Finite volume schemes for two phase incompressible Darcy flows

- *Two phase Darcy flow models*
  - *Saturations*
  - *Relative permeabilities: two phase Darcy velocities*

- *Scalar hyperbolic conservation laws*
  - *Two points monotone fluxes*
  - *Explicit scheme and CFL condition*

- *IMPES discretization of two phase flows*
  - *Pressure computation: elliptic equation*
  - *Saturation computation: scalar hyperbolic equation*

OIL AND WATER SATURATIONS

In a given pore volume V_p, let V_w be the water phase volume and V_o the oil phase volume with $V_p = V_w + V_o$.

The oil and water saturations are the volume fractions of the oil and water phases

$$S_w = \frac{V_w}{V_p}$$

$$S_o = \frac{V_o}{V_p}$$

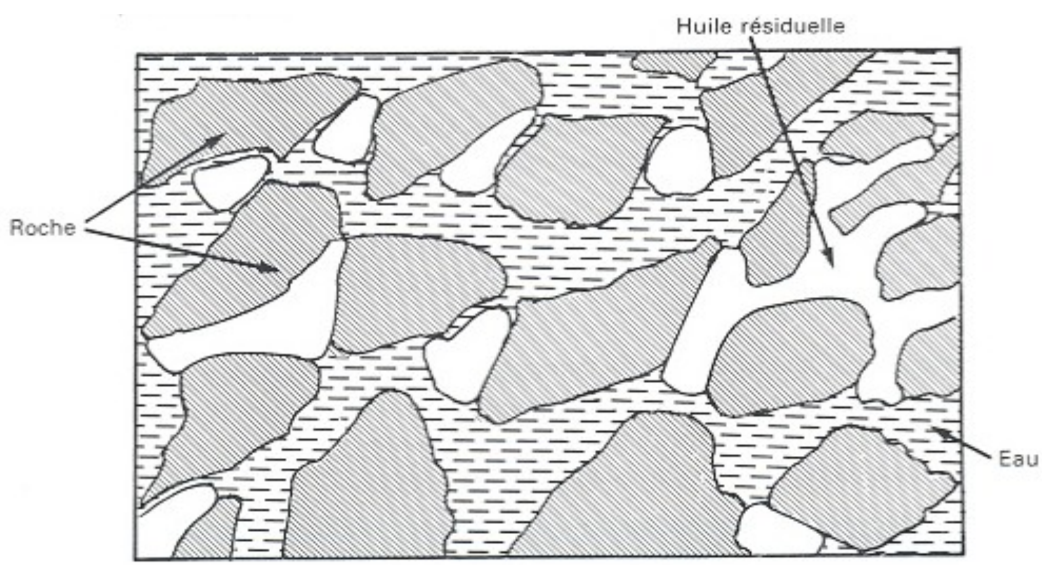
with $$S_w + S_o = 1$$

Oil residual Saturation

and Irreducible Water Saturation

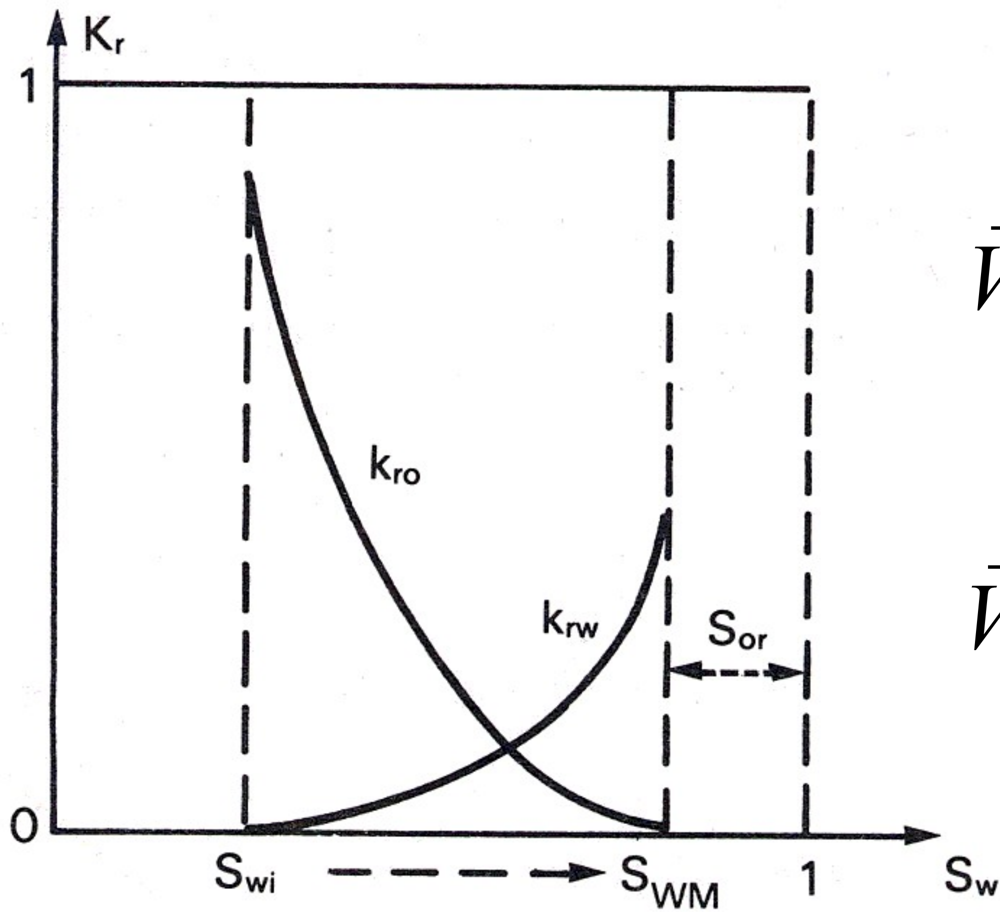
Sor = Volume fraction of oil trapped by capillarity which cannot be displaced

Swi = Volume fraction of water trapped by capillarity which cannot be displaced



RELATIVE PERMEABILITIES

AND TWO PHASE DARCY VELOCITIES



$$\vec{V}_w = -\frac{kr_w(S_w)}{\mu_w} K\nabla P$$

$$\vec{V}_o = -\frac{kr_o(S_o)}{\mu_o} K\nabla P$$

TWO PHASE DARCY FLOW

Mass conservation for α=*o,w*

$$\frac{\partial\left(\phi\rho_\alpha S_\alpha\right)}{\partial t}+div\left(\rho_\alpha\vec{V}_\alpha\right)=0$$

Darcy velocities for α=*o,w*

$$\vec{V}_\alpha=-\frac{kr_\alpha}{\mu_\alpha}K\nabla P$$

Pore volume conservation

$$\sum_{\alpha=w,o}S_\alpha=1$$

TWO PHASE INCOMPRESSIBLE DARCY FLOWS

$$S_w = 1 - S_o$$

$$\begin{cases} -\phi \partial_t S_o + div\left(-M_w(1-S_o)K\nabla P\right) = 0 \\ \phi \partial_t S_o + div\left(-M_o(S_o)K\nabla P\right) = 0 \end{cases}$$

Water and oil mobilities

$$M_\alpha(S_\alpha) = \frac{kr_\alpha(S_\alpha)}{\mu_\alpha}, \ \alpha = w, o$$

Total mobility

$$M_T(S_o) = M_w(1-S_o) + M_o(S_o)$$

TWO PHASE INCOMPRESSIBLE DARCY FLOWS

Pressure equation

$$div\left(-M_T(S_o)K\nabla P\right)=0$$

Total velocity

$$\vec{V}_T=-M_T(S_o)K\nabla P$$

Oil saturation equation

$$\phi\ \partial_t S_o+div\left(f_o(S_o)\vec{V}_T\right)=0$$

with

$$f_o(S_o)=\frac{M_o(S_o)}{M_T(S_o)}=\frac{M_o(S_o)}{M_o(S_o)+M_w(1-S_o)}$$

IMPES DISCRETIZATION

OF TWO PHASE INCOMPRESSIBLE DARCY FLOWS

Elliptic Pressure equation

$$div\left(-M_T(S_o^n)K\nabla P^{n+1}\right)=0$$

Total Velocity

$$\vec{V}_T^{n+1}=-M_T(S_o^n)K\nabla P^{n+1}$$

Hyperbolic oil saturation equation

$$\phi\frac{S_o^{n+1}-S_o^n}{t^{n+1}-t^n}+div\left(f_o(S_o^n)\vec{V}_T^{n+1}\right)=0$$

Total Mobility

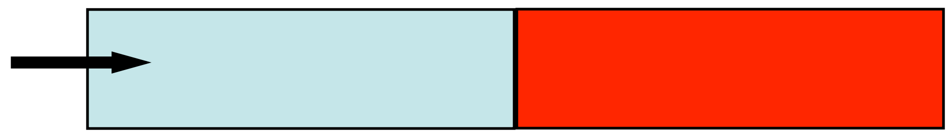
$$M_T(S_o^{n+1})=M_w(1-S_o^{n+1})+M_o(S_o^{n+1})$$

Scalar hyperbolic equation
Buckley leverett example

Water injection in a 1D reservoir

$S_o = 0$

$V_T = V$



$p = p_0$

$$\partial_x V_T = 0$$

$$\phi \partial_t S_o + \partial_x \left(\frac{M_o(S_o)}{M_w(1-S_o) + M_o(S_o)} V_T \right) = 0$$

1D hyperbolic equation
$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \ \ on\ \Re \times (0,T) \\ u_{t=0} = u_0 \ \ on\ \Re \end{cases}$$

Scalar hyperbolic equation in dimension d

$$\begin{cases} \partial_t u + div\left(\vec{F}(x,t,u)\right) = 0 \;\; on \; \Re^d \times (0,T) \\ u_{t=0} = u_0 \;\; on \; \Re^d \end{cases}$$

Ex: Two phase Darcy flow

$$\phi \partial_t S + div\left(\frac{M_w}{M_w + M_o} \vec{V}_T \right) = 0$$

$$div(\vec{V}_T) = 0$$

Advection equation in 1D

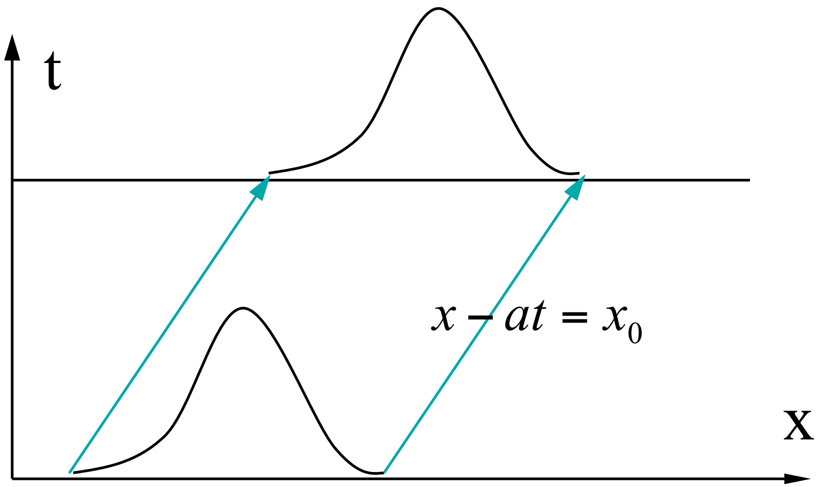
$$\begin{cases} \partial_t u + a\partial_x u = 0 \ \ on\ \Re \times (0,T) \\ u_{t=0} = u_0 \ \ on\ \Re \end{cases}$$

$$u(x,t) = u_0(x - at)$$

The solution is constant along characteristics:

$$\begin{cases} \dfrac{dx}{dt} = a \\ x(0) = x_0 \end{cases}$$

$$x - at = x_0$$



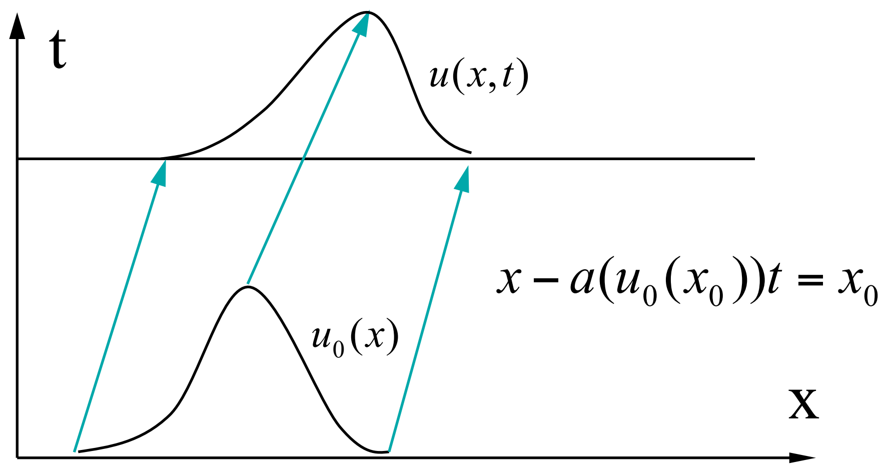
Scalar non linear hyperbolic equation

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \;\; on \; \Re \times (0,T) \\ u_{t=0} = u_0 \;\; on \; \Re \end{cases}$$

Characteristics: $$\begin{cases} \dfrac{dx}{dt}(t) = a\left[u\left(x(t),t\right)\right] \\ x(0) = x_0 \end{cases} \qquad a(u) = f'(u)$$

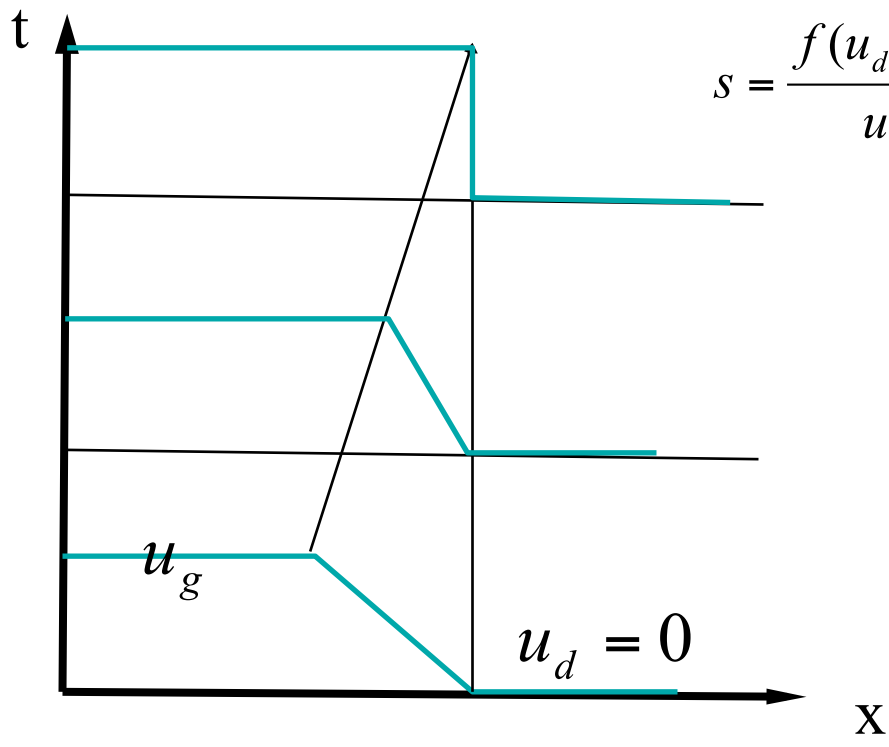
$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= \partial_t u + x'(t)\partial_x u \\ &= \partial_t u + a(u)\partial_x u \\ &= 0 \end{aligned}$$

$$u(x(t),t) = u_0(x_0)$$



Formation of a shock

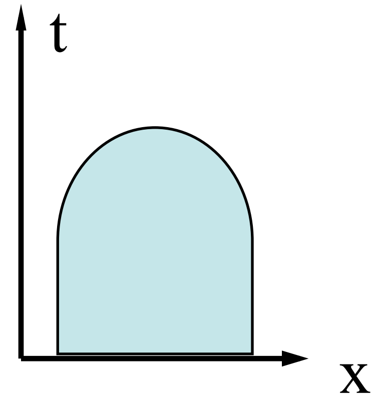
$$\begin{cases} f(u) = \dfrac{u^2}{2} \\ a(u) = u \end{cases}$$



$$s = \frac{f(u_d) - f(u_g)}{u_d - u_g} = \frac{u_g}{2}$$

Weak solutions

Weak formulation: $\varphi \in C_c^1$ compactly supported in $(-\infty, +\infty) \times [0, \infty)$



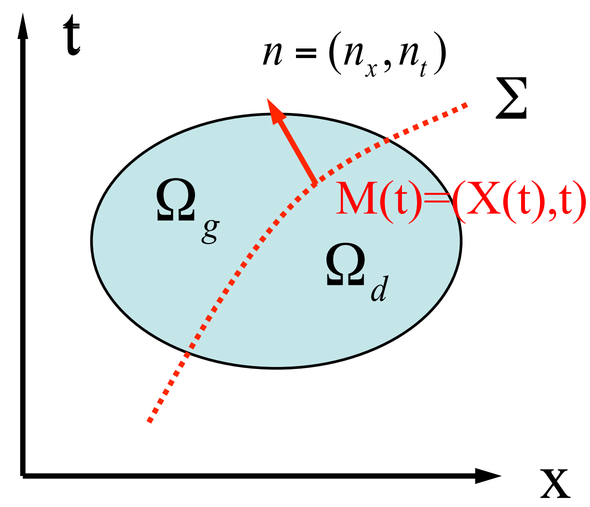
$$\int_0^{\infty} \int_{-\infty}^{+\infty} \varphi(x,t) \left[\partial_t u + \partial_x f(u) \right] dx dt = 0$$

$$\int_0^{\infty} \int_{-\infty}^{+\infty} \left[u \partial_t \varphi + f(u) \partial_x \varphi \right] dx dt = -\int_{-\infty}^{+\infty} \varphi(x,0) u_0(x) dx$$

Rankine-Hugoniot relations for piecewise smooth solutions

φ compactly supported in $\Omega = \Omega_g \cup \Omega_d \cup \Sigma$

$$\int_{\Omega_g} \left[u \partial_t \varphi + f(u) \partial_x \varphi \right] dxdt + \int_{\Omega_d} \left[u \partial_t \varphi + f(u) \partial_x \varphi \right] dxdt = 0$$



$$\int_{\Omega_g} \left[u \partial_t \varphi + f(u) \partial_x \varphi \right] dxdt = -\int_{\Omega_g} \varphi \left[\partial_t u + \partial_x f(u) \right] dxdt + \int_{\Sigma} \varphi \left[u_g(t) n_t + f(u_g(t)) n_x \right] d\sigma$$

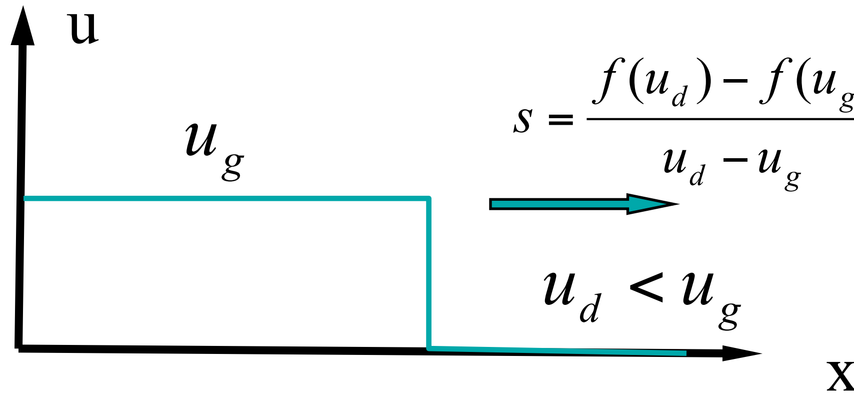
$$n = \frac{\left(-1, X'(t) \right)}{\left[1 + (X'(t))^2 \right]^{1/2}}$$

$$\int_{\Sigma} \varphi \left[(u_d - u_g) n_t + (f(u_d) - f(u_g)) n_x \right] d\sigma = 0$$

$$s = X'(t) = -\frac{n_t}{n_x} = \frac{f(u_d) - f(u_g)}{u_d - u_g}$$

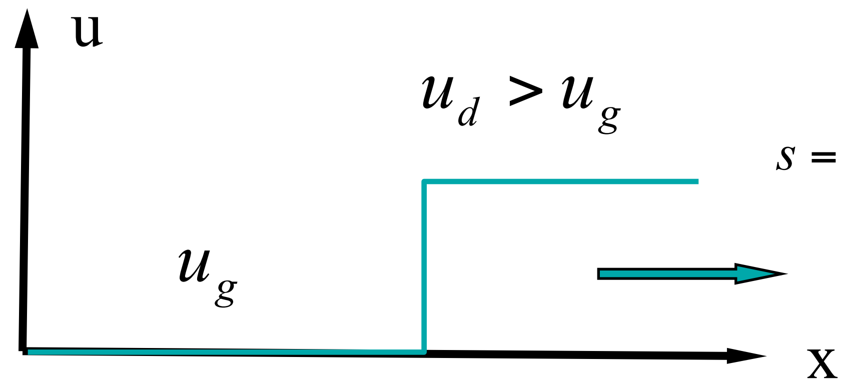
Non uniqueness of weak solutions

$$\begin{cases} f(u) = \dfrac{u^2}{2} \\ a(u) = u \end{cases}$$



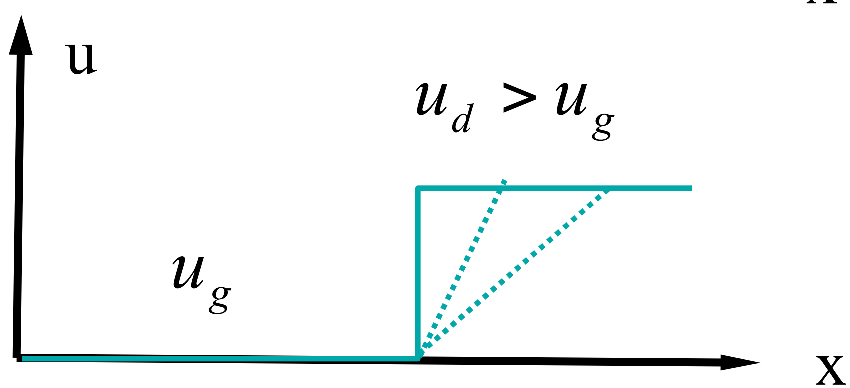
$$s = \frac{f(u_d) - f(u_g)}{u_d - u_g} = \frac{u_g + u_d}{2}$$

Non physical shock



$$s = \frac{f(u_d) - f(u_g)}{u_d - u_g} = \frac{u_g + u_d}{2}$$

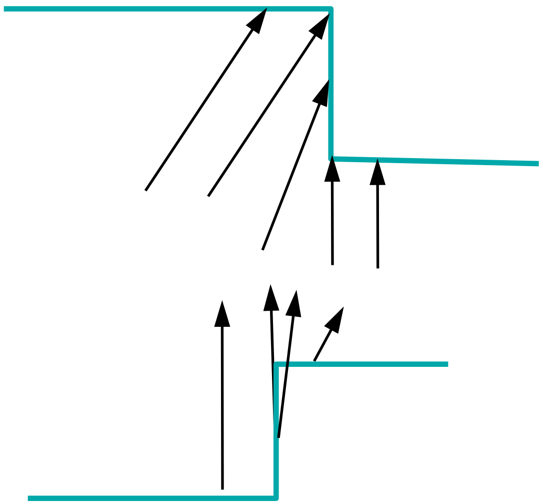
Rarefaction wave



Physical solutions

- If f is a convex function, a schock (u_g,u_d) is admissible iff u_g>u_d

- Lax criterium: the characteristics focus to the shock



- Physical solutions are vanishing viscosity solutions

$$\partial_t u_\nu + \partial_x f(u_\nu) - \nu \partial_{x^2} u_\nu = 0$$

- The entropy is increasing when crossing the shock

Entropy solutions to the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \;\; on \;\Re \times (0,T) \\ u_{t=0} = u_0 \;\; on \;\Re \end{cases} \qquad u_0 \in L^\infty(\Re),\;\; f \in C^1(\Re)$$

Existence and uniqueness of the entropy solution $u \in L^\infty(\Re)$

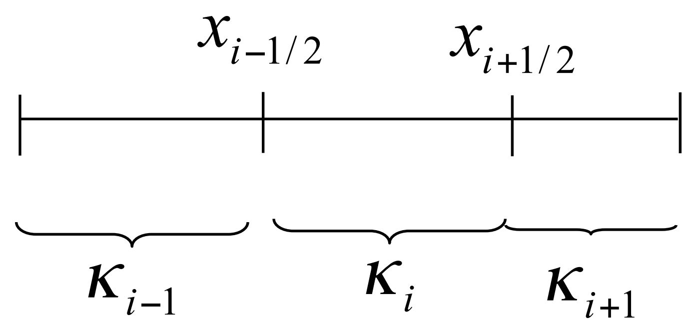
It satisfies the maximum principle:

$$\text{if} \quad m \le u_0(x) \le M \;\; for\; all\;\; x \in \Re$$

$$\text{then} \quad m \le u(x,t) \le M \;\; for\; all\;\; x \in \Re,\;\; t > 0$$

1D Case

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \;\; on \; \Re \times (0,T) \\ u_{t=0} = u_0 \;\; on \; \Re \end{cases}$$



$$|\kappa_i| \frac{\left(u_i^{n+1} - u_i^n\right)}{\Delta t} + f_{i+1/2} - f_{i-1/2} = 0$$

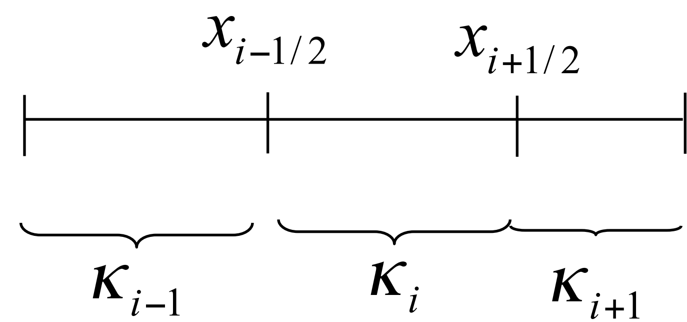
Numerical flux: g

$$f_{i+1/2} = g(u_i^n, u_{i+1}^n) \qquad f_{i-1/2} = g(u_{i-1}^n, u_i^n) = 0$$

$$g(u,u) = f(u)$$ Flux consistency

Advection equation: centered scheme

$$\begin{cases} \partial_t u + \partial_x au = 0 \;\; on \; \Re \times (0,T) \\ u_{t=0} = u_0 \;\; on \; \Re \end{cases}$$



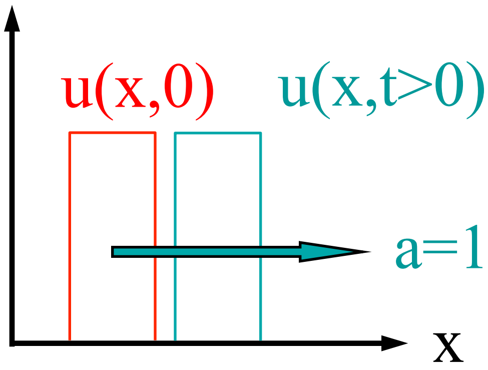
$$f_{i+1/2} = g(u_i, u_{i+1}) = \frac{f(u_i) + f(u_{i+1})}{2} = a\frac{u_i + u_{i+1}}{2}$$

$$|\kappa_i|\frac{\left(u_i^{n+1} - u_i^n\right)}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2} = 0$$

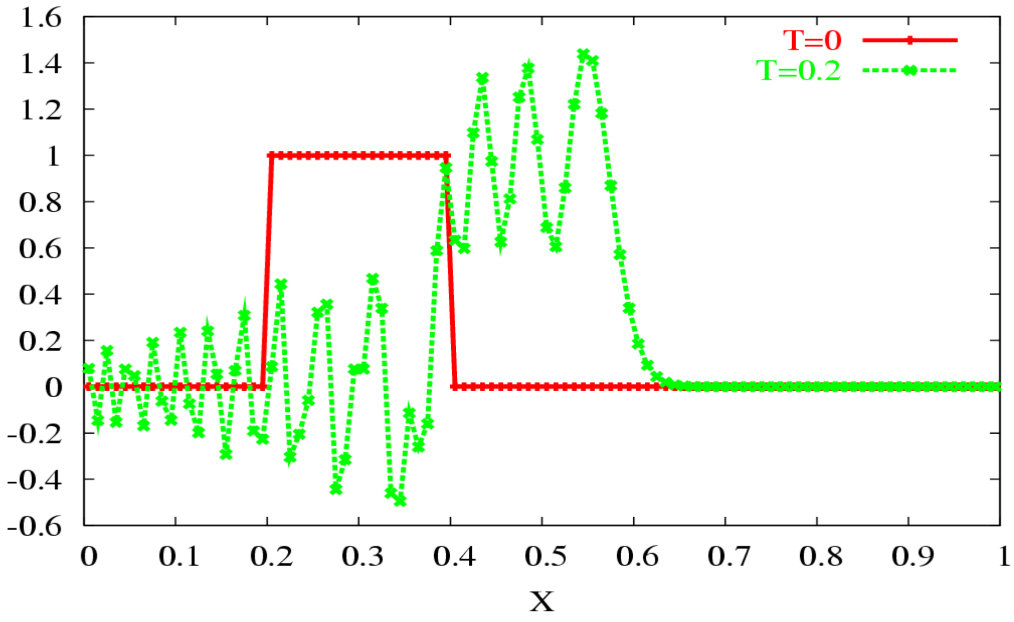
The centered scheme is always unstable

The centered scheme is unstable

$$\begin{cases} \partial_t u + \partial_x a u = 0 \;\; on \;\; \Re \times (0,T) \\ u_{t=0} = u_0 \;\; on \;\; \Re \end{cases}$$

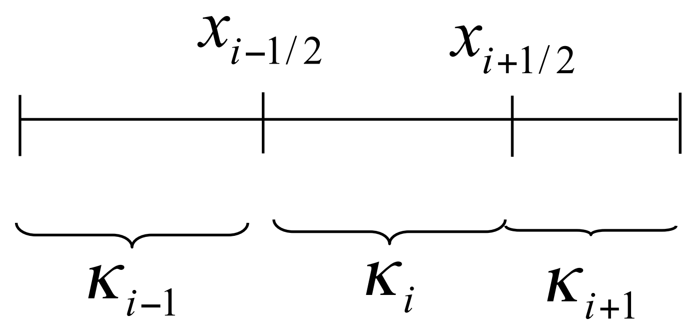


Centered scheme with dt=0.001



Advection equation: upwind scheme

$$\begin{cases} \partial_t u + \partial_x au = 0 \;\; on \; \Re \times (0,T) \\ u_{t=0} = u_0 \;\; on \; \Re \end{cases}$$



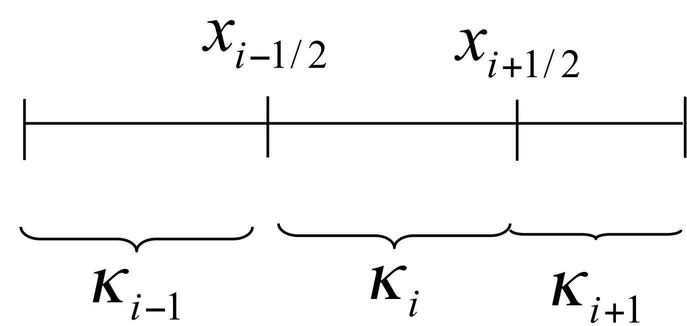
$$f(u) = au \;\; with \;\; a > 0$$

$$f_{i+1/2} = g(u_i^n, u_{i+1}^n) = f(u_i^n) = au_i^n$$

$$|\kappa_i| \frac{\left(u_i^{n+1} - u_i^n\right)}{\Delta t} + a(u_i^n - u_{i-1}^n) = 0$$

Advection equation: upwind scheme

$$|\kappa_i| \frac{\left(u_i^{n+1} - u_i^n\right)}{\Delta t} + a(u_i^n - u_{i-1}^n) = 0$$



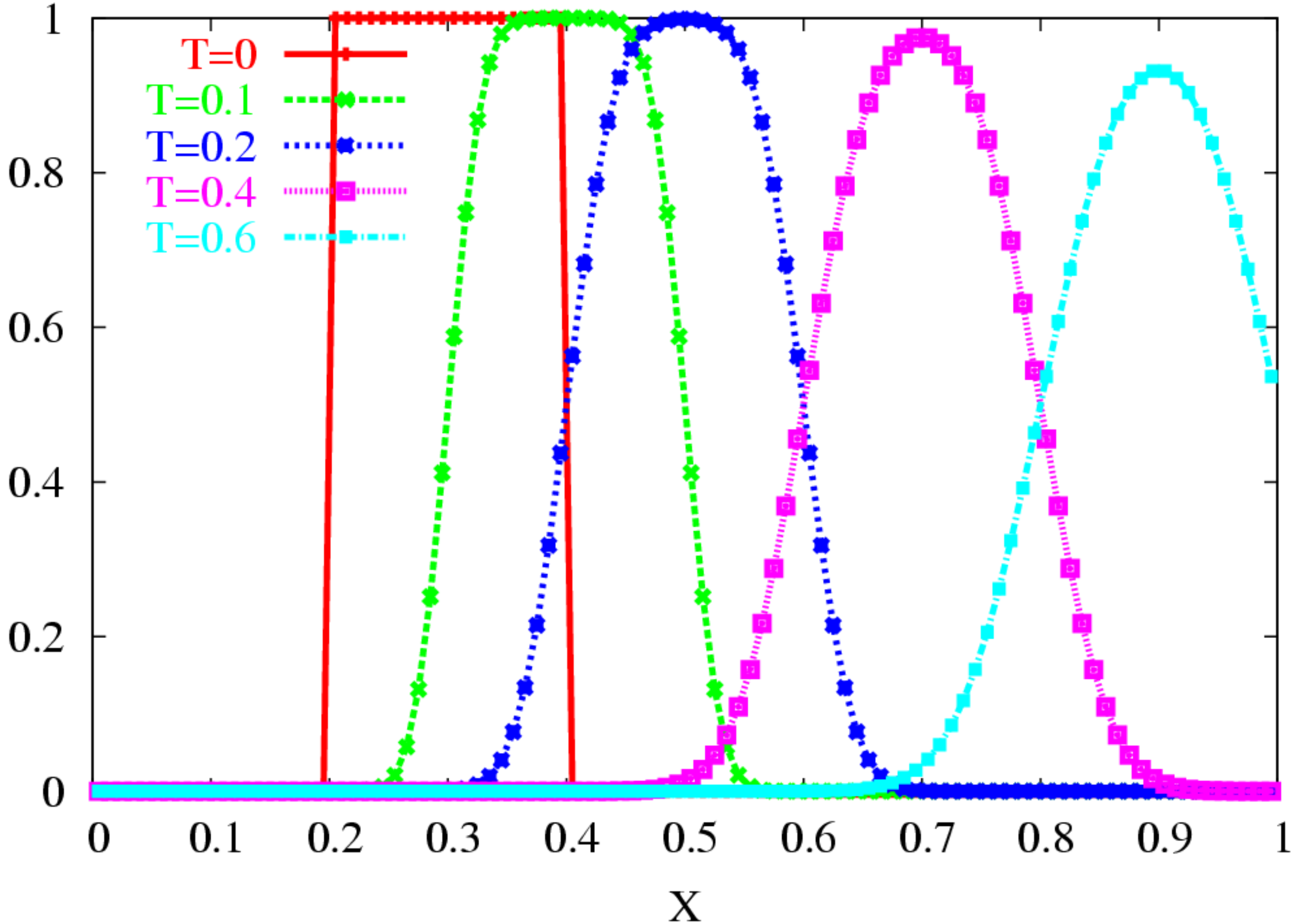
$$u_i^{n+1} = (1 - \xi_i) u_i^n + \xi_i u_{i-1}^n \qquad \xi_i = \frac{a \Delta t}{|\kappa_i|}$$

The scheme is stable under the CFL condition

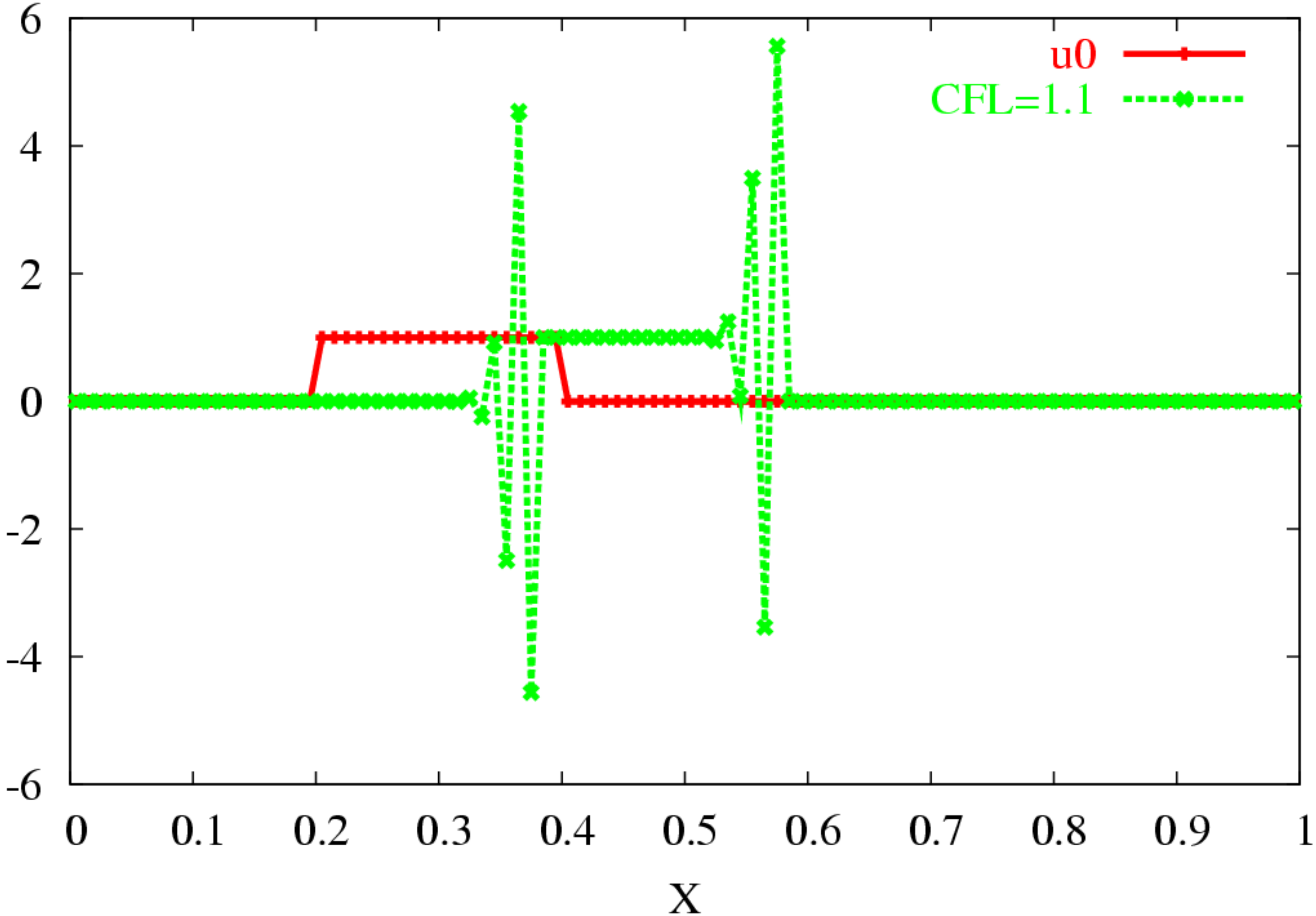
$$\boxed{\Delta t \leq \frac{\inf_i |\kappa_i|}{a}}$$

= advection of at most one cell in one time step

Advection equation, upwind scheme with CFL = 0.5

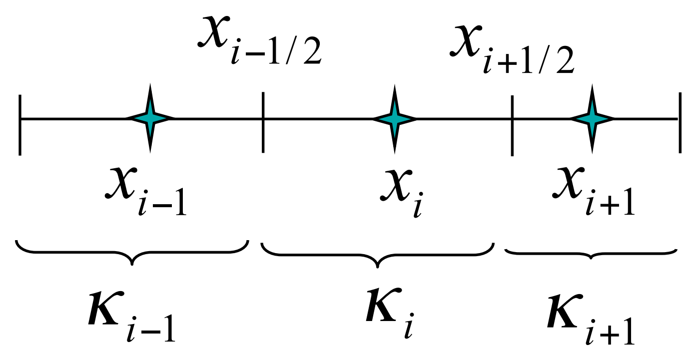


Advection equation: upwind scheme with CFL =1.1



Advection equation: interpretation of the upwind flux

$$a > 0$$



$$f_{i+1/2} = g(u_i, u_{i+1}) = f(u_i) = au_i$$

- The upwind flux is first order consistent

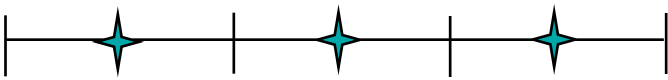
- We look for the flux which is second order consistent

$$g(u_i, u_{i+1}) = a\frac{(u_i + u_{i+1})}{2} + \frac{\left(\frac{a|x_{i+1} - x_i|}{2}\right)}{|x_{i+1} - x_i|}(u_i - u_{i+1})$$

Centered flux $f = au$ — diffusion of viscosity $f = -K(x)\partial_x u$

$$K_{i+1/2} = \frac{a|x_{i+1} - x_i|}{2}$$

Advection equation: the explicit time integration reduce diffusion



uniform mesh $|\kappa_i| = \Delta x \ \forall i$

We look for the equation satisfied by $u_i^n = u(x_i, t^n)$ $\quad \partial_t u + a \partial_x u = 0$

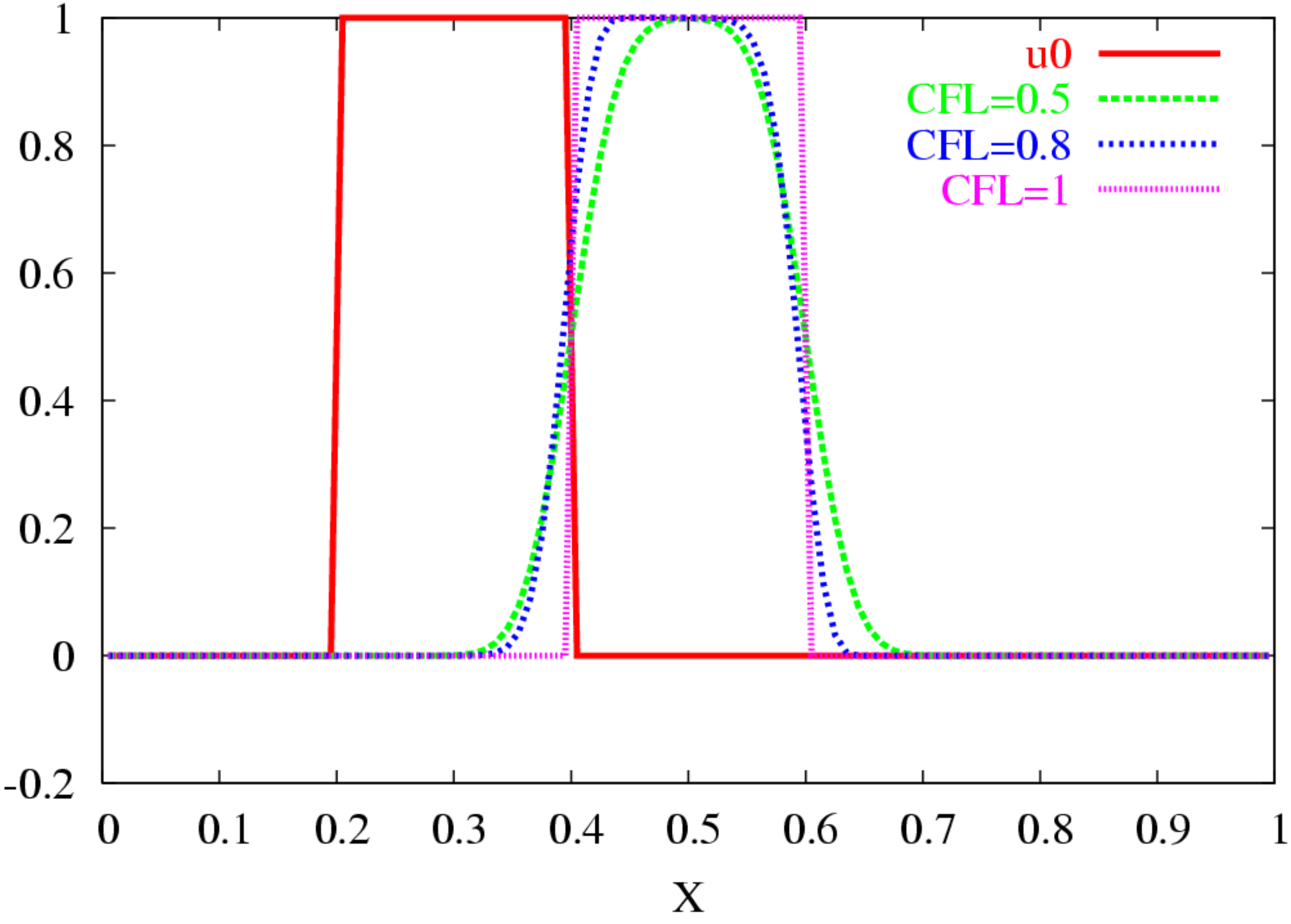
at second order in $O(\Delta x^2 + \Delta t^2)$

$$\frac{\left(u_i^{n+1} - u_i^n\right)}{\Delta t} + a \frac{\left(u_i^n - u_{i-1}^n\right)}{\Delta x} =$$

$$\left[\partial_t u + a \partial_x u - \frac{a \Delta x}{2}(1 - CFL) \partial_{x^2} u\right](x_i, t^{n+1/2}) + O(\Delta x^2 + \Delta t^2)$$

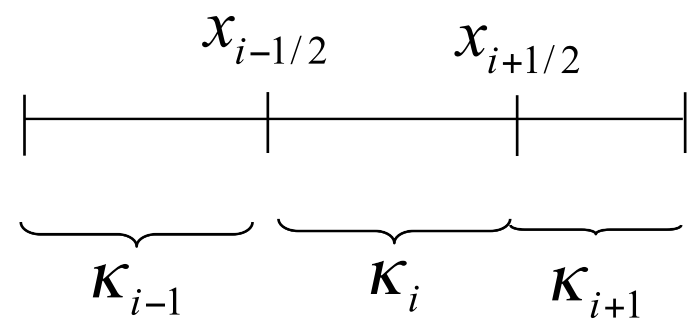
$$CFL = \frac{\Delta t}{\Delta x / a} \leq 1$$

Advection equation: explicit upwind scheme on a uniform mesh



Monotonous two point schemes for 1D scalar hyperbolic equations

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \ \ on\ \Re \times (0,T) \\ u_{t=0} = u_0 \ \ on\ \Re \end{cases}$$



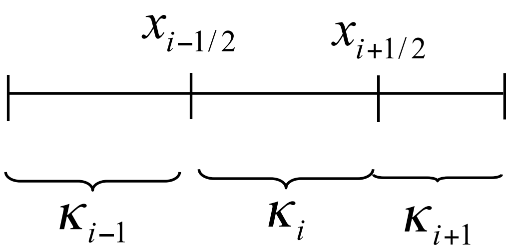
$$|\kappa_i| \frac{\left(u_i^{n+1} - u_i^n\right)}{\Delta t} + g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n) = 0$$

$g(u,u) = f(u)$ Flux consistency

$g(u,v)$ increasing in u and decreasing in v

stable scheme (under CFL <=1) and convergent

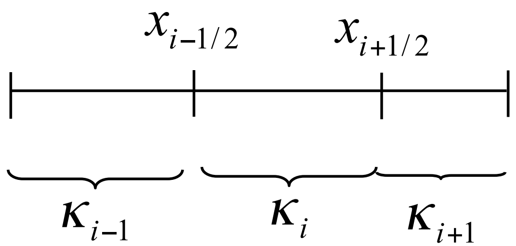
Monotonous two points schemes in 1D proof of stability



$$|\kappa_i| \frac{\left(u_i^{n+1} - u_i^n\right)}{\Delta t} + g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n) = 0$$

$$|\kappa_i| \frac{\left(u_i^{n+1} - u_i^n\right)}{\Delta t} + \frac{g(u_i^n, u_{i+1}^n) - g(u_i^n, u_i^n)}{u_{i+1}^n - u_i^n}(u_{i+1}^n - u_i^n)$$
$$+ \frac{g(u_i^n, u_i^n) - g(u_{i-1}^n, u_i^n)}{u_i^n - u_{i-1}^n}(u_i^n - u_{i-1}^n) = 0$$

Monotonous two points schemes in 1D proof of stability



$$u_i^{n+1} = u_i^n \left[1 - \frac{\Delta t}{|\kappa_i|} \left(d_{i+1/2} + d_{i-1/2} \right) \right] + \frac{\Delta t}{|\kappa_i|} \left(d_{i+1/2} u_{i+1}^n + d_{i-1/2} u_{i-1}^n \right)$$

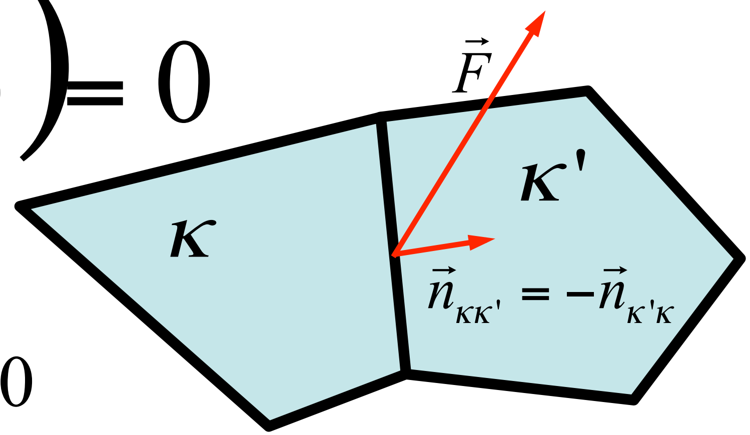
$$0 \le d_{i-1/2} = \frac{g(u_{i-1}^n, u_i^n) - g(u_i^n, u_i^n)}{u_{i-1}^n - u_i^n} \le d_1 \qquad \left(for\; u_0 \in [m, M] \right)$$

$$0 \le d_{i+1/2} = \frac{g(u_i^n, u_i^n) - g(u_i^n, u_{i+1}^n)}{u_{i+1}^n - u_i^n} \le d_2$$

$$CFL\ condition: \;\; \Delta t \le \frac{\inf_i |\kappa_i|}{d_1 + d_2} \quad \Longrightarrow \quad m \le u_i^{n+1} \le M$$

Finite Volume Scheme for scalar hyperbolic conservation laws in dimension d

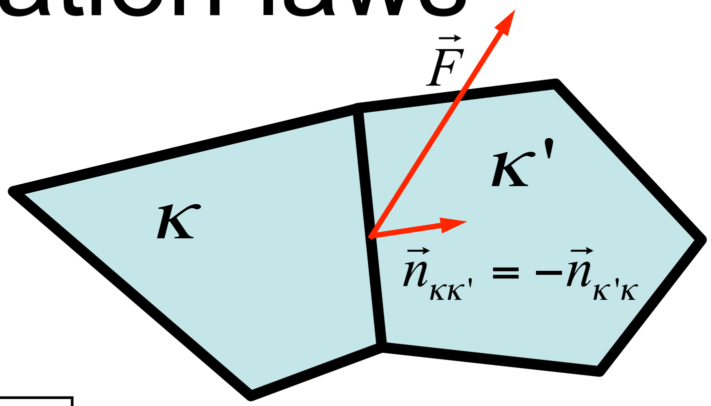
$$\partial_t u + div\left(\vec{F}(x,t,u)\right) = 0$$



$$\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{\kappa}\left[\partial_t u + div\left(\vec{F}(x,t,u)\right)\right]dxdt = 0$$

$$\frac{1}{\Delta t}\left(\underbrace{\int_\kappa u(x,t^{n+1})dx}_{|\kappa|u_\kappa^{n+1}} - \underbrace{\int_\kappa u(x,t^n)dx}_{|\kappa|u_\kappa^n}\right) + \sum_{\kappa'}\underbrace{\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{\kappa\kappa'}\vec{F}(x,t,u).\vec{n}_{\kappa\kappa'}d\sigma}_{F_{\kappa\kappa'} = -F_{\kappa'\kappa}} = 0$$

Finite Volume Scheme for scalar hyperbolic conservation laws



Discrete conservation law

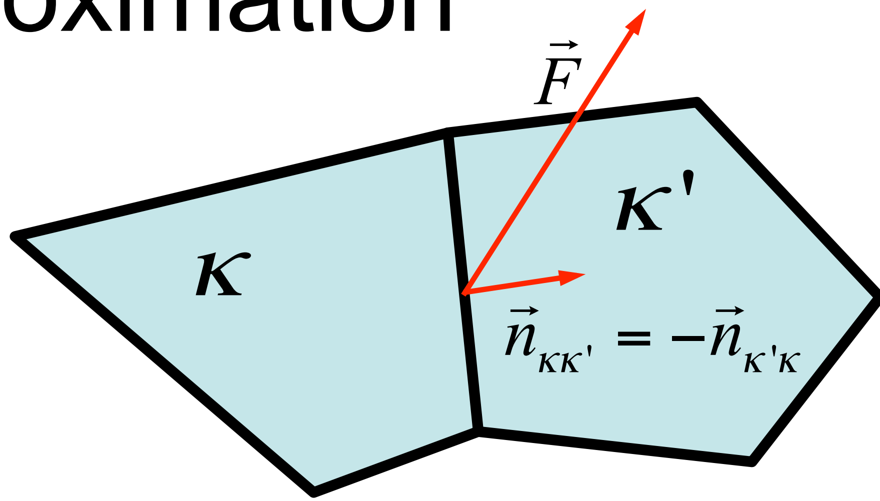
$$|\kappa| \frac{\left(u_\kappa^{n+1} - u_\kappa^n\right)}{\Delta t} + \sum_{\kappa'} F_{\kappa\kappa'} = 0$$

$$F_{\kappa\kappa'} = -F_{\kappa'\kappa}$$

$$F_{\kappa\kappa'} \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_{\kappa\kappa'} \vec{F}(x,t,u).\vec{n}_{\kappa\kappa'} d\sigma$$

Two point Flux Approximation

$$|\kappa| \frac{\left(u_\kappa^{n+1} - u_\kappa^n\right)}{\Delta t} + \sum_{\kappa'} F_{\kappa\kappa'}(u_\kappa^*, u_{\kappa'}^*) = 0$$



$$F_{\kappa\kappa'}(u_\kappa^*, u_{\kappa'}^*) \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_{\kappa\kappa'} \vec{F}(x,t,u).\vec{n}_{\kappa\kappa'} d\sigma$$

- Explicit scheme $* = n$
- Implicit scheme $* = n+1$

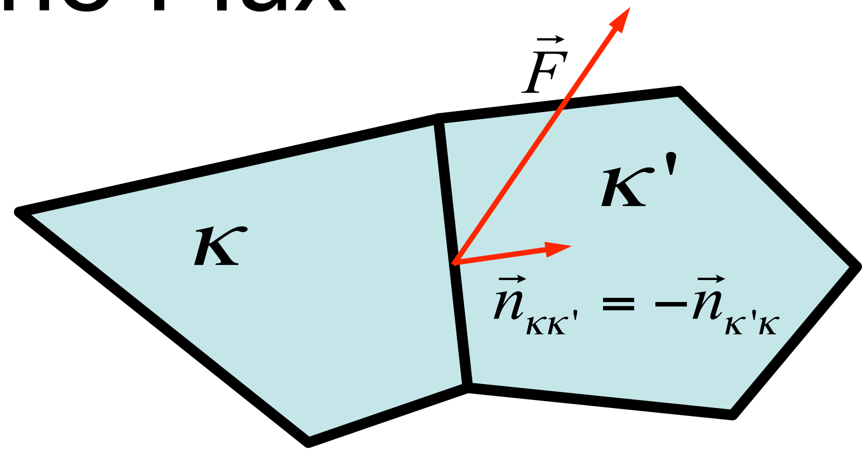
$$F_{\kappa\kappa'}(u_\kappa, u_{\kappa'}) = -F_{\kappa'\kappa}(u_{\kappa'}, u_\kappa)$$

Flux Conservativity

$$F_{\kappa\kappa'}(u,u) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_{\kappa\kappa'} \vec{F}(x,t,u).\vec{n}_{\kappa\kappa'} d\sigma$$

Flux Consistency

Monotone Flux



$$|\kappa| \frac{\left(u_\kappa^{n+1} - u_\kappa^n\right)}{\Delta t} + \sum_{\kappa'} F_{\kappa\kappa'}(u_\kappa^*, u_{\kappa'}^*) = 0$$

$F_{\kappa\kappa'}(u, v)$ Increasing with u and decreasing with v

Maximum principle for $div_x \vec{F} = 0$

$$\partial_t u + div\left(\vec{F}(x,t,u)\right) = 0$$

$$div_x \vec{F} = 0 \quad \longrightarrow \quad \sum_{\sigma=\kappa\kappa'} F_{\kappa\kappa'}(v,v) = 0$$

$$|\kappa| \frac{\left(u_\kappa^{n+1} - u_\kappa^n\right)}{\Delta t} + \sum_{\sigma=\kappa\kappa'} \underbrace{\frac{\left[F_{\kappa\kappa'}(u_\kappa^n, u_{\kappa'}^n) - F_{\kappa\kappa'}(u_\kappa^n, u_\kappa^n)\right]}{u_\kappa^n - u_{\kappa'}^n}}_{d \;\geq\; d_{\kappa\kappa'} \;\geq\; 0} (u_\kappa^n - u_{\kappa'}^n) = 0$$

$$u_\kappa^{n+1} = u_\kappa^n \left(1 - \frac{\Delta t}{|\kappa|} \sum_{\kappa'} d_{\kappa\kappa'}\right) + \frac{\Delta t}{|\kappa|} \sum_{\kappa'} d_{\kappa\kappa'} u_{\kappa'}^n = 0$$

$$CFL\ condition \quad \Delta t \leq \frac{\inf_\kappa |\kappa|}{(Nbneighbors)d}$$

Saturation Equation for two phase Darcy flow

$$\phi \partial_t S_o + div\left(f_o(S_o)\vec{V}_T\right) = 0$$

$$\vec{F}(x,t,S_o) = \vec{V}(x,t) f_o(S_o)$$ with

$$f_o(S_o) = \frac{M_o(S_o)}{M_w(1-S_o) + M_o(S_o)}$$

Increasing function

$$\vec{F}(x,t,u)=\vec{V}(x,t)f(u)$$ with f increasing

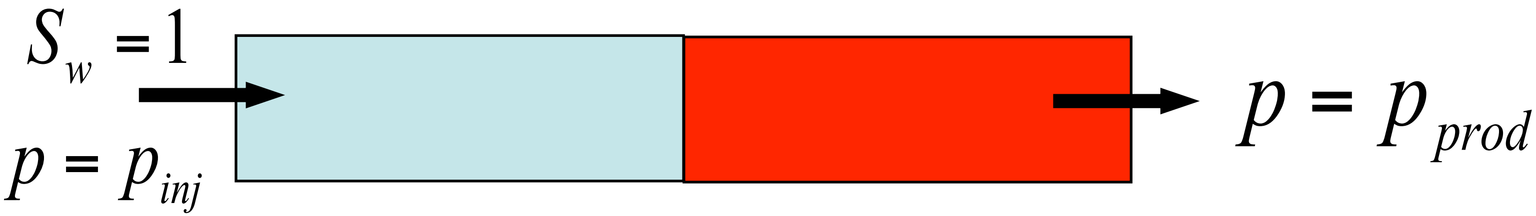
$$\boxed{\partial_t u+div\left(\vec{V}(x,t)f(u)\right)=0}$$

$$F_{\kappa\kappa'}(u_\kappa,u_{\kappa'})=(V_{\kappa\kappa'}^{n+1})^+f(u_\kappa)+(V_{\kappa\kappa'}^{n+1})^-f(u_{\kappa'})$$

with $$V_{\kappa\kappa'}^{n+1}=-V_{\kappa'\kappa}^{n+1}=\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{\kappa\kappa'}\vec{V}(x,t).\vec{n}_{\kappa\kappa'}ds$$

$$\boxed{|\kappa|\frac{\left(u_\kappa^{n+1}-u_\kappa^n\right)}{\Delta t}+\sum_{\kappa'}(V_{\kappa\kappa'}^{n+1})^+f(u_\kappa^n)+(V_{\kappa\kappa'}^{n+1})^-f(u_{\kappa'}^n)=0}$$

1D test case
Injection of water in a reservoir



IMPES DISCRETIZATION

OF TWO PHASE INCOMPRESSIBLE DARCY FLOWS

Elliptic Pressure equation

$$div\left(-M_T(S_o^n)K\nabla P^{n+1}\right)=0$$

Total Velocity

$$\vec{V}_T^{n+1}=-M_T(S_o^n)K\nabla P^{n+1}$$

Hyperbolic oil saturation equation

$$\phi\frac{S_o^{n+1}-S_o^n}{t^{n+1}-t^n}+div\left(f_o(S_o^n)\vec{V}_T^{n+1}\right)=0$$

Total Mobility

$$M_T(S_o^{n+1})=M_w(1-S_o^{n+1})+M_o(S_o^{n+1})$$

Pressure Equation and Total Velocities

$$\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\text{int}}} V_{T,\kappa\kappa'}^{\text{int},n+1} + \sum_{\sigma\in\partial\kappa\cap\Sigma_{bound}} V_{T,\sigma}^{bound,n+1} = 0$$

$$\begin{cases} V_{T,\kappa\kappa'}^{\text{int},n+1} = -V_{T,\kappa'\kappa}^{\text{int},n+1} = M_{T,\sigma}^{\text{int},n} \; T_{\sigma}^{\text{int}} \left(p_{\kappa}^{n+1} - p_{\kappa'}^{n+1}\right) \sigma = \kappa\kappa' \in \Sigma_{\text{int}} \\ V_{T,\sigma}^{bound,n+1} = M_{T,\sigma}^{bound,n} \; T_{\sigma}^{bound} \left(p_{\kappa(\sigma)}^{n+1} - p_{\sigma}\right) \sigma \in \Sigma_{bound} \end{cases}$$

Oil Saturation Equation

$$|\kappa|\phi\frac{\left(S_\kappa^{n+1}-S_\kappa^{n}\right)}{\Delta t}+\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\text{int}}}\overbrace{f_o(S_\kappa^n)(V_{T,\kappa\kappa'}^{\text{int},n+1})^{+}+f_o(S_{\kappa'}^n)(V_{T,\kappa\kappa'}^{\text{int},n+1})^{-}}^{F_{o,\kappa\kappa'}^{\text{int}}=-F_{o,\kappa'\kappa}^{\text{int}}}$$

$$+\sum_{\sigma\in\partial\kappa\cap\Sigma_{bound}}\underbrace{f_o(S_\kappa^n)(V_{T,\sigma}^{bound,n+1})^{+}}_{F_{o,\sigma}^{bound}}=0$$

Total Mobilities

$$M_{T,\sigma}^{\text{int},n+1} = \begin{cases} M_w(1 - S_\kappa^{n+1}) + M_o(S_\kappa^{n+1}) \quad if \quad V_{T,\kappa\kappa'}^{\text{int},n+1} \geq 0 \\ M_w(1 - S_{\kappa'}^{n+1}) + M_o(S_{\kappa'}^{n+1}) \quad if \quad V_{T,\kappa\kappa'}^{\text{int},n+1} < 0 \end{cases} \quad \sigma = \kappa\kappa' \in \Sigma_{\text{int}}$$

$$M_{T,\sigma}^{bound,n+1} = \begin{cases} M_w(1) \quad if \quad V_{T,\sigma}^{bound,n+1} < 0 \\ M_w(1 - S_{\kappa(\sigma)}^{n+1}) + M_o(S_{\kappa(\sigma)}^{n+1}) \quad if \quad V_{T,\sigma}^{bound,n+1} \geq 0 \end{cases} \quad \sigma \in \Sigma_{bound}$$

Initialization at n=0

- p arbitrary
- So=1-Swi
- Total mobilities

$$M_{T,\sigma}^{\text{int},n=0} = M_w(S_{wi}) + M_o(1 - S_{wi}) \;\; \sigma = \kappa\kappa' \in \Sigma_{\text{int}}$$

$$M_{T,\sigma}^{bound,n=0} = \begin{cases} M_w(1) \;\; for \;\; INPUT \;\; BOUNDARY \\ M_w(S_{wi}) + M_o(1 - S_{wi}) \;\; for \;\; OUPUT \;\; BOUNDARY \end{cases} \sigma \in \Sigma_{bound}$$

Computation of Saturations

- $S = S^n$
- Loop over inner faces: i=1,...,Nint
  - m1 = cellint(i,1)
  - m2 = cellint(i,2)
  - VTint(i) = MTint(i)*Tint(i)*(p(m1)-p(m2))
  - Voint(i) = fo(S(m1)) $VTint(i)^{+}$ + fo(S(m2)) $VTint(i)^{-}$
  - S(m1) = S(m1) - Voint(i)*dt/(volume(m1)*phi)
  - S(m2) = S(m2) + Voint(i)*dt/(volume(m2)*phi)
- Loop over boundary faces: i=1:Nbord
  - m = cellbound(i)
  - VTbound(i) = MTbound(i)*Tbound(i)*(p(m)-pbound(i))
  - Vobound(i) = fo(S(m)) $VTbound(i)^{+}$
  - S(m) = S(m) - Vobound(i)*dt/(volume(m)*phi)

CFL Condition

$$|\kappa|\phi \frac{\left(S_\kappa^{n+1}-S_\kappa^n\right)}{\Delta t} + \sum_{\kappa'\in\partial\kappa\cap\Sigma_{\mathrm{int}}} (V_{T,\kappa\kappa'}^{\mathrm{int},n+1})^+ f_o(S_\kappa^n) + (V_{T,\kappa\kappa'}^{n+1})^- f_o(S_{\kappa'}^n) + \sum_{\sigma\in\partial\kappa\cap\Sigma^{bound}} (V_{T,\sigma}^{bound,n+1})^+ f_o(S_\kappa^n) = 0$$

$$-\left(\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\mathrm{int}}} V_{T,\kappa\kappa'}^{\mathrm{int},n+1} + \sum_{\sigma\in\partial\kappa\cap\Sigma^{bord}} V_{T,\sigma}^{bound,n+1}\right)\times f_o(S_\kappa^n) = 0$$

$$0 \le d_{\kappa\kappa'} \le \sup_{0\le S\le 1} f_o'(S) \qquad 0 \le d_{\kappa 0} \le \sup_{0\le S\le 1} f_o'(S)$$

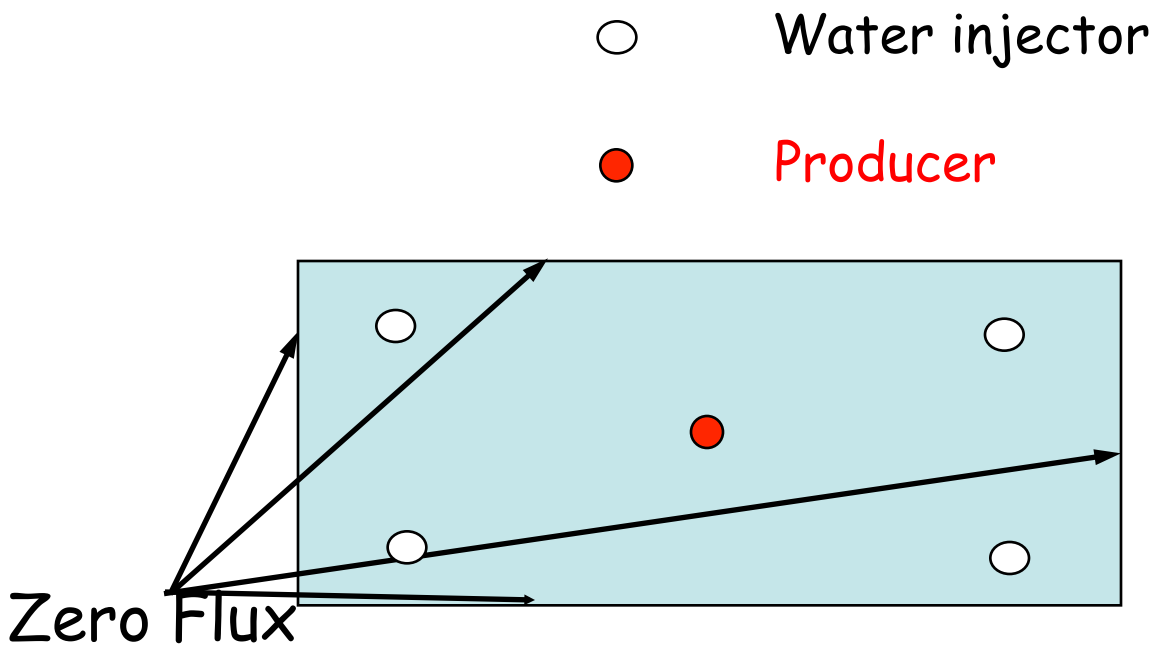
$$|\kappa|\phi \frac{\left(S_\kappa^{n+1}-S_\kappa^n\right)}{\Delta t} + \sum_{\kappa'} (-V_{T,\kappa\kappa'}^{\mathrm{int},n+1})^+ \left(\frac{f_o(S_\kappa^n)-f_o(S_{\kappa'}^n)}{S_\kappa^n - S_{\kappa'}^n}\right)(S_\kappa^n - S_{\kappa'}^n) + \sum_{\sigma\in\partial\kappa\cap\Sigma^{bound}} -(V_{T,\sigma}^{bound,n+1})^- \left(\frac{f_o(S_\kappa^n)-f_o(0)}{S_\kappa^n - 0}\right)(S_\kappa^n - 0) = 0$$

$$S_\kappa^{n+1} = S_\kappa^n\left(1-\frac{\Delta t}{|\kappa|\phi}\left[\sum_{\kappa'}(-V_{T,\kappa\kappa'}^{\mathrm{int},n+1})^+ d_{\kappa\kappa'} + \sum_{\sigma\in\partial\kappa\cap\Sigma^{bound}} -(V_{T,\sigma}^{bound,n+1})^- d_{\kappa 0}\right]\right) + \frac{\Delta t}{|\kappa|\phi}\sum_{\kappa'}(-V_{T,\kappa\kappa'}^{\mathrm{int},n+1})^+ d_{\kappa\kappa'} S_{\kappa'}^n + \frac{\Delta t}{|\kappa|\phi}\sum_{\sigma\in\partial\kappa\cap\Sigma^{bound}} -(V_{T,\sigma}^{bound,n+1})^- d_{\kappa 0}\times 0$$

$$\boxed{\Delta t^{n+1} \le \frac{\inf_\kappa |\kappa|\phi}{\sup_\kappa\left(\sum_{\kappa'}(-V_{T,\kappa\kappa'}^{\mathrm{int},n+1})^+ + \sum_{\sigma\in\partial\kappa\cap\Sigma^{bound}} -(V_{T,\sigma}^{bound,n+1})^-\right)\sup_{0\le S\le 1} f_o'(S)}}$$

Two phase flow
Water injector wells
Producer well (oil + water)

- Five spots 2D test case



Two phase flow
IMPES scheme with producer wells and water injector wells

Scheme for the pressure equation and computation of the total velocities

$$\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\text{int}}} V_{T,\kappa\kappa'}^{\text{int},n+1} + \sum_{i\in\Pi^{prod} \,|\, \kappa(i)=\kappa} V_{T,i}^{prod,n+1} + \sum_{i\in\Pi^{inj} \,|\, \kappa(i)=\kappa} V_{T,i}^{inj,n+1} = 0$$

$$\begin{cases} V_{T,\kappa\kappa'}^{\text{int},n+1} = -V_{T,\kappa'\kappa}^{\text{int},n+1} = M_{T,\sigma'}^{\text{int},n}\, T_{\sigma}^{\text{int}} \left(p_{\kappa}^{n+1} - p_{\kappa'}^{n+1}\right) \sigma = \kappa\kappa' \in \Sigma_{\text{int}} \\ V_{T,i}^{prod,n+1} = M_{T,i}^{prod,n}\, IP_{i} \left(p_{\kappa(i)}^{n+1} - p_{w,i}\right)^{+} i \in \Pi^{prod} \\ V_{T,i}^{inj,n+1} = M_{T,i}^{inj,n}\, IP_{i} \left(p_{\kappa(i)}^{n+1} - p_{w,i}\right)^{-} i \in \Pi^{inj} \end{cases}$$

Two phase flow
IMPES scheme with producer wells and water injector wells

Monotonous explicit scheme
for the saturation equation

$$F^{\text{int}}_{o,\kappa\kappa'} = -F^{\text{int}}_{o,\kappa'\kappa}$$

$$|\kappa|\phi\frac{\left(S_\kappa^{n+1}-S_\kappa^n\right)}{\Delta t}+\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\text{int}}}\overbrace{f_o(S_\kappa^n)(V_{T,\kappa\kappa'}^{\text{int},n+1})^+ + f_o(S_{\kappa'}^n)(V_{T,\kappa\kappa'}^{\text{int},n+1})^-}$$

$$+\sum_{i\in\Pi^{prod}\,|\,\kappa(i)=\kappa} f_o(S_\kappa^n)V_{T,i}^{prod,n+1}$$

$$+\sum_{i\in\Pi^{inj}\,|\,\kappa(i)=\kappa} 0\times V_{T,i}^{inj,n+1} = 0$$

Two phase flow
IMPES scheme with producer wells and water injector wells

Update of the total Mobilities with upwinding in space

$$\begin{cases} M_{T,\sigma}^{\text{int},n+1} = \begin{cases} M_w(1-S_\kappa^{n+1}) + M_o(S_\kappa^{n+1}) \; if \quad p_\kappa^{n+1} \geq p_{\kappa'}^{n+1} \\ M_w(1-S_{\kappa'}^{n+1}) + M_o(S_{\kappa'}^{n+1}) \; if \quad p_\kappa^{n+1} < p_{\kappa'}^{n+1} \end{cases} \quad \sigma = \kappa\kappa' \in \Sigma_{\text{int}} \\ M_{T,i}^{prod,n+1} = M_w(1-S_{\kappa(i)}^{n+1}) + M_o(S_{\kappa(i)}^{n+1}) \;\; i \in \Pi^{prod} \\ M_{T,i}^{inj,n+1} = M_w(1) \;\; i \in \Pi^{inj} \end{cases}$$

Initialization at n=0

- p=p0 (arbitrary)
- So=1-Swi, Sw=Swi
- Total Mobilities

$$\begin{cases} M_{T,\sigma}^{\text{int},n=0} = M_w(S_{wi}) + M_o(1-S_{wi}) \ \ \sigma = \kappa\kappa' \in \Sigma_{\text{int}} \\ M_{T,i}^{prod,n=0} = M_w(S_{wi}) + M_o(1-S_{wi}) \ \ i \in \Pi^{prod} \\ M_{T,i}^{inj,n=0} = M_w(1) \ \ i \in \Pi^{inj} \end{cases}$$

Computation of the saturation with a loop on inner faces and wells

- $S = S^n$
- Loop over inner faces: i=1,...,Nint
  - m1=cellint(i,1)
  - m2=cellint(i,2)
  - VTint(i)=MTint(i)*Tint(i)*(p(m1)-p(m2))
  - Foint(i)=fo(S(m1)) $VTint(i)^{+}$ + fo(S(m2)) $VTint(i)^{-}$
  - S(m1) = S(m1) - Foint(i)*dt/(volume(m1)*phi)
  - S(m2) = S(m2) + Foint(i)*dt/(volume(m2)*phi)

- Loop over producers i=1:Pprod
  - m=cellwell(i)
  - VTwell(i)=MTwell(i)*IP(i)*(p(m)-pwell(i))+
  - Fowell(i)=fo(S(m))*VTwell(i)
  - S(m) = S(m) - Fowell(i)*dt/(volume(m)*phi)

CFL condition

$$|\kappa|\phi\frac{\left(S_\kappa^{n+1}-S_\kappa^n\right)}{\Delta t}+\sum_{\kappa'\in\partial\kappa\cap\Sigma_{\rm int}}(V_{T,\kappa\kappa'}^{{\rm int},n+1})^+f_o(S_\kappa^n)+(V_{T,\kappa\kappa'}^{n+1})^-f_o(S_{\kappa'}^n)+\sum_{i\in\Pi^{prod}|\kappa(i)=\kappa}V_{T,i}^{prod,n+1}f_o(S_\kappa^n)=0$$

$$-\left(\sum_{\sigma=\kappa\kappa'\in\partial\kappa\cap\Sigma_{\rm int}}V_{T,\kappa\kappa'}^{{\rm int},n+1}+\sum_{i\in\Pi^{prod}|\kappa(i)=\kappa}V_{T,i}^{prod,n+1}+\sum_{i\in\Pi^{inj}|\kappa(i)=\kappa}V_{T,i}^{inj,n+1}\right)\times f_o(S_\kappa^n)=0$$

$$0\le d_{\kappa\kappa'}\le\sup_{0\le S\le1}f_o'(S)\qquad 0\le d_{\kappa 0}\le\sup_{0\le S\le1}f_o'(S)$$

$$|\kappa|\phi\frac{\left(S_\kappa^{n+1}-S_\kappa^n\right)}{\Delta t}+\sum_{\kappa'}(-V_{T,\kappa\kappa'}^{{\rm int},n+1})^+\left(\frac{f_o(S_\kappa^n)-f_o(S_{\kappa'}^n)}{S_\kappa^n-S_{\kappa'}^n}\right)(S_\kappa^n-S_{\kappa'}^n)+\sum_{i\in\Pi^{inj}|\kappa(i)=\kappa}(-V_{T,i}^{inj,n+1})\left(\frac{f_o(S_\kappa^n)-f_o(0)}{S_\kappa^n-0}\right)(S_\kappa^n-0)=0$$

$$S_\kappa^{n+1}=S_\kappa^n\left(1-\frac{\Delta t}{|\kappa|\phi}\left[\sum_{\kappa'}(-V_{T,\kappa\kappa'}^{{\rm int},n+1})^+d_{\kappa\kappa'}+\sum_{i\in\Pi^{inj}|\kappa(i)=\kappa}(-V_{T,i}^{inj,n+1})d_{\kappa 0}\right]\right)+\frac{\Delta t}{|\kappa|\phi}\sum_{\kappa'}(-V_{T,\kappa\kappa'}^{{\rm int},n+1})^+d_{\kappa\kappa'}S_{\kappa'}^n+\frac{\Delta t}{|\kappa|\phi}\sum_{i\in\Pi^{inj}|\kappa(i)=\kappa}(-V_{T,i}^{inj,n+1})d_{\kappa 0}\times 0$$

$$\boxed{\Delta t^{n+1}\le\frac{\inf_\kappa|\kappa|\phi}{\sup_\kappa\left(\sum_{\kappa'}(-V_{T,\kappa\kappa'}^{{\rm int},n+1})^++\sum_{i\in\Pi^{inj}|\kappa(i)=\kappa}(-V_{T,i}^{inj,n+1})\right)\sup_{0\le S\le1}f_o'(S)}}$$