- Numerical simulation of a stratigraphic model - single lithology case

Single lithology stratigraphic model

Our objective is to simulate the infill of sedimentary basins at large space and time scales.

- $\Omega = (0, L_x)$: horizontal extension of the basin
- $(0, T)$ is the time interval of the simulation with $T > 0$
- $h(x,t)$ is the sediment thickness for $(x,t) \in \Omega \times (0,T)$
- $h_{\text{sea}}(t)$ is the given sea level for $t \in (0,T)$
- $b(x,t) = h_{\text{sea}}(t) - h(x,t)$ is the bathymetry

The model accounts for the following sediment thickness conservation and boundary and initial conditions

$$\left\{\begin{array}{l} \partial_t h(x,t) + \operatorname{div}\, (\nabla\psi(b(x,t))) = 0 \text{ on } \Omega \times (0,T) \\ \\ h(x,0) = h_{\text{init}}(x) \text{ on } \Omega \\ \\ \nabla\psi(b(x,t)) \cdot \boldsymbol{n} = g_0 \text{ at } x = 0 \\ \nabla\psi(b(x,t)) \cdot \boldsymbol{n} = g_1 \text{ at } x = L_x \end{array}\right. \tag{1}$$

where $\psi(b) = \int_0^b k(u)\mathrm{d}u$ with $k(b) > 0$ the diffusion coefficient of the sediments measuring their ability to be transported by gravity. This coefficient is modeled by a nonlinear function of the bathymetry as follows

$$k(b) = \left\{\begin{array}{l} k^m \text{ if } b \geqslant 0 \\ k^c \text{ otherwise} \end{array}\right.$$

Finite Volume discretization

The model is discretized using a Two Points Flux Approximation (TPFA) on an orthogonal mesh. We obtain at each time step $n > 0$ and for each cell K

$$\left\{\begin{array}{l} |K|\dfrac{h_K^n - h_K^{n-1}}{\Delta t^n} + \displaystyle\sum_{\sigma = K|L \in \Sigma_K \cap \Sigma_{\text{int}}} T_\sigma(\psi(b_L^n) - \psi(b_K^n)) + \sum_{\sigma \in \Sigma_K \cap \Sigma_{\text{b}}} g_\sigma = 0 \\ \\ \text{with } \; T_{\sigma = K|L} = \dfrac{|\sigma|}{\operatorname{dist}(x_K, x_L)} \\ \\ b_K^n = h_{\text{sea}}(t^n) - h_K^n. \end{array}\right. \tag{2}$$

The initial condition is computed by

$$h_K^0 = h_{\text{init}}(x_K).$$

Scilab implementation

The data structure needed for the implementation of the scheme for the given uniform 1D mesh with N cells of the domain $(0, L_x)$ is already given.

You need to implement the scheme (2) by completing the given Scilab file.

At each time step (inside the time loop), solve using a Newton algorithm the nonlinear system (2) denoted by

$$R(h^n) = 0$$

where $R : \mathbb{R}^N \to \mathbb{R}^N$ is the so-called "residual" function representing the finite volume scheme conservation equations in all cells. To do this, the Newton algorithm is applied: set $\epsilon = 10^{-6}$, $h^{0,n} = h^{n-1}$, and for $l \geq 0$ until $\|R(h^{l,n})\| \leq \epsilon \|R(h^{0,n})\|$ compute

$$\frac{\partial R}{\partial h}(h^{l,n})dh = -R(h^{l,n}),$$

$$h^{l+1,n} = h^{l,n} + dh.$$

We underline that the Newton algorithm has a quadratic convergence if the initial solution $h^{0,n}$ is closed enough to the solution h^n, ie there exist $\alpha > 0$ and $\beta > 0$ such that if $\|h^{0,n} - h^n\| \leq \alpha$ then

$$\|R(h^{l+1,n})\| \leqslant \beta \; \|R(h^{l,n})\|^2. \tag{3}$$

Note also that if the Newton algorithm is not converged in less than $Newtmax$ iterations, then the time step is restarted using a reduced time step by a factor 2. If the time step is converged in less than $Newtmax$ iterations we can increase the time step by a factor 1.2 until the maximum time step is reached.

(1) Write the Scilab function computing the `residual` $R(h^n)$ given h^{n-1} and $h_{sea}(t^n)$. This computation is acheived using one loop on the cells, one loop on the inner faces, and one loop on the boundary faces.

(2) Write the Scilab function computing the `jacobian` $\frac{\partial R}{\partial h}(h^n)$ given $h_{sea}(t^n)$. This computation is acheived using one loop over the cells, one loop over the inner faces, and one loop over the boundary faces.

(3) Complete the Newton and the time loops. By using a logarithm scale, check graphically the quadratic convergence of the Newton algorithm.

(4) Plot h and b function of x for the computed discrete times (one plot for h and one plot for b).

(5) Write the algorithm to compute

$$h_s(x,t) = \min_{\{t \,\leqslant\, q \,\leqslant\, T\}} h(x,q).$$

Comment on the geological meaning of h_s?

Then, plot $h_s(x,t)$ function of x for all the discrete times t^n, $n = 0, \cdots$, nb time steps, of the simulation (in a single plot).

(6) (More difficult) We want to compute the paleobathymetry $b_p(x,z,t)$ in the basin defined as the bathymetry of the sediment at the time it has been deposited.

It is the solution of the equation:

$$\begin{cases} \partial_t b_p(x,z,t) = 0 \text{ on } \mathcal{B} \\ b_p(x,h(x,t),t) = b(x,t) \text{ if } \partial_t h(x,t) > 0 \text{ (sedimentation)} \\ b_p(x,z,0) = b_{p,\text{init}}(x) \text{ for } z \in (-\infty, h_{\text{init}}(x)). \end{cases} \tag{4}$$

with

$$\mathcal{B} = \{(x,z,t) \mid (x,t) \in \Omega \times (0,T), z \in (-\infty, h(x,t))\}.$$

The initial paleobathymetry $b_{p,\text{init}}(x)$ will be assumed constant in the following to fix ideas.

The above equation is discretized using the following algorithm: let at time $t = 0$ $b^0_{p,K}(z) = b_{p,\text{init}}$ (constant) for all $z < h^0_K$. Then we compute at each time step:
If $h^n_K \geq h^{n-1}_K$ one has

$$\begin{cases} b^n_{p,K}(z) = b^{n-1}_{p,K}(z) \text{ for all } z < h^{n-1}_K, \\ \\ b^n_{p,K}(z) = b^n_K = h_{sea}(t^n) - h^n_K \text{ for all } h^{n-1}_K < z < h^n_K, \end{cases}$$

Else if $h^n_K < h^{n-1}_K$ one has

$$b^n_{p,K}(z) = b^{n-1}_{p,K}(z) \text{ for all } z < h^n_K.$$

Code the algorithm in scilab (initialization before the time loop and at each time step in the time loop after the computation of h^n) using the following arrays:

The paleobathymetry $b^n_{p,K}(z)$ is piecewise constant on the intervals

$$]-\infty, hcol(K,1)],\ ]hcol(K,1), hcol(K,2)], \cdots]hcol(K, ncol(K)-1), hcol(K, ncol(K))],$$

where $ncol(K)$ is the number of intervals. The value of the concentration $b^n_{p,K}(z)$ in each interval is stored in $bcol(K,l)$ for $l = 1, \cdots, ncol(K)$.

The arrays $ncol$, $hcol$ and $bcol$ have to be updated according to the above algorithm at each time step and in each cell K.

Question: Write the corresponding code in scilab and plot the paleobathymetry at final time T in 3 wells located at one third, one half and two third of the basin as functions of z.

(8) Comment the physical results that you obtain. What do you think of this model? Why is it used in oil exploration?

(9) Look at the plots of the solution obtained for different mesh sizes $N = 10, 20, 50, 100, 200, 400$ and $\Delta t = 0.02$ and comment the results.

(10) Look at the plot of the solutions obtained for different time steps and $N = 100$ and comment the results.

(11) Comment the behavior of the Newton algorithm for the different mesh sizes and time steps values