

- Darcy law and conservation equation
- Two Point Flux Discretization (TPFA) of diffusion fluxes on admissible meshes
- Exercice: single phase incompressible Darcy flow in 1D

Oil recovery by water injection

Mass conservation equations

$$\begin{cases} \dfrac{\partial(\phi\rho_w S_w)}{\partial t} + div(\rho_w \vec{V}_w) = 0 \\ \dfrac{\partial(\phi\rho_o S_o)}{\partial t} + div(\rho_o \vec{V}_o) = 0 \end{cases}$$

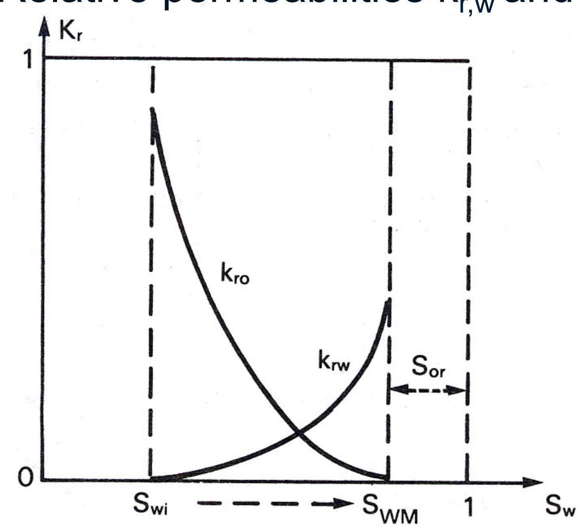
Two phase Darcy laws

$$\begin{cases} \vec{V}_w = -\dfrac{k_{r,w}(S_w)}{\mu_w} K(\nabla P_w - \rho_w \vec{g}) \\ \vec{V}_o = -\dfrac{k_{r,o}(S_o)}{\mu_o} K(\nabla P_w + \nabla P_c(S_w) - \rho_o \vec{g}) \end{cases}$$

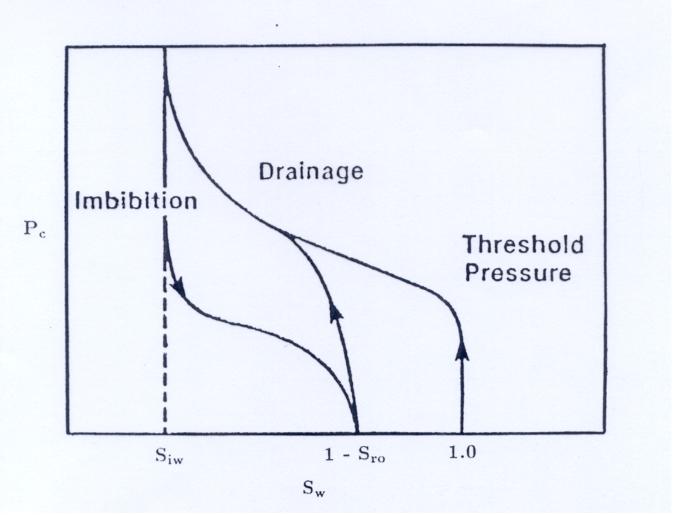
Pore volume conservation

$$S_w + S_o = 1$$

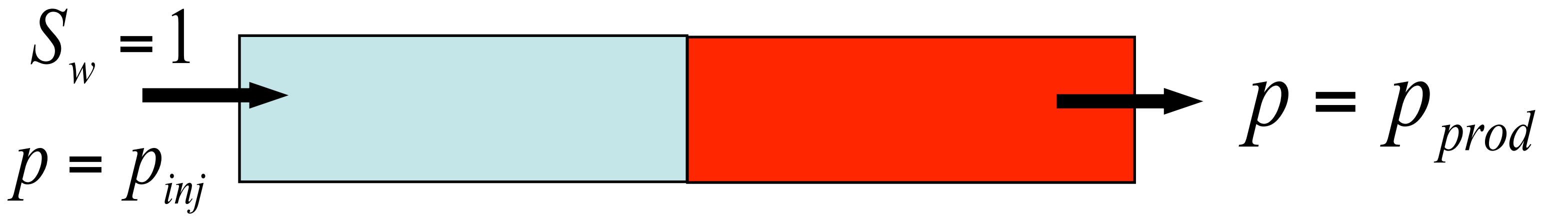
Relative permeabilities $k_{r,w}$ and $k_{r,o}$



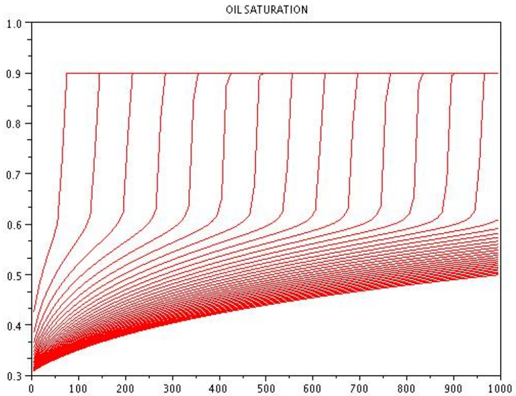
Capillary pressure Pc

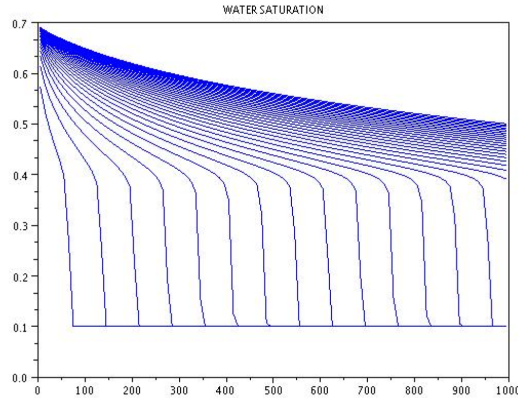


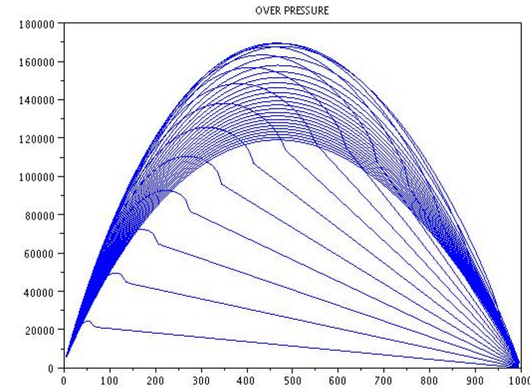
1D test case
Injection of water in a reservoir

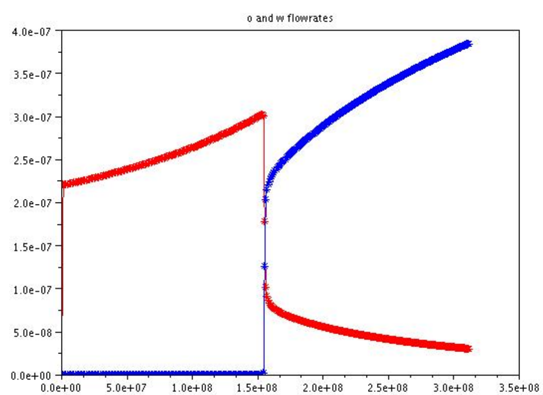


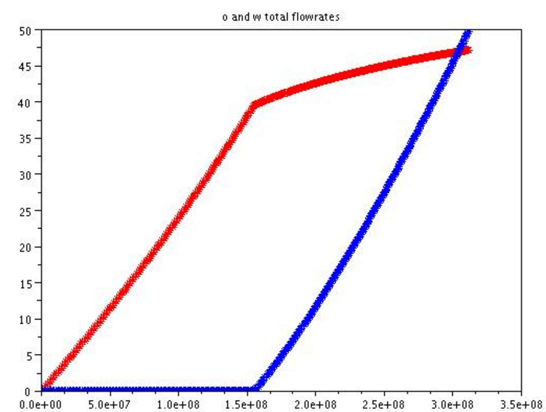
Water injection in a 1D reservoir



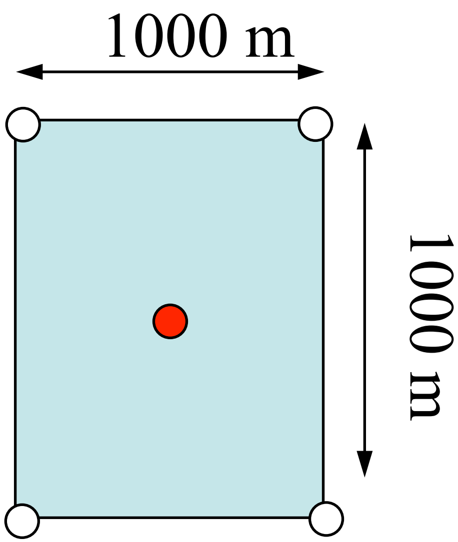




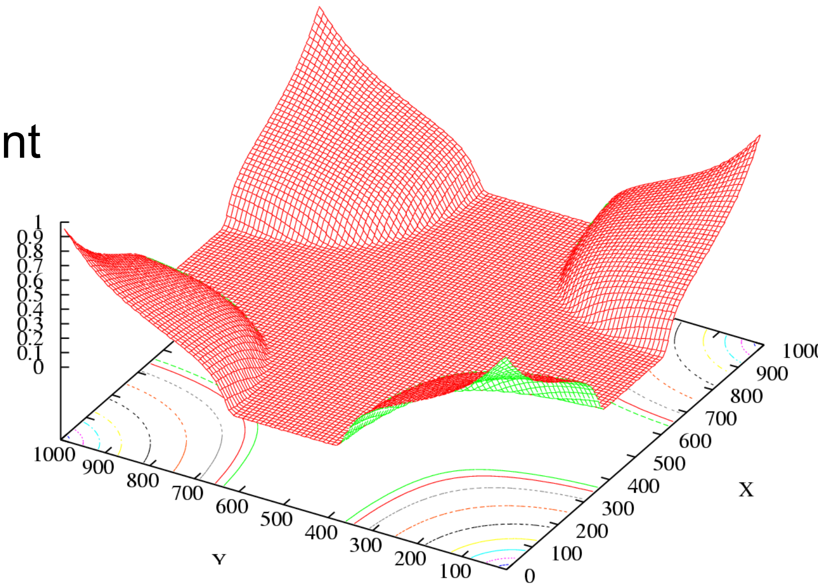




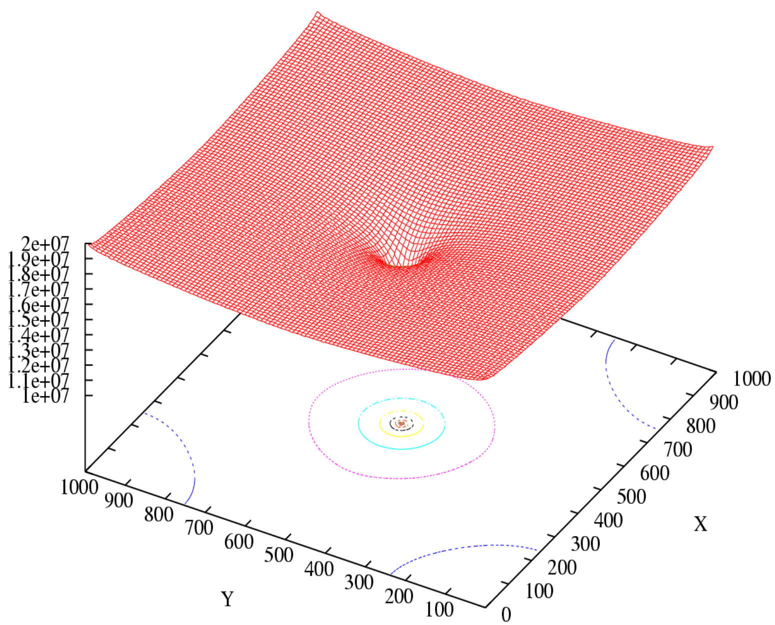
Five Spots simulation in 2D



Water front



Pressure

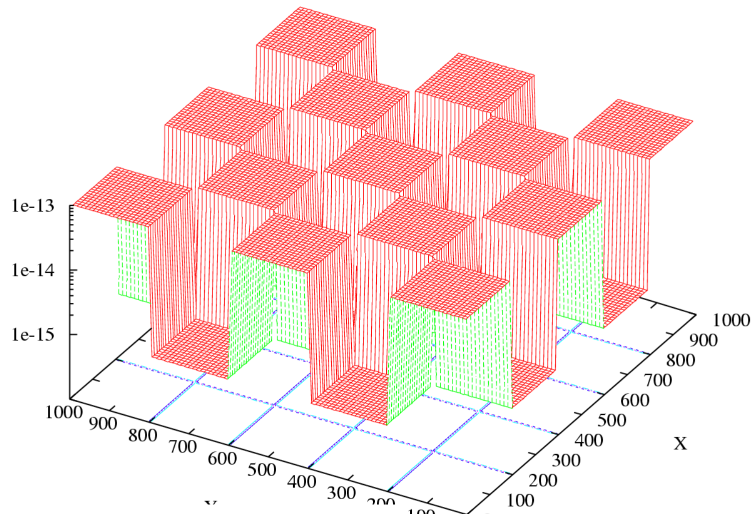


○ Injector wells

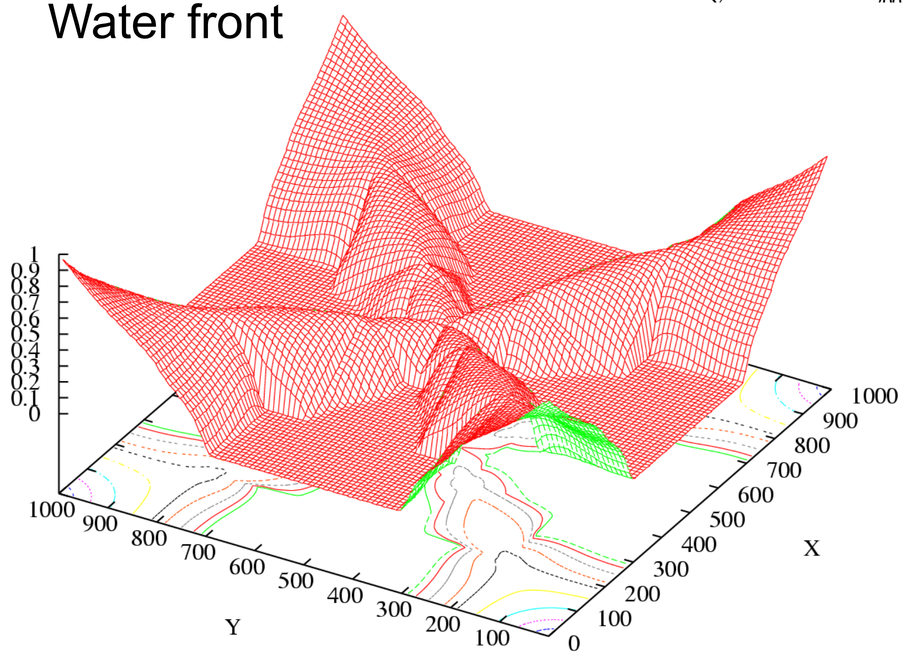
● Producer well

Heterogeneities

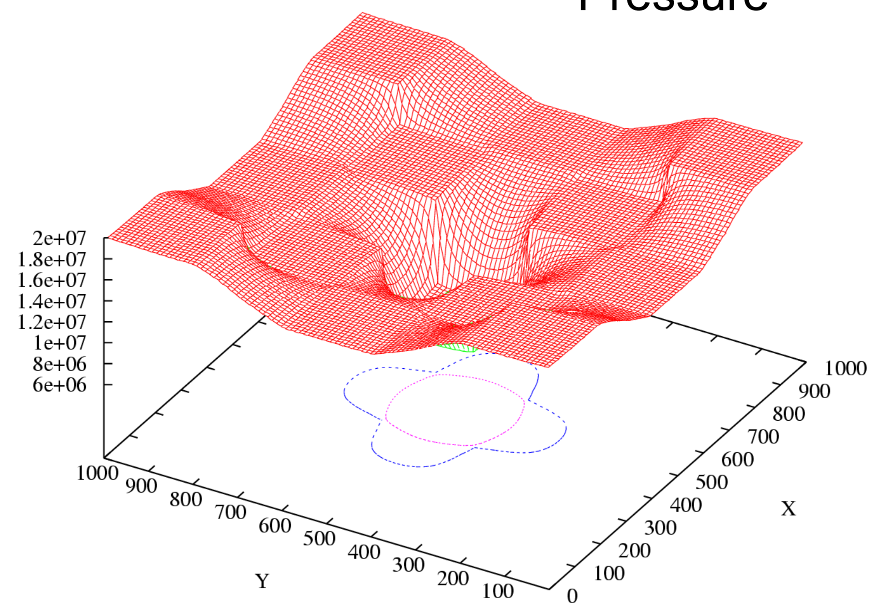
Permeability



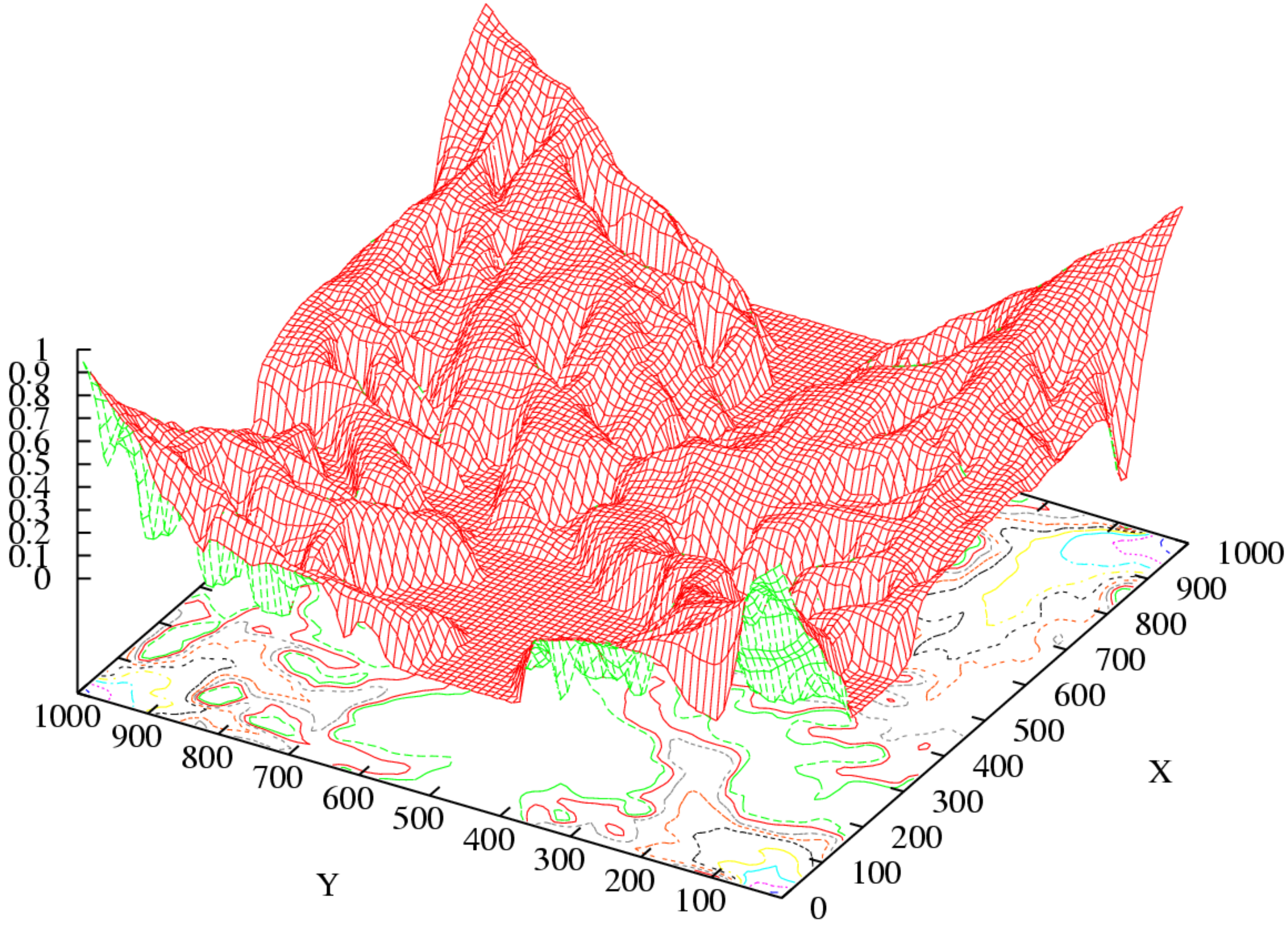
Water front



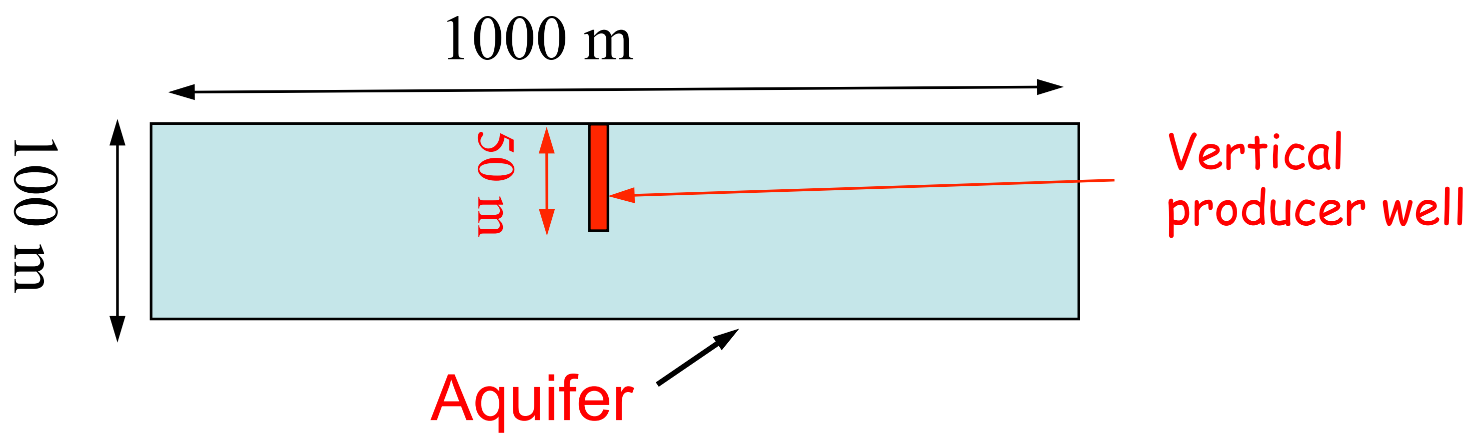
Pressure



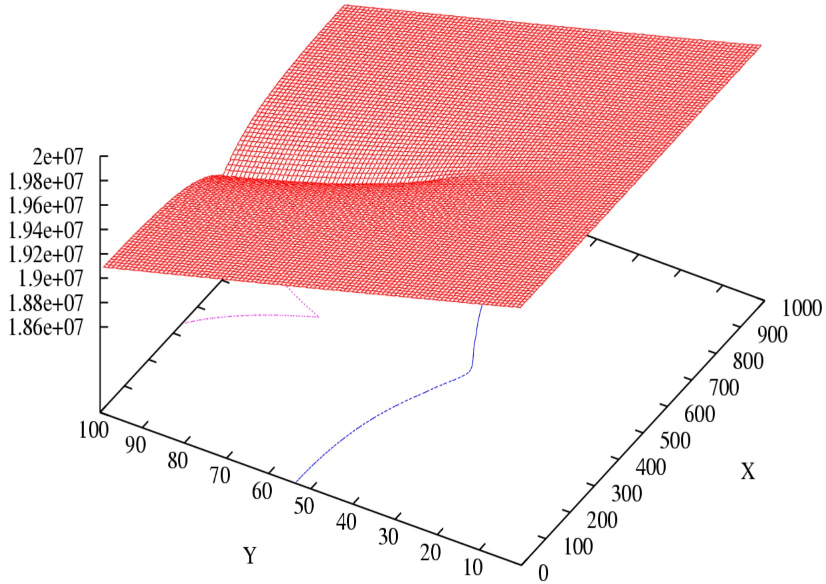
Heterogeneities



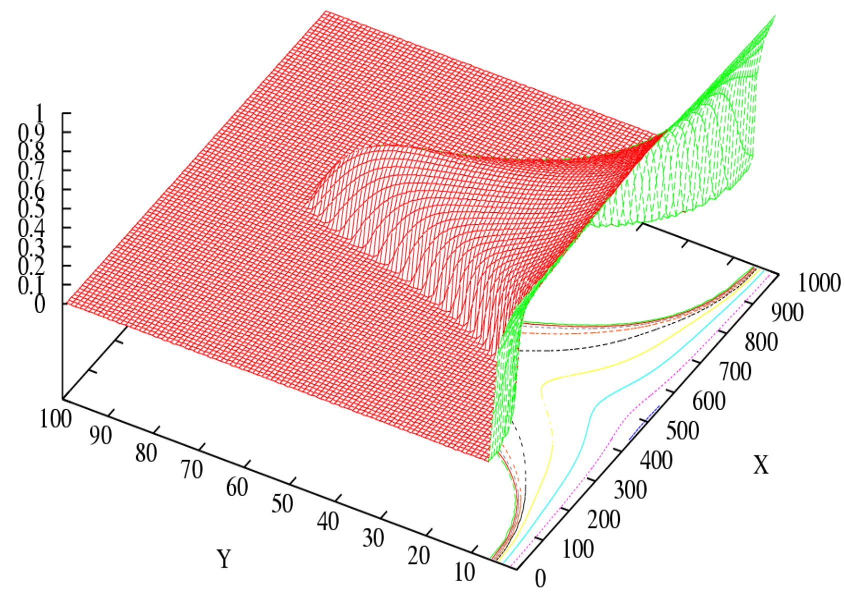
Coning: aquifer and vertical well



Pressure

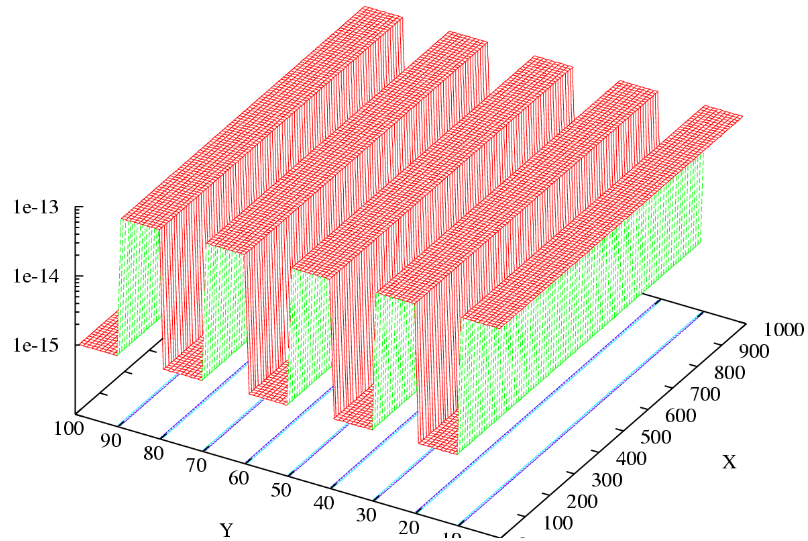


Water front

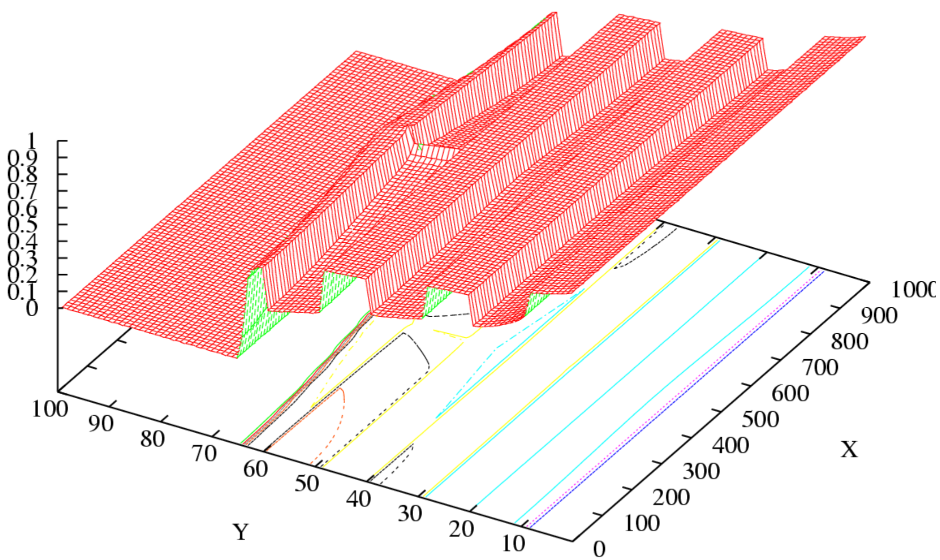


Coning: stratified reservoir

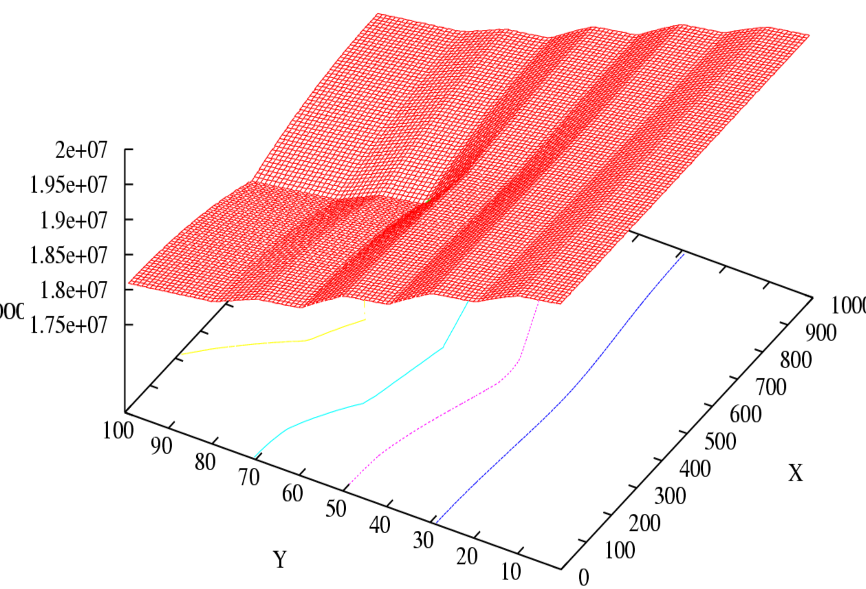
Permeability



Water front



Pressure



SINGLE PHASE DARCY FLOW

$$\frac{\partial(\phi\rho)}{\partial t} + div\left(\rho\vec{V}\right) = q$$

Mass conservation equation

$$\vec{V} = -\frac{K}{\mu}(\nabla P - \rho\vec{g})$$

Darcy law

ϕ Porosity of the porous media

K Permeability of the porous media (tensor)

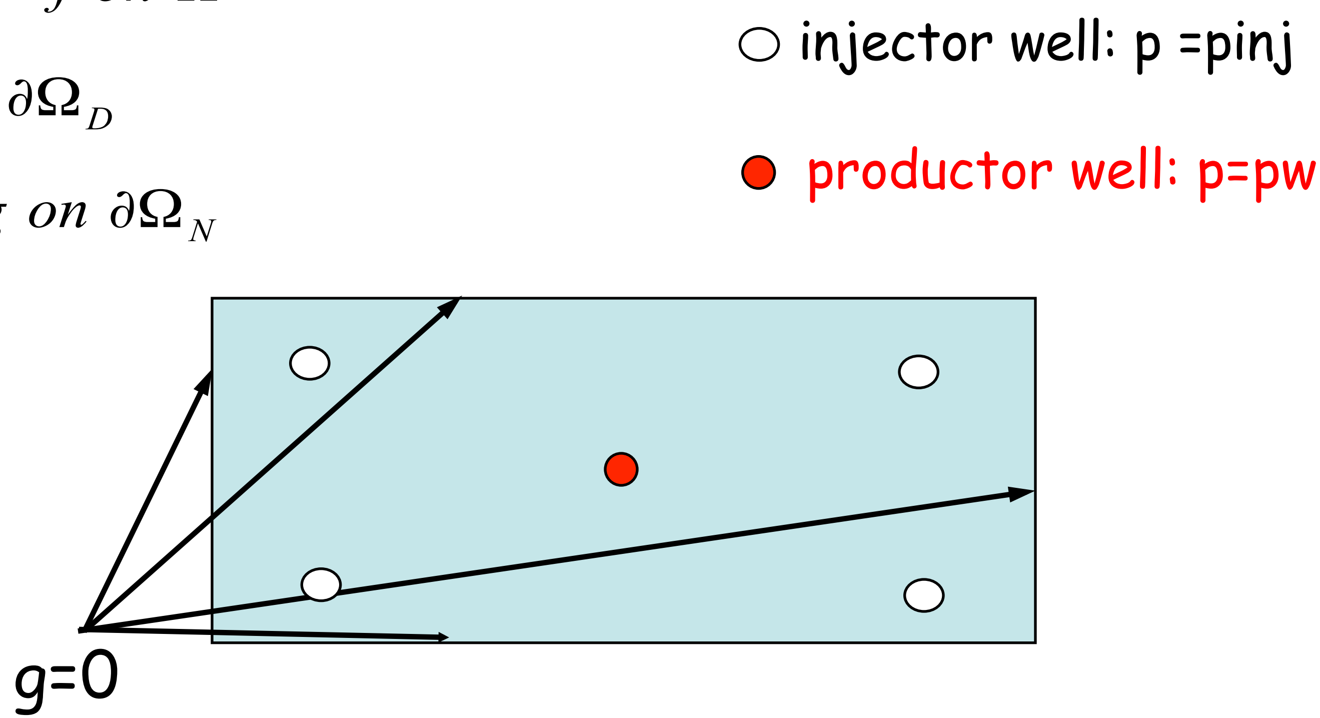
ρ Density of the fluid

μ Viscosity of the fluid

Incompressible Darcy single phase flow

- Diffusion equation

$$\begin{cases} div(-\rho \frac{K}{\mu} \nabla p) = f \ on \ \Omega \\ p = p_D \ on \ \partial\Omega_D \\ -\rho \frac{K}{\mu} \nabla p.n = g \ on \ \partial\Omega_N \end{cases}$$

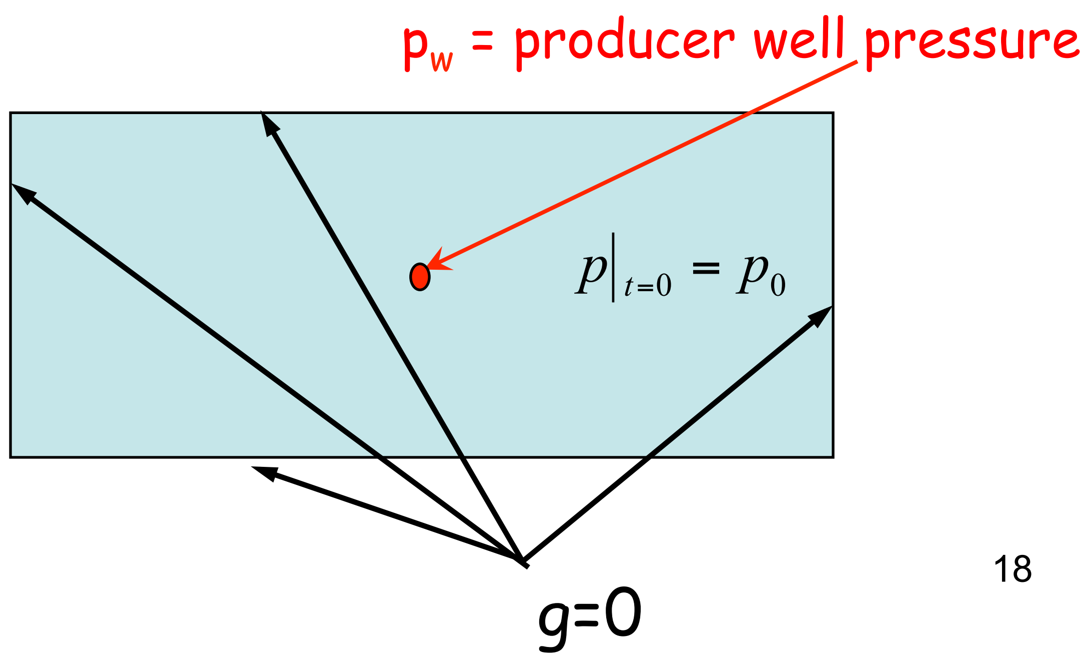


Compressible Darcy single phase flow

- Parabolic equation (linearized)

$$
\begin{cases}
(\frac{1}{\rho_0}\frac{d\rho}{dp})\rho_0 \partial_t p + div(-\rho_0 \frac{K}{\mu}\nabla p) = 0 \;\; on \;\; \Omega \times (0,T) \\
p = p_D \;\; on \;\; \partial\Omega_D \times (0,T) \\
-\rho_0 \frac{K}{\mu}\nabla p.n = g \;\; on \;\; \partial\Omega_N \times (0,T) \\
p_{t=0} = p_0 \;\; on \;\; \Omega
\end{cases}
$$

Ex: well test



NOTATIONS

$\left| geometrical\ object \right|$ d-dimensional measure of the geometrical object of dimension d

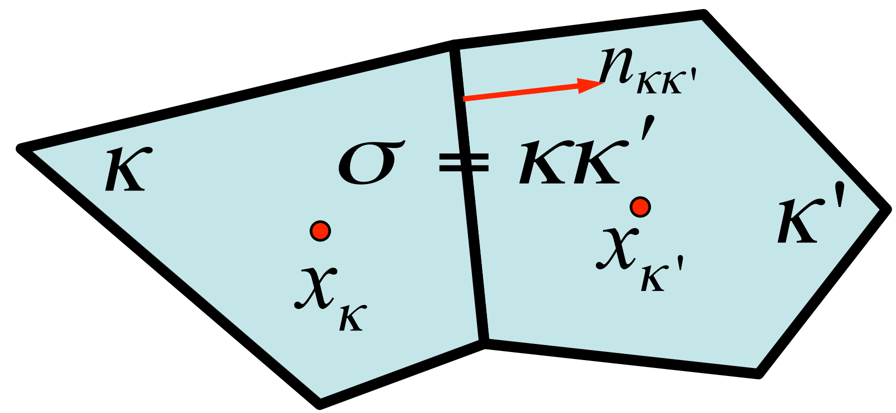
$|\kappa|$ **Cell**: volume for d=3, surface for d=2, lengh for d=1

$|\sigma|$ **Face**: surface for d=2, lengh for d=1, 1 for a point

$|x_1 x_2|$ **Segment**: lengh for d=1

Finite Volume Discretization

- Finite volume mesh
  - Cells
  - Cell centers
  - Faces



- Degrees of freedom: u_κ

- Discrete conservation law

$$\int_\kappa -\Delta u dx = \sum_{\sigma=\kappa\kappa'} \int_\sigma -\nabla u . n_{\kappa\kappa'} ds = \int_\kappa f dx$$

Two Point Flux Approximation (TPFA)

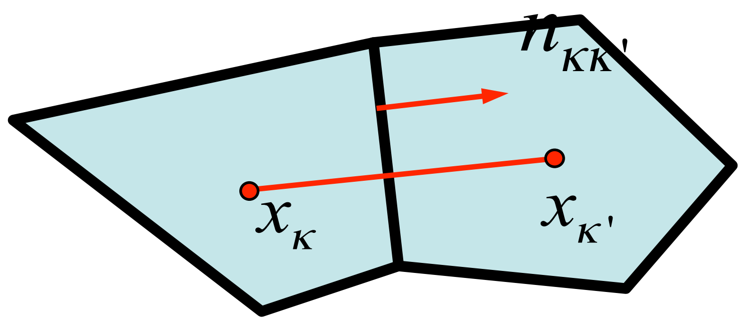
- TPFA $\int_{\sigma} -\nabla u . n_{\kappa\kappa'} ds \approx F_{\kappa\kappa'}(u_{\kappa}, u_{\kappa'})$

- Flux Conservativity

$$F_{\kappa\kappa'}(u_{\kappa}, u_{\kappa'}) + F_{\kappa'\kappa}(u_{\kappa'}, u_{\kappa}) = 0$$

- Flux Consistency

$$F_{\kappa\kappa'}(u_{\kappa}, u_{\kappa'}) = \frac{|\sigma|}{|x_{\kappa} x_{\kappa'}|}(u_{\kappa} - u_{\kappa'}) = \int_{\sigma} -\nabla u . n_{\kappa\kappa'} ds + O(|\sigma| h)$$

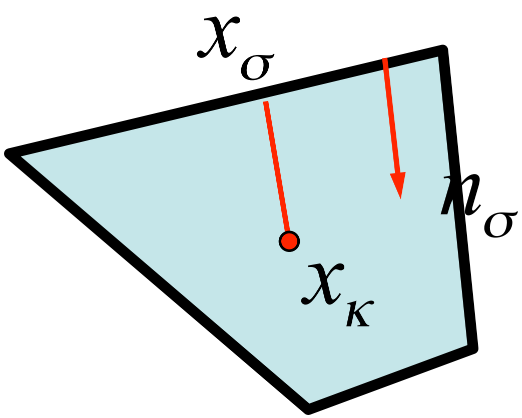


$$x_{\kappa'} x_{\kappa'} \perp \kappa\kappa'$$

Two Point Flux Approximation

- Boundary faces

$$F_\sigma(u_\kappa, u_\sigma) = \frac{|\sigma|}{|x_\kappa x_\sigma|}(u_\kappa - u_\sigma) = \int_\sigma -\nabla u . n_\sigma ds + O(|\sigma| h)$$



$$x_{\kappa'} x_\sigma \perp \sigma$$

Two Point Flux Approximation

- Finite Volume Scheme

$$\begin{cases} -\Delta u = f \ sur \ \Omega \\ u = g \ sur \ \partial\Omega \end{cases}$$

$$\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\text{int}}} \frac{|\sigma|}{|x_\kappa x_{\kappa'}|}(u_\kappa - u_{\kappa'}) + \sum_{\sigma\in\partial\kappa\cap\Sigma_{bord}} \frac{|\sigma|}{|x_\kappa x_\sigma|}(u_\kappa - g_\sigma) = |\kappa| f_\kappa$$

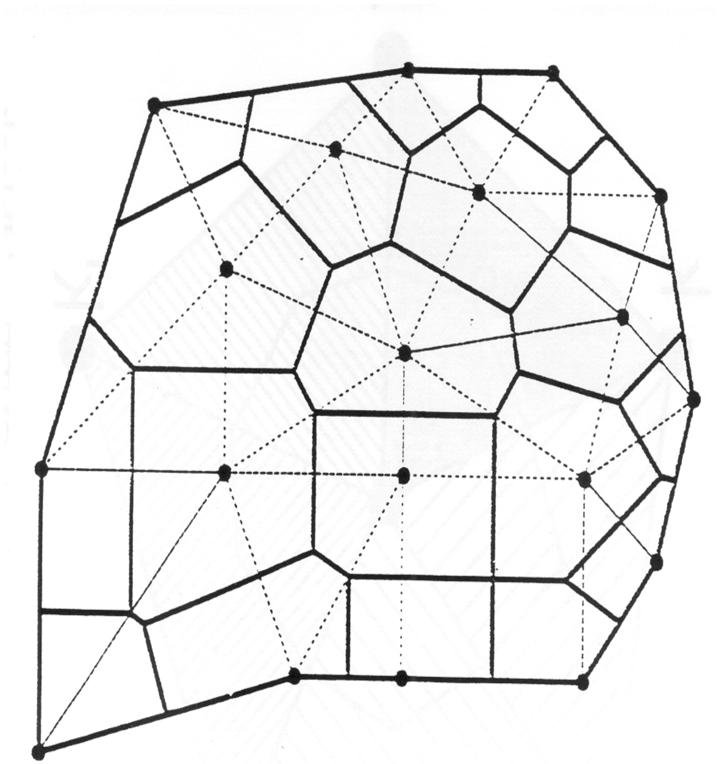
$$T_{\kappa\kappa'} = \frac{|\kappa\kappa'|}{|x_\kappa x_{\kappa'}|}$$

$$T_{\kappa\sigma} = \frac{|\sigma|}{|x_\kappa x_\sigma|}$$

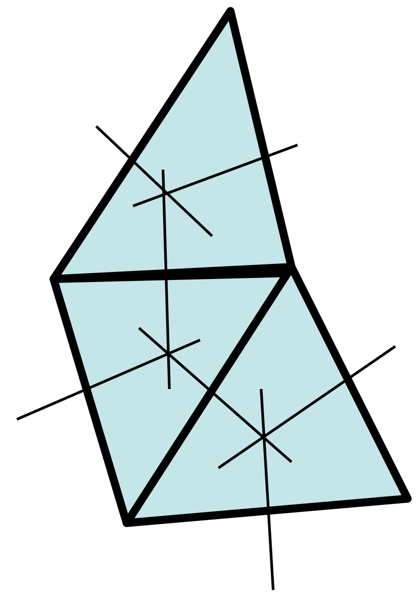
Transmissibilities of interior and boundary faces

Exemples of admissible meshes

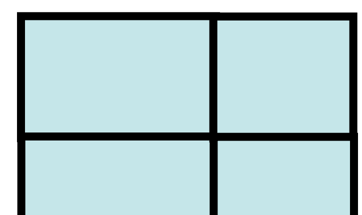
Voronoi



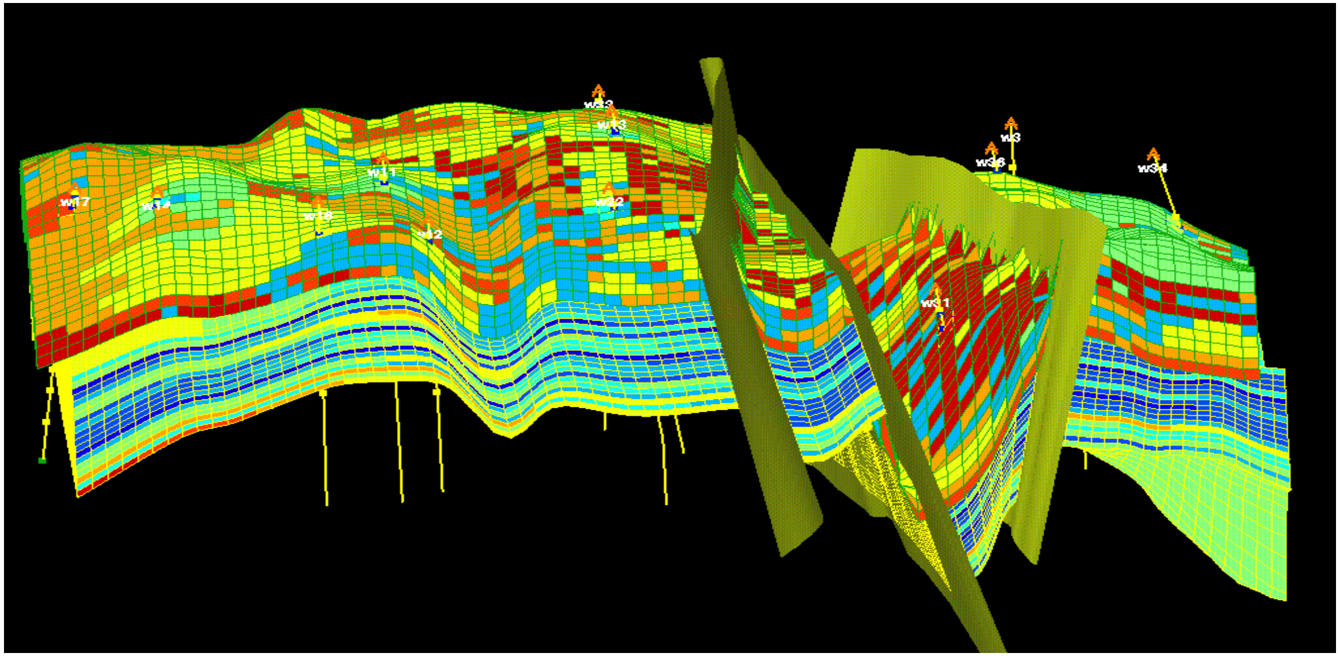
Triangles: angles $\leq \pi / 2$



Cartesian:



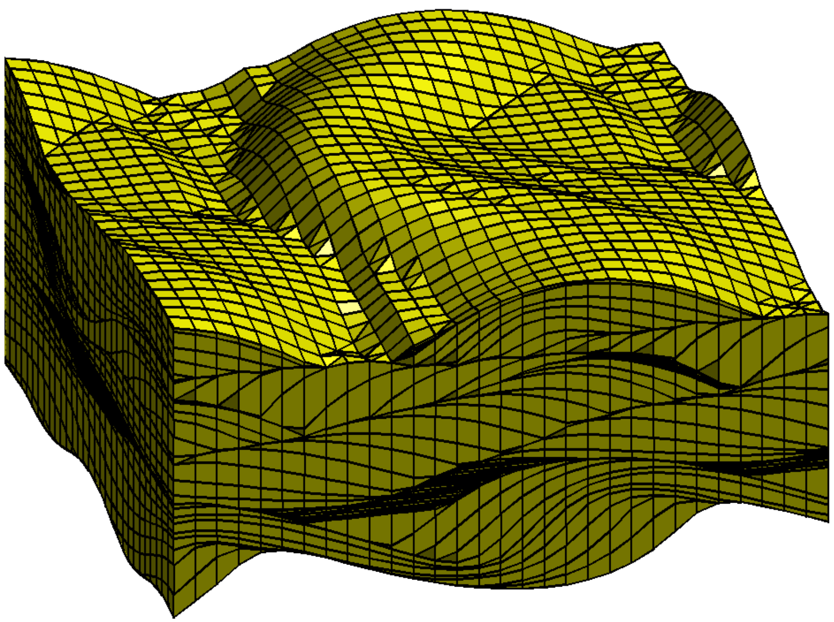
Corner Point Geometries and TPFA



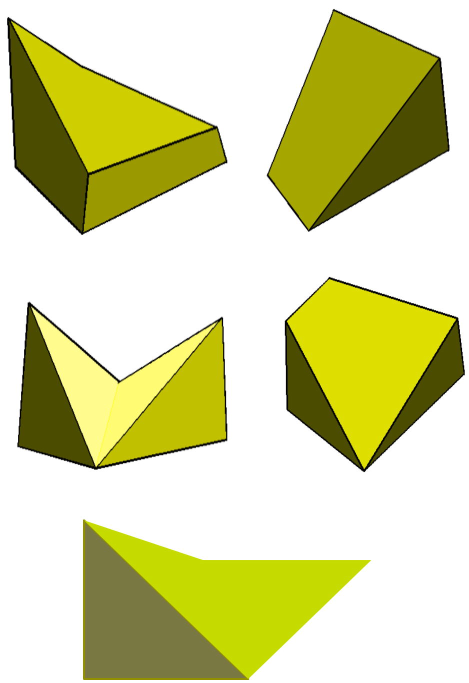
Assumption that the directions of the CPG are aligned with the principal directions of the permeability field

Corner Point Geometries
Stratigraphic grids with erosions

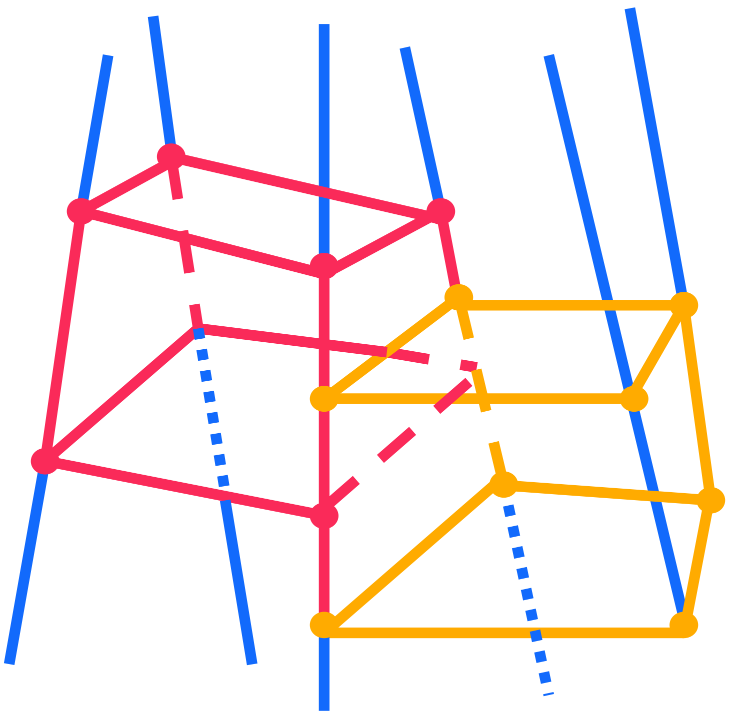
- Hexahedra
- Topologicaly Cartesian
- Dead cells
- Erosions
- Local Grid Refinement (LGR)

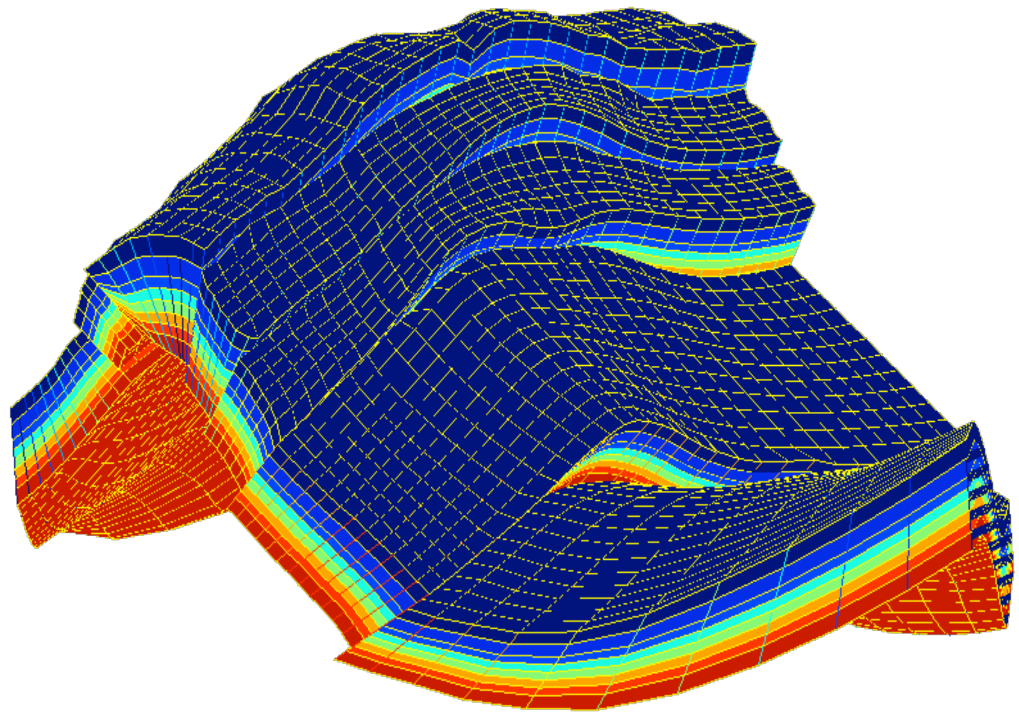


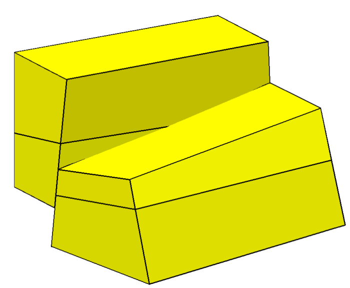
Examples of degenerate cells (erosions)



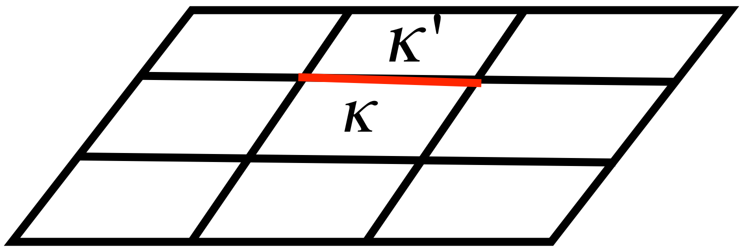
CPG faults







Cell Centered FV: MultiPoint Flux Approximation (MPFA)



$$F_{\kappa\kappa'} = \sum_L T_{\kappa\kappa'}^L u_L$$

$$\sum_L T_{\kappa\kappa'}^L = 0, \quad T_{\kappa\kappa'}^L = -T_{\kappa'\kappa}^L$$

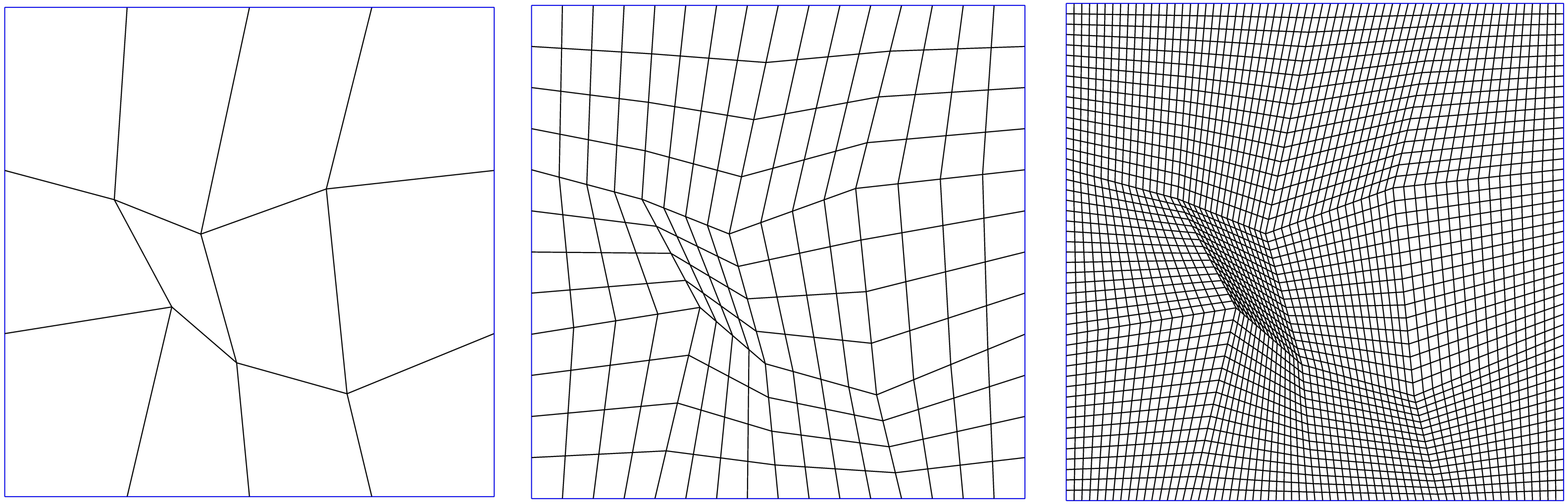
- Example of the "O" scheme
  - Exact on piecewise linear functions
  - Account for discontinuous diffusion tensors
  - Account for anisotropic diffusion tensors

2D example

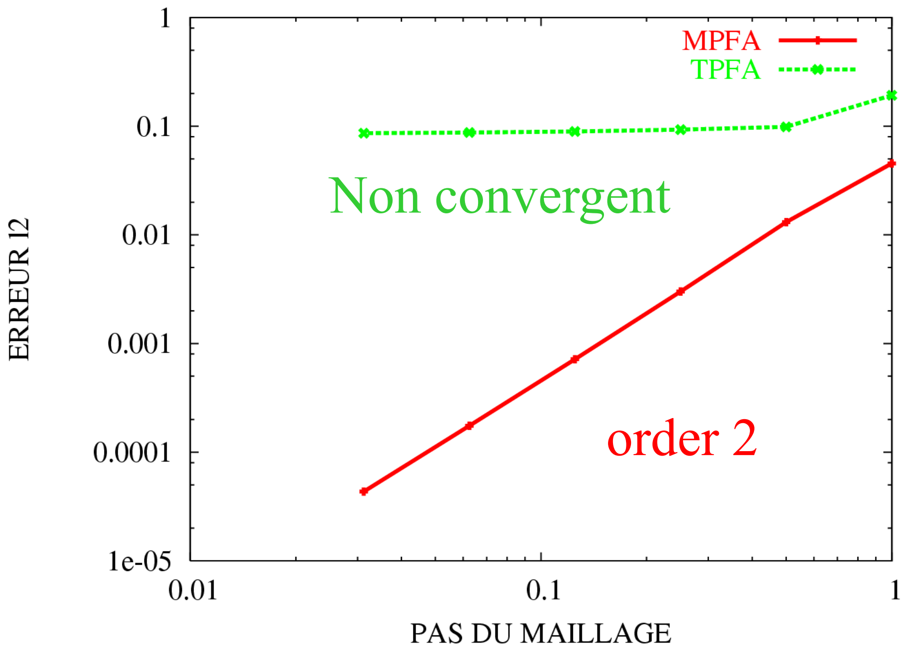
$$\begin{cases} -\Delta u = f \ sur \ \Omega \\ u = g \ sur \ \partial\Omega \end{cases}$$

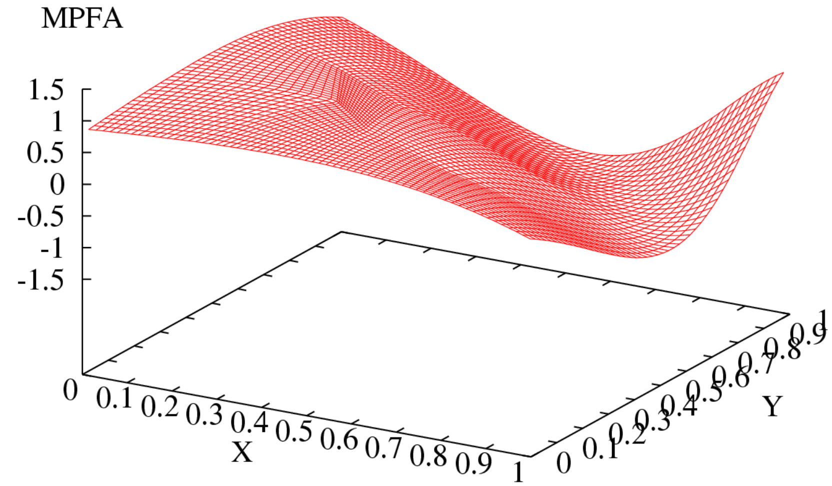
Smooth solution $u = \sin\left(e^{x+y}\right)$

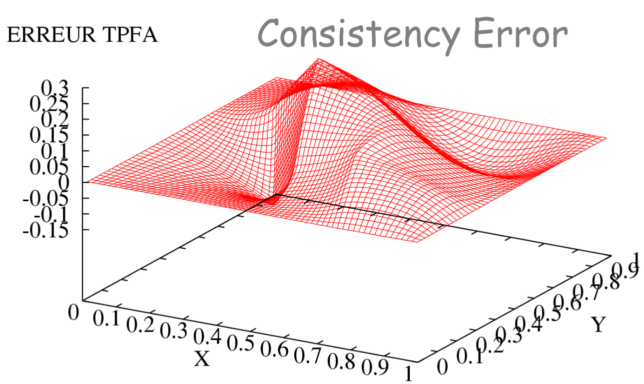
Uniformly refined quadrangular mesh

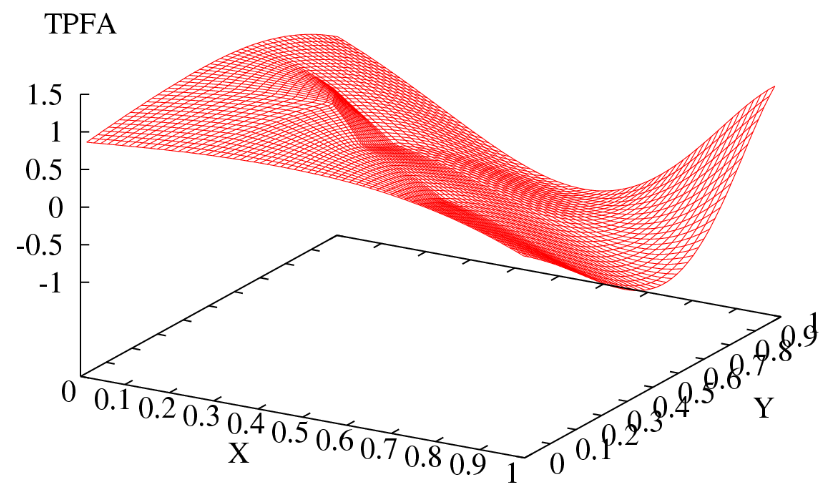


Comparison of MPFA "O" scheme and TPFA



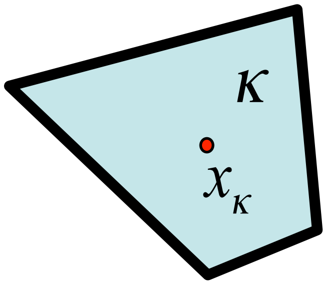




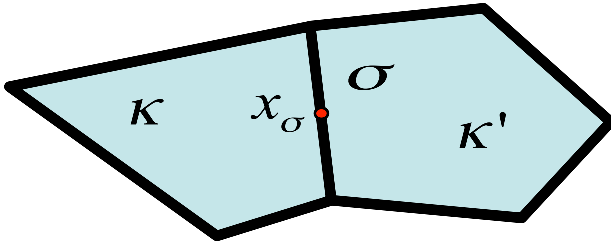


Cell-Face data structure

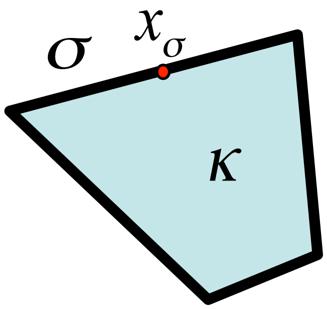
- List of cells: m=1,...,N
  - Volume(m)
  - Cell center X(m)



- List of interior faces: i=1,...,Nint
  - cellint(i,1) = m1, cellint(i,2)=m2
  - surfaceint(i)
  - Xint(i)



- List of boundary faces: i=1,...,Nbound
  - cellbound(i)
  - surfacebound(i)
  - Xbound(i)



Computation of interior and boundary face transmissibilities

- Interior faces: i=1,...,Nint
  - m1 = cellint(i,1)
  - m2 = cellint(i,2)
  - Tint(i) = surfaceint(i)/|X(m2)-X(m1)|

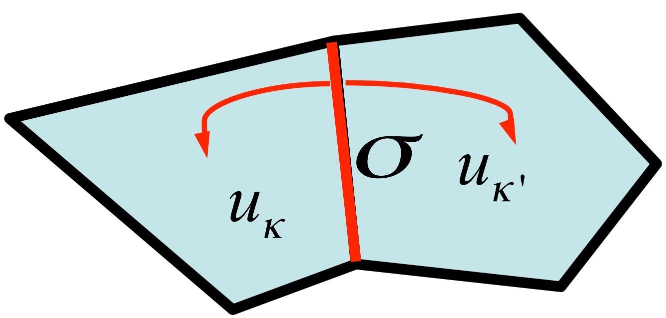
- Boundary faces: i=1,...,Nbound
  - m = cellbound(i)
  - Tbound(i) = surfacebound(i)/|X(m)-Xbound(i)|

Computation of the Jacobian sparse matrix and the right hand side JU = B

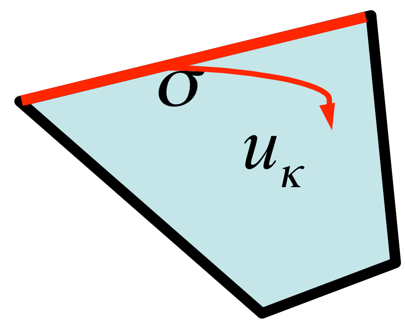
$$\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\text{int}}} T_{\kappa\kappa'}\left(u_\kappa - u_{\kappa'}\right) + \sum_{\sigma\in\partial\kappa\cap\Sigma_{bound}} T_\sigma\left(u_\kappa - g_\sigma\right) = |\kappa| f_\kappa$$

Loop on interior faces

Loop on boundary faces



$$\begin{cases} line\ \kappa : T_{\sigma=\kappa\kappa'}\left(u_\kappa - u_{\kappa'}\right) \\ line\ \kappa' : T_{\sigma=\kappa\kappa'}\left(u_{\kappa'} - u_\kappa\right) \end{cases}$$



$$line\ \kappa : T_\sigma\left(u_\kappa - g_\sigma\right)$$

Loop on cells

$$line\ \kappa : |\kappa| f_\kappa$$

Computation of the Jacobian sparse matrix and the right hand side: JU = B

$$\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\text{int}}} T_{\kappa\kappa'}\left(u_\kappa - u_{\kappa'}\right) + \sum_{\sigma\in\partial\kappa\cap\Sigma_{bound}} T_\sigma\left(u_\kappa - g_\sigma\right) = |\kappa| f_\kappa$$

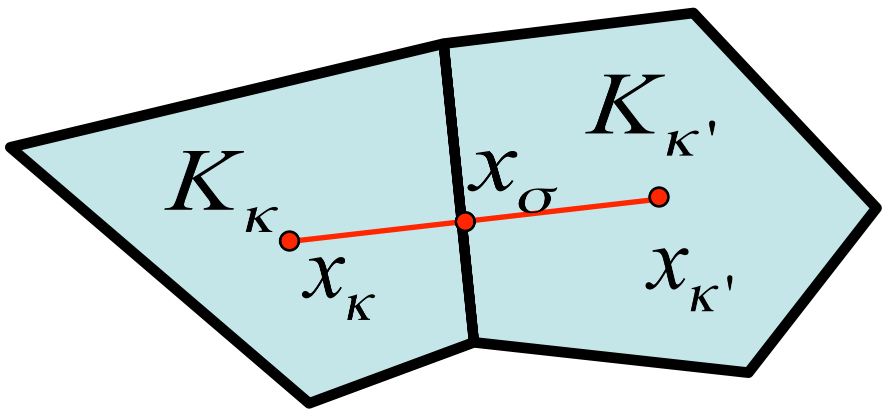
- Cell loop: m=1,...,N
  - B(m) = Volume(m)*f(X(m))

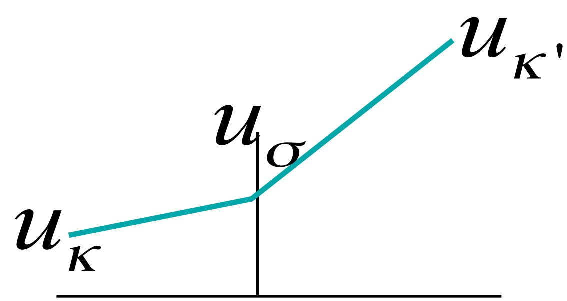
- Interior face loop: i=1,...,Nint
  - m1 = cellint(i,1), m2 = cellint(i,2)
  - J(m1,m1) = J(m1,m1) +Tint(i)
  - J(m2,m2) = J(m2,m2) +Tint(i)
  - J(m1,m2) = J(m1,m2) -Tint(i)
  - J(m2,m1) = J(m2,m1) -Tint(i)

- Boundary face loop: i=1,...,Nbound
  - m = cellbound(i)
  - J(m,m) = J(m,m) +Tbound(i)
  - B(m) = B(m) + Tbound(i)*g(Xbound(i))

TPFA
Isotropic Heterogeneous media

- FV scheme $\begin{cases} div(-K\nabla u) = f \; sur \; \Omega \\ u = g \; sur \; \partial\Omega \end{cases}$

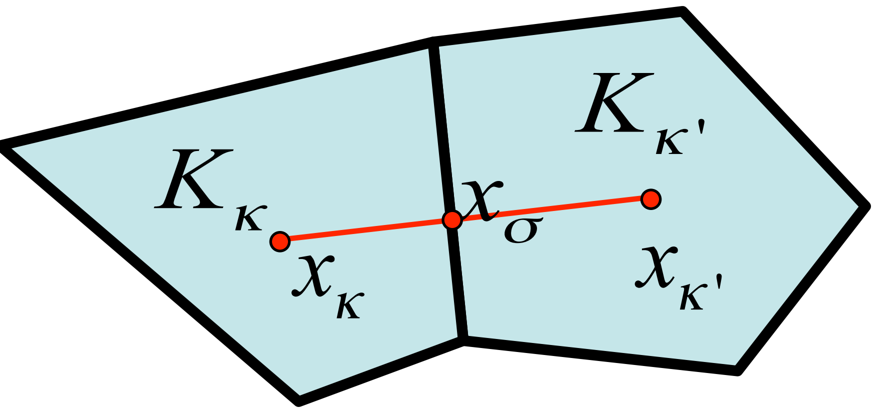


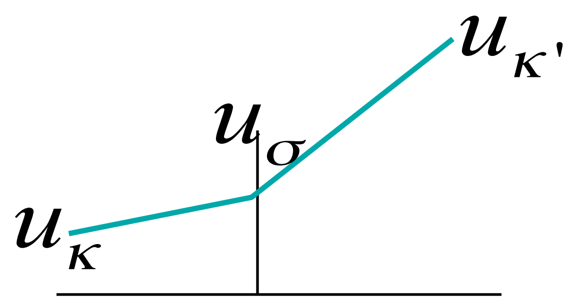


$$F_{\kappa\kappa'} = K_{\kappa} \frac{|\kappa\kappa'|}{|x_{\kappa}x_{\sigma}|}(u_{\kappa} - u_{\sigma}) = K_{\kappa'} \frac{|\kappa\kappa'|}{|x_{\kappa'}x_{\sigma}|}(u_{\sigma} - u_{\kappa'}) = T_{\kappa\kappa'}(u_{\kappa} - u_{\kappa'})$$

Transmissibility: $$\frac{1}{T_{\kappa\kappa'}} = \frac{|x_{\kappa}x_{\sigma}|}{K_{\kappa}|\kappa\kappa'|} + \frac{|x_{\kappa'}x_{\sigma}|}{K_{\kappa'}|\kappa\kappa'|}$$

TPFA
Isotropic heterogeneous permeability





Transmissibility:

$$\frac{1}{T_{\kappa\kappa'}} = \frac{|x_\kappa x_\sigma|}{K_\kappa |\kappa\kappa'|} + \frac{|x_{\kappa'} x_\sigma|}{K_{\kappa'} |\kappa\kappa'|}$$

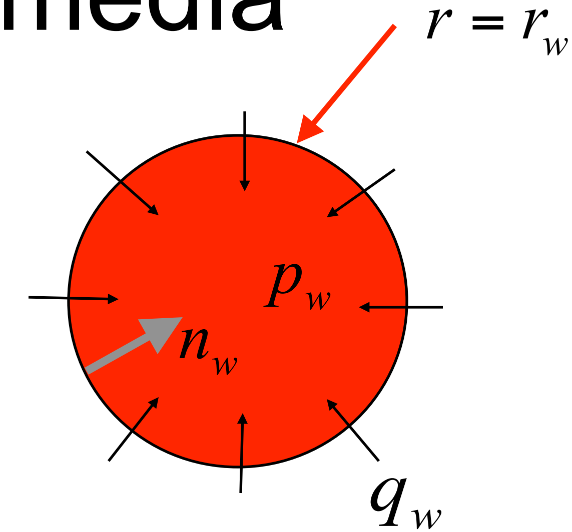
$$T_{\kappa\kappa'} = \left(\frac{|x_\kappa x_{\kappa'}|}{\dfrac{|x_\kappa x_\sigma|}{K_\kappa} + \dfrac{|x_{\kappa'} x_\sigma|}{K_{\kappa'}}} \right) \frac{|\kappa\kappa'|}{|x_\kappa x_{\kappa'}|} = K_{\kappa\kappa'} \frac{|\kappa\kappa'|}{|x_\kappa x_{\kappa'}|}$$

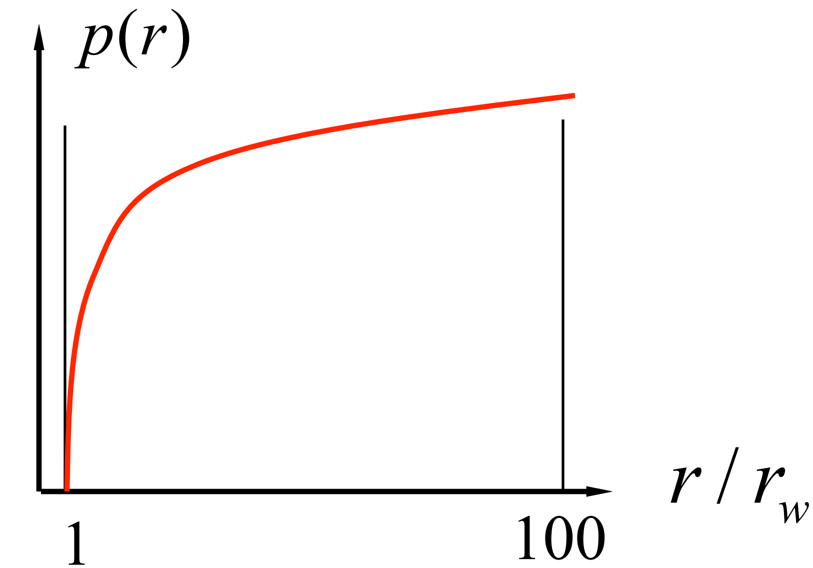
Well discretization

- Radial stationary analytical solution for vertical wells in homogeneous porous media
- Numerical Peaceman well index for well discretization with imposed pressure
- Proof of Peaceman formula for uniform cartesian meshes
- Pressure drop for vertical single phase wells

Stationary radial analytical solution in homegeneous media

$$
\begin{cases}
-K\Delta\overline{p} = 0 & r > r_w \\
\overline{p} = p_w & r = r_w \\
\displaystyle\int_{r=r_w} -(K\nabla\overline{p}.n_w)ds = q_w &
\end{cases}
$$





$$\overline{p}(r) - p_w = \frac{q_w}{2\pi K}\ln(r / r_w)$$

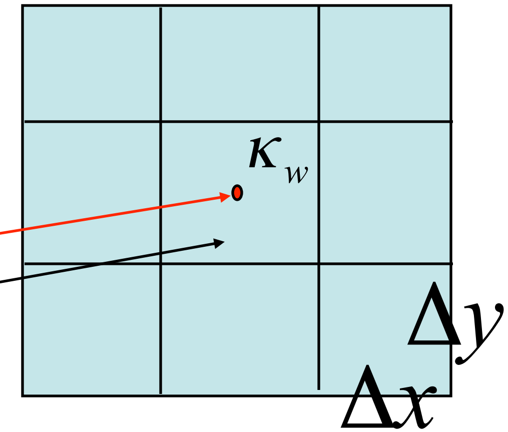
$$\overline{q}(r) = -K\nabla\overline{p}(r).n_r = \frac{q_w}{2\pi\ r}$$

Numerical well index

- Cartesian mesh
  $\Delta x, \Delta y >> r_w$

Well w

Well cell κ_w



Pressure Numerical computation with specified well flow rate and pressure boundary condition given by the analytical solution

$$\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\text{int}}} T_{\kappa\kappa'}(p_\kappa - p_{\kappa'}) + \sum_{\sigma\in\partial\kappa\cap\Sigma_{bord}} T_\sigma(p_\kappa - \overline{p}_\sigma) + \sum_{w|\kappa_w=\kappa} q_w = 0$$

analytical solution

$$p_{\kappa_w} - p_w = \frac{q_w}{2\pi K}\ln(r_0 / r_w)$$

with $\quad r_0 \approx 0.14(\Delta x^2 + \Delta y^2)^{1/2}$

Well flow rate with specified pressure

$$q_w = \frac{2\pi K}{\ln(r_0 / r_w)} (p_{\kappa_w} - p_w)$$

$$\boxed{WI = \frac{2\pi K}{\ln(r_0 / r_w)}}$$

Well index

Finite Volume discretization with well specified pressures

$$\sum_{\sigma = \kappa\kappa' \in \partial\kappa \cap \Sigma_{\text{int}}} T_{\kappa\kappa'} \left(p_\kappa - p_{\kappa'}\right) + \sum_{i \in \Pi, \kappa(i) = \kappa} WI_i \left(p_\kappa - p_{w,i}\right) = 0$$

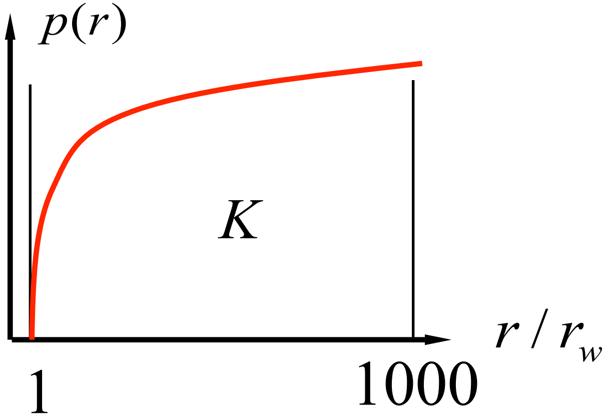
Computation of the Jacobian matrix and right hand side JU = B with wells

$$\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\text{int}}} T_{\kappa\kappa'}\left(p_\kappa - p_{\kappa'}\right) + \sum_{i\in\Pi,\kappa(i)=\kappa} WI_i\left(p_\kappa - p_{w,i}\right) = 0$$

- Loop on interior faces: i=1,...,Nint
  - m1 = cellint(i,1), m2 = cellint(i,2)
  - J(m1,m1) = J(m1,m1) +Tint(i)
  - J(m2,m2) = J(m2,m2) +Tint(i)
  - J(m1,m2) = J(m1,m2) -Tint(i)
  - J(m2,m1) = J(m2,m1) -Tint(i)

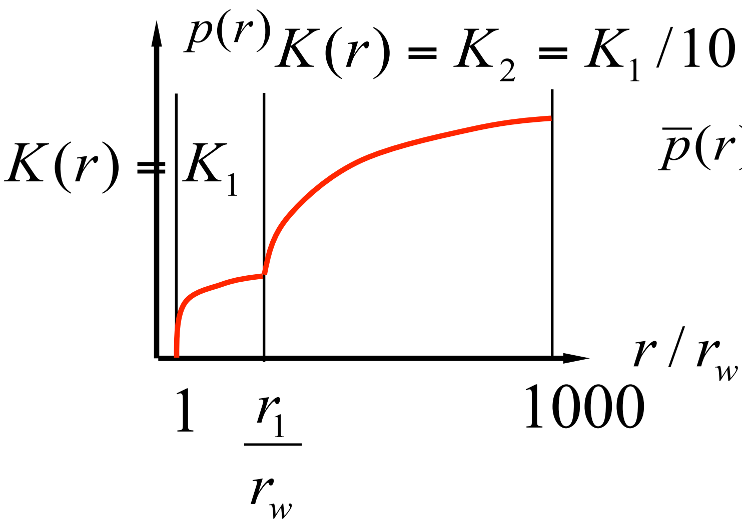
- Loop on wells: i=1,...,Nwell
  - m = cellwell(i)
  - J(m,m) = J(m,m) + WI(i)
  - B(m) = B(m) + WI(i)*pw(i)

Exercice: convergence of the scheme to an analytical well solution



$$\overline{p}(r) - p_w = \frac{q_w}{2\pi K}\ln(r / r_w)$$

$$\overline{q}(r) = -K\nabla\overline{p}(r).n_r = \frac{q_w}{2\pi\ r}$$

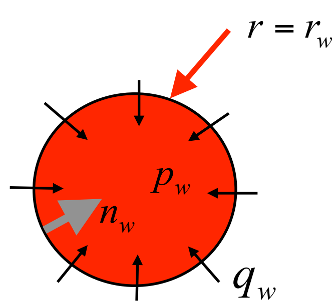


$$\overline{p}(r) - p_w = \begin{cases} \dfrac{q_w}{2\pi K_1}\ln(r / r_w) & if \;\; r_w \le r \le r_1 \\ \dfrac{q_w}{2\pi K_1}\ln(r_1 / r_w) + \dfrac{q_w}{2\pi K_2}\ln(r / r_1) & if \;\; r \ge r_1 \end{cases}$$

$$\overline{q}(r) = -K(r)\nabla\overline{p}(r).n_r = \frac{q_w}{2\pi\ r}$$

Proof of Peaceman well index: uniform cartesian mesh, well at the center of the cell

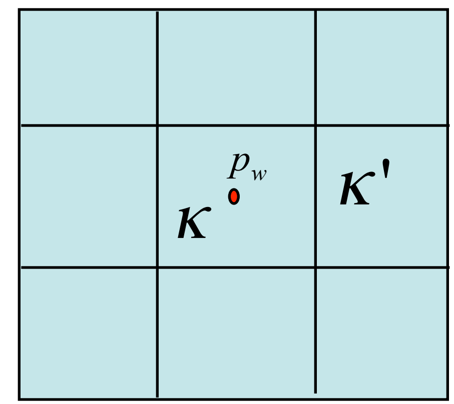
$$\Delta y = \Delta x >> r_w$$



$$\overline{p}(r) - p_w = \frac{q_w}{2\pi K} \ln(r / r_w)$$

$$\overline{q}(r) = -K\nabla\overline{p}(r).n_r = \frac{q_w}{2\pi\ r}$$

p_w specified well pressure



Hypothesis: p radial in the well cell κ

$$q_w = \int_{r=r_w} -K\nabla p.n_w ds \quad \text{unknown}$$

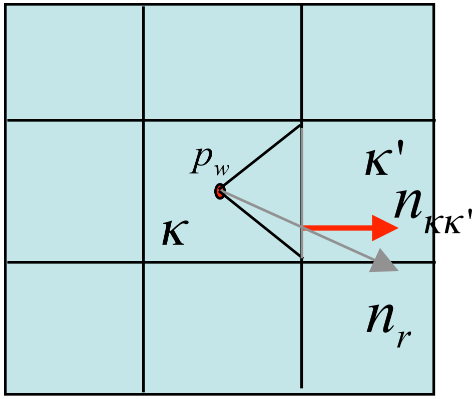
Equation in the well cell κ

$$\sum_{\kappa'} \int_{\sigma=\kappa\kappa'} -K\nabla p.n_{\kappa\kappa'} ds + q_w = 0$$

$$\begin{cases} u = p - \overline{p} & r > r_w \\ u = 0 & r < r_w \end{cases}$$

$$\Rightarrow \quad -K\Delta u = 0 \quad \forall r$$

Proof of Peaceman well index formula



$$\int_{\sigma=\kappa\kappa'} -K\nabla p . n_{\kappa\kappa'} ds = \int_{\sigma=\kappa\kappa'} -K\nabla u . n_{\kappa\kappa'} ds + \int_{\sigma=\kappa\kappa'} -\frac{q_w}{2\pi\ r} n_r . n_{\kappa\kappa'} ds$$

$$\int_{\sigma=\kappa\kappa'} -K\nabla p . n_{\kappa\kappa'} ds \approx \frac{|\sigma|}{|x_\kappa x_{\kappa'}|}(u_\kappa - u_{\kappa'}) + \frac{q_w}{4}$$

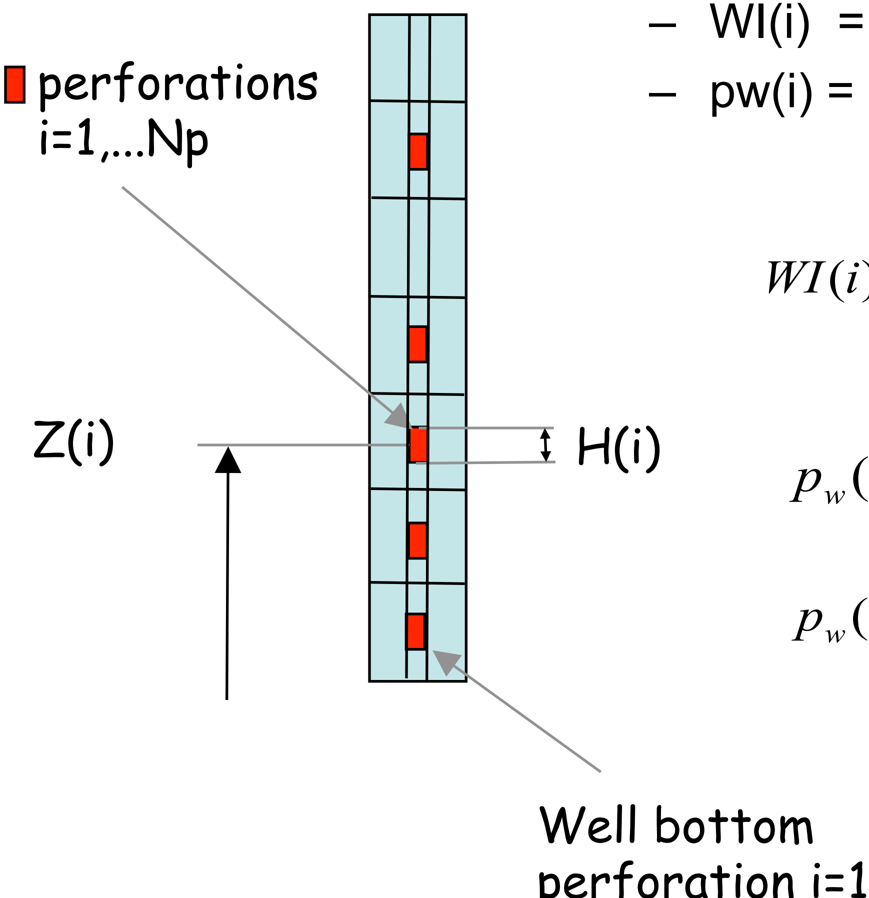
$$\int_{\sigma=\kappa\kappa'} -K\nabla p . n_{\kappa\kappa'} ds \approx \frac{|\sigma|}{|x_\kappa x_{\kappa'}|}\left(0 - (p_{\kappa'} - p_w - \frac{q_w}{2\pi K}\ln(\Delta x / r_w))\right) + \frac{q_w}{4}$$

$$\boxed{\int_{\sigma=\kappa\kappa'} -K\nabla p . n_{\kappa\kappa'} ds \approx \frac{|\sigma|}{|x_\kappa x_{\kappa'}|}\left(p_\kappa - p_{\kappa'}\right)}$$

setting $$\boxed{p_\kappa = p_w + \frac{q_w}{2\pi K}\ln\left(\exp(-\pi/2)\Delta x / r_w\right)}$$

Vertical well with hydrostatic pressure drop

- List of well perforations from bottom to top: i=1,...,Np
  - m(i) = cell of perforation i
  - WI(i) = Well index of perforation i
  - pw(i) = pressure of perforation i



$$WI(i) = \frac{2\pi \ K(m(i))}{\ln(r_0 / r_w)} H(i), \ \ r_0 = 0.14\sqrt{\Delta x^2 + \Delta y^2}$$

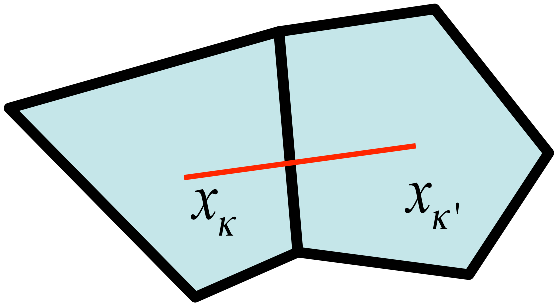
$$p_w(1) = p_{BHP}$$

Specified bottom hole pressure

$$p_w(i) = p_w(i-1) - \underbrace{\rho_{i-1/2} g\big(Z(i) - Z(i-1)\big)}$$

Hydrostatic pressure drop

Analysis of TPFA discretization



– Discrete norms: $u_h = u_\kappa$ on each cell

$$\|u_h\|_{l^2} = \left(\sum_{\kappa \in K} |\kappa| |u_\kappa|^2 \right)^{1/2}$$

$$\|u_h\|_{h_0^1(T_h)} = \left(\sum_{\sigma = \kappa\kappa' \in \Sigma_{\text{int}}} \frac{|\sigma|}{|x_\kappa x_{\kappa'}|} |u_\kappa - u_{\kappa'}|^2 + \sum_{\sigma \in \Sigma_{bound}} \frac{|\sigma|}{|x_{\kappa(\sigma)} x_\sigma|} |u_{\kappa(\sigma)}|^2 \right)^{1/2}$$

– Discrete Poincaré Inequality

$$\|u_h\|_{l^2} \leq D(\Omega) \|u_h\|_{h_0^1}$$

Analysis of TPFA discretization

- A priori estimate: $\begin{cases} -\Delta u = f \ sur \ \Omega \\ u = 0 \ \ sur \ \partial\Omega \end{cases}$

$$\sum_{\kappa} u_{\kappa} \left(\sum_{\sigma=\kappa\kappa'} \frac{|\sigma|}{|x_{\kappa}x_{\kappa'}|}(u_{\kappa} - u_{\kappa'}) + \sum_{\sigma\in\partial\kappa\cap\Sigma_{bound}} \frac{|\sigma|}{|x_{\kappa}x_{\sigma}|}(u_{\kappa} - 0) \right) = \sum_{\kappa} |\kappa| f_{\kappa} u_{\kappa}$$

$$\sum_{\sigma=\kappa\kappa'} \frac{|\sigma|}{|x_{\kappa}x_{\kappa'}|}(u_{\kappa} - u_{\kappa'})^{2} + \sum_{\sigma\in\Sigma_{bound}} \frac{|\sigma|}{|x_{\kappa}x_{\sigma}|}|u_{\kappa}|^{2} \leq \left(\sum_{\kappa} |\kappa| |f_{\kappa}|^{2} \right)^{1/2} \left(\sum_{\kappa} |\kappa| |u_{\kappa}|^{2} \right)^{1/2}$$

$\Rightarrow$ $$\|u_h\|_{h_0^1(T_h)} \leq D(\Omega) \|f_h\|_{l^2}$$

$\Rightarrow$ Uniqueness, existence, stability: well posed problem

Analysis of TPFA discretization

- Error estimate $e_\kappa = u(x_\kappa) - u_\kappa$

$$\sum_{\sigma=\kappa\kappa'} \frac{|\sigma|}{|x_\kappa x_{\kappa'}|}\left(u_\kappa - u_{\kappa'}\right) = |\kappa| f_\kappa$$

$$\sum_{\sigma=\kappa\kappa'} \int_{\kappa\kappa'} -\nabla u . n_{\kappa\kappa'} ds = |\kappa| f_\kappa$$

$$\sum_{\kappa'} \left(\frac{|\kappa\kappa'|}{|x_\kappa x_{\kappa'}|} (e_\kappa - e_{\kappa'}) + |\kappa\kappa'| R_{\kappa\kappa'} \right) = 0$$

$$R_{\kappa\kappa'} = \frac{u(x_\kappa) - u(x_{\kappa'})}{|x_\kappa x_{\kappa'}|} - \frac{1}{|\kappa\kappa'|} \int_{\kappa\kappa'} -\nabla u . n_{\kappa\kappa'} ds$$

$$R_{\kappa\kappa'} = -R_{\kappa'\kappa}, \quad |R_{\kappa\kappa'}| = O(h)$$

Analysis of TPFA discretization

- Error estimate $e_\kappa = u(x_\kappa) - u_\kappa$

$$\sum_{\kappa'} \left(\frac{|\kappa\kappa'|}{|x_\kappa x_{\kappa'}|}(e_\kappa - e_{\kappa'}) + |\kappa\kappa'| R_{\kappa\kappa'} \right) = 0 \qquad R_{\kappa\kappa'} = -R_{\kappa'\kappa},\ |R_{\kappa\kappa'}| = O(h)$$

$$\|e_h\|^2_{h_0^1(T_h)} = -\sum_{\kappa} e_\kappa \sum_{\kappa'} |\kappa\kappa'| R_{\kappa\kappa'} = -\sum_{\sigma} |\kappa\kappa'| (e_\kappa - e_{\kappa'}) R_{\kappa\kappa'}$$

$$\|e_h\|^2_{h_0^1(T_h)} \leq C \|e_h\|_{h_0^1(T_h)} \left(\sum_{\sigma} |\kappa\kappa'| |x_\kappa x_{\kappa'}| \right) h$$

$$\boxed{\|e_h\|_{h_0^1(T_h)} \leq Ch}$$

TPFA discretization

- Discrete linear system: $A_h U_h = F_h$
  - Coercivity: $(A_h U_h, U_h) \geq K_{\min} \|u_h\|^2_{h_0^1(T_h)}$
  - Symmetry: $A_h = A_h^T$
  - Monotonicity: $A_h^{-1} \geq 0$ (A_h=M-Matrice)

M- Matrice → monotonicity

$$\sum_{\sigma=\kappa\kappa'} \frac{|\sigma|}{|x_\kappa x_{\kappa'}|}(u_\kappa - u_{\kappa'}) + \sum_{\sigma\in\partial\kappa\cap\Sigma_{bord}} \frac{|\sigma|}{|x_{\kappa(\sigma)} x_\sigma|}(u_{\kappa(\sigma)} - g_\sigma) = |\kappa| f_\kappa$$

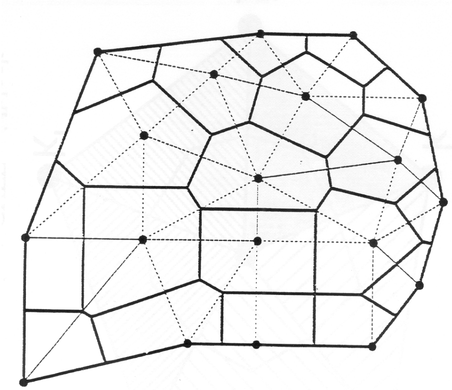
$$A_{i,i} > 0, \quad A_{i,j\neq i} \leq 0$$

$$\sum_j A_{i,j} \geq 0$$

$$\exists i \text{ such that } \sum_j A_{i,j} > 0$$

Graph of A strongly connex

$$\Rightarrow \quad A^{-1} \geq 0$$



Proof: $$A_{ii}U_i + \sum_{j\neq i} A_{ij}U_j = S_i \geq 0$$

$$\text{if } U_{i_0} = \min_i U_i < 0$$

$$\left(\sum_j A_{i_0 j}\right) U_{i_0} = \sum_{j\neq i_0} A_{i_0 j}(U_{i_0} - U_j) + S_{i_0}$$

Propagate using neighboring cells j from i_0 to a cell i such that $\left(\sum_j A_{ij}\right) > 0$

Finite volume schemes

- *Parabolic Equations: time discretization*
  - *Implicit Euler integration in time*
  - *Stability analysis*

Parabolic model

$$\begin{cases} \partial_t u + div(-K\nabla u) = f \ on \ \Omega\times(0,T) \\ -K\nabla u.n = 0 \ on \ \partial\Omega \ \times(0,T) \\ u_{t=0} = u_0 \ on \ \Omega \end{cases}$$

Finite volume space and time discretizations

$$t^0 = 0, \quad t^{n+1} - t^n = \Delta t$$

Discrete space and time conservation law:

$$\int_{t^n}^{t^{n+1}} \int_{\kappa} \left[\partial_t u - div\left(K \nabla u(t)\right) - f \right] dx dt = 0$$

$$\int_{\kappa} u(t^{n+1}) dx - \int_{\kappa} u(t^n) dx + \int_{t^n}^{t^{n+1}} \underbrace{\left[f(t) + \sum_{\sigma = \kappa\kappa'} \int_{\sigma} - K \nabla u(t).n_{\kappa\kappa'} ds \right]}_{Y(t)} dt = 0$$

Forward Euler time integration scheme

$$\int_{t^n}^{t^{n+1}} Y(t) dt \approx \Delta t Y(t^{n+1})$$

Consistency of order Δt

Finite volume space and time discretizations

$$\frac{u_\kappa^{n+1} - u_\kappa^n}{\Delta t}|\kappa| + \sum_{\sigma=\kappa\kappa'} T_{\kappa\kappa'}\left(u_\kappa^{n+1} - u_{\kappa'}^{n+1}\right) = |\kappa| f_\kappa$$

Stability analysis: discrete energy estimate

$$\sum_{\kappa} \left(u_{\kappa}^{n+1}\right) \left[\frac{u_{\kappa}^{n+1} - u_{\kappa}^{n}}{\Delta t} |\kappa| + \sum_{\sigma = \kappa\kappa'} T_{\kappa\kappa'} \left(u_{\kappa}^{n+1} - u_{\kappa'}^{n+1}\right) = |\kappa| f_{\kappa} \right]$$

$$2a(a-b) = a^2 - b^2 + (a-b)^2$$

$$\left\| u_h^{n+1} \right\|_{l^2}^2 - \left\| u_h^{n} \right\|_{l^2}^2 + \left\| u_h^{n+1} - u_h^{n} \right\|_{l^2}^2 + 2\Delta t \left\| u_h^{n+1} \right\|_{h_0^1}^2 \leq$$
$$2\Delta t \left\| f_h \right\|_{l^2} \left\| u_h^{n+1} \right\|_{l^2}$$

Stability: discrete energy estimate

$$\left\| u_h^{n+1} \right\|_{l^2}^2 - \left\| u_h^n \right\|_{l^2}^2 + \left\| u_h^{n+1} - u_h^n \right\|_{l^2}^2 \leq \gamma \Delta t \left\| f_h \right\|_{l^2}^2$$

$$\left\| u_h^N \right\|_{l^2}^2 \leq \left\| u_h^0 \right\|_{l^2}^2 + \gamma t^N \left\| f_h \right\|_{l^2}^2$$

Unconditionally stable scheme in L^2 norm

Stability analysis: discrete maximum principle (f=0, zero flux BC)

$$u_\kappa^{n+1}\left(1+\sum_{\sigma=\kappa\kappa'}\frac{\Delta t}{|\kappa|}T_{\kappa\kappa'}\right)= \sum_{\sigma=\kappa\kappa'}\frac{\Delta t}{|\kappa|}T_{\kappa\kappa'}u_{\kappa'}^{n+1}+u_\kappa^n$$

$$m \le u_\kappa^n \le M \ \ for\ all\ \kappa$$

Then $$m \le u_\kappa^{n+1} \le M \ \ for\ all\ \kappa$$

Stability analysis: discrete maximum principle (f=0, zero flux BC)

Proof: $$if\ u_{\kappa_0}^{n+1} = \sup_{\kappa} u_{\kappa}^{n+1} > M$$

$$u_{\kappa_0}^{n+1} - M = \sum_{\sigma=\kappa_0\kappa'} \frac{\Delta t}{|\kappa_0|} T_{\kappa_0\kappa'} \left(u_{\kappa'}^{n+1} - u_{\kappa_0}^{n+1}\right) + \left(u_{\kappa_0}^{n} - M\right)$$

$\Rightarrow$ lead to a contradiction

Exercise: well test with compressible Darcy single phase flow

- Parabolic equation (linearized)

$$
\begin{cases}
\left(\frac{1}{\rho_0}\frac{d\rho}{dp}\right)\rho_0 \partial_t p + div\left(-\rho_0 \frac{K}{\mu}\nabla p\right) = 0 \;\; on \;\; \Omega \times (0,T) \\
p = p_D \;\; on \;\; \partial\Omega_D \times (0,T) \\
-\rho_0 \frac{K}{\mu}\nabla p.n = g \;\; on \;\; \partial\Omega_N \times (0,T) \\
p_{t=0} = p_0 \;\; on \;\; \Omega
\end{cases}
$$

Ex: well test