Simulation of oil recovery with miscible gas injection using a black oil model

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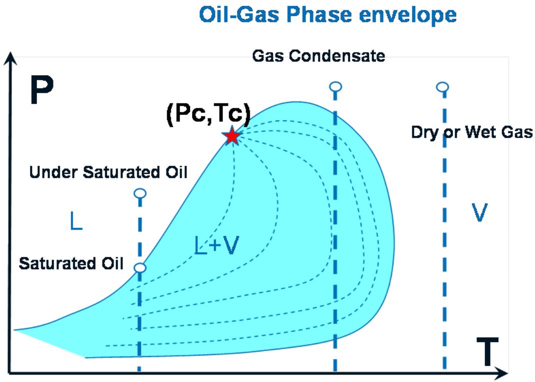
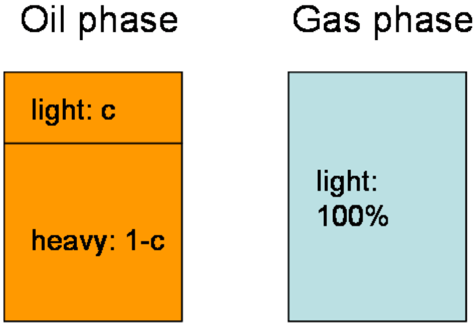
Motivations of compositional two phase oil gas models

- Saturated or under saturated oil reservoirs: volatile Hydrocarbon Components (HC) evaporate with pressure drop
- Oil recovery by miscible gas injection
- Gas condensate reservoirs: liquid phase appears with pressure drop



Black Oil Model: definition

- Two phases: oil and gas
- Two components: heavy HC (h), and light HC: (l)
- The light component can dissolve into the oil phase
- The heavy component cannot evaporate into the gas phase



Black Oil Model: conservation equations

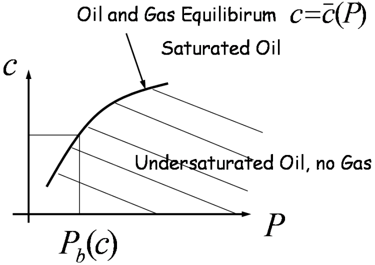
$$
\begin{cases}
\partial_t\Big(\phi\ \rho^o\ (1-c)\ S^o\Big) + \mathrm{div}\Big((1-c)\ \rho^o\ \frac{k_{r,o}(S^o)}{\mu^o}\ \mathbf{V}^o\Big) = 0, \\
\partial_t\Big(\phi\Big[\rho^o\ c\ S^o + \rho^g\ S^g\Big]\Big) + \mathrm{div}\Big(c\ \rho^o\ \frac{k_{r,o}(S^o)}{\mu^o}\ \mathbf{V}^o\Big) \\
\qquad\qquad + \mathrm{div}\Big(\rho^g\ \frac{k_{r,g}(S^g)}{\mu^g}\ \mathbf{V}^g\Big) = 0, \\
\mathbf{V}^o = -K\Big(\nabla P - \rho^o \mathbf{g}\Big),\ \ \mathbf{V}^g = -K\Big(\nabla P - \rho^g \mathbf{g}\Big), \\
S^o + S^g = 1.
\end{cases}
$$

- Oil phase properties: $\rho^o(P,c),\ \mu^o(P,c)$
- Gas phase properties: $\rho^g(P),\ \mu^g(P)$

Black Oil Model: thermodynamical equilibrium

If the oil and gas phases are both present, the thermodynamical equilibrium imposes that $c = \bar{c}(P)$.

$$\begin{cases} S^g \, (\bar{c}(P) - c) = 0, \\ S^g \geq 0, \\ (\bar{c}(P) - c) \geq 0. \end{cases}$$



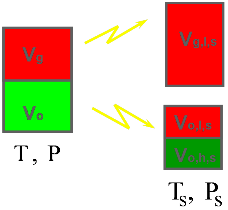
Black Oil Model

$$
\begin{cases}
\partial_t\Big(\phi\ \rho^o(P,c)\ (1-c)\ S^o\Big) + \mathrm{div}\Big((1-c)\ \rho^o(P,c)\ \frac{k_{r,o}(S^o)}{\mu^o(P,c)}\ \mathbf{V}^o\Big) = 0, \\
\partial_t\Big(\phi\Big[\rho^o(P,c)\ c\ S^o + \rho^g(P)\ S^g\Big]\Big) + \mathrm{div}\Big(c\ \rho^o(P,c)\ \frac{k_{r,o}(S^o)}{\mu^o(P,c)}\ \mathbf{V}^o\Big) \\
\qquad + \mathrm{div}\Big(\rho^g(P)\ \frac{k_{r,g}(S^g)}{\mu^g(P)}\ \mathbf{V}^g\Big) = 0, \\
\mathbf{V}^o = -K\Big(\nabla P - \rho^o(P,c)\mathbf{g}\Big),\ \ \mathbf{V}^g = -K\Big(\nabla P - \rho^g(P)\mathbf{g}\Big), \\
S^o + S^g = 1, \\
S^g\ (\bar{c}(P) - c) = 0, \\
S^g \geq 0, \\
(\bar{c}(P) - c) \geq 0.
\end{cases}
$$

Remarks:

- Only one set of unknowns P, S^o, S^g, c and equations because the oil phase is always present.
- No flash needed.

Black Oil Model: B_o, B_g and R_s



$$B_o = \frac{V_o}{V_{o,h,s}} = \frac{\rho_{h,s}}{\rho^o(P,c)\,(1-c)}, \quad B_g = \frac{V_g}{V_{g,l,s}} = \frac{\rho_{l,s}}{\rho^g(P)}$$

$$R_s = \frac{V_{o,l,s}}{V_{o,h,s}} = \frac{1-c}{c}\frac{\rho_{l,s}}{\rho_{h,s}}$$

$\rho_{h,s}$ and $\rho_{l,s}$ are the "oil" and "gas" densities at surface conditions. T is fixed.

Black Oil Model: phase properties in terms of B_o, B_g and R_s

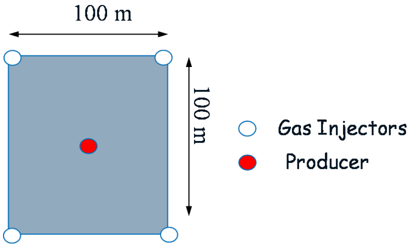
$\bar{B}_o(P)$, $\bar{R}_s(P)$, $\bar{\mu}^o(P)$ are given at equilibrium (saturated oil).
$B_g(P)$ and $\mu^g(P)$ are given.

For saturated oil ($c = \bar{c}(P)$ or $P = P_b(c)$)
and undersaturated oil ($c < \bar{c}(P)$ or $P > P_b(c)$)

$$B_o(P, P_b) = \bar{B}_o(P_b) + c_{B_o}(P - P_b), \quad \mu^o(P, P_b) = \bar{\mu}^o(P_b) + c_{\mu_o}(P - P_b)$$

$$\begin{cases} \rho^o(P, c) = \dfrac{\rho_{h,s}}{(1 - c)\left(\bar{B}_o(P_b(c)) + c_{B_o}(P - P_b(c))\right)}, \\ \mu^o(P, c) = \bar{\mu}^o(P_b(c)) + c_{\mu_o}(P - P_b(c)), \\ \bar{c}(P) = \dfrac{\rho_{l,s}}{\rho_{h,s}\ \bar{R}_s(P) + \rho_{l,s}}, \\ P_b(c) = \bar{c}^{-1}(c), \\ \rho^g(P) = \dfrac{\rho_{l,s}}{B_g(P)}. \end{cases}$$

Discretization of Black Oil model: 2D fivespots test case



- 2D horizontal reservoir with 4 gas injectors at the corners and 1 producer at the center.
- Prescribed pressures P^{inj} for the injection wells and P^{prod} for the production well
- Initial state: under saturated oil at $P^{\text{prod}} < P_{\text{init}} < P^{\text{inj}}$ and $P_b \in \left(P^{prod}, P_{\text{init}}\right)$
- Gas will appear around the producer due to pressure drop
- Gas front will propagate from injectors to producer and saturate the oil phase behind the front

Discretization of Black Oil model: diffusion fluxes and upwinding (without gravity)

Diffusion fluxes (without gravity for simplicity)

$$\left\{\begin{array}{r} F^{\rm int}_{\kappa_1(\sigma),\sigma} = -F^{\rm int}_{\kappa_2(\sigma),\sigma} = T^{\rm int}_\sigma\Big(p_{\kappa_1(\sigma)} - p_{\kappa_2(\sigma)}\Big),\ \sigma\in\Sigma_{\rm int},\\ F^{\rm prod}_{\sigma_w} = WI^{\rm prod}_{\sigma_w}\Big(p_{\kappa_{\rm prod}(\sigma_w)} - p^{\rm prod}_{\sigma_w}\Big),\ \sigma_w\in\Sigma^{\rm prod},\\ F^{\rm inj}_{\sigma_w} = WI^{\rm inj}_{\sigma_w}\Big(p_{\kappa_{\rm inj}(\sigma_w)} - p^{\rm inj}_{\sigma_w}\Big),\ \sigma_w\in\Sigma^{\rm inj}, \end{array}\right.$$

Upwinding (without gravity): for all $\sigma \in \Sigma_{\rm int}$

$$\kappa_{\rm up}(\sigma) = \left\{\begin{array}{l} \kappa_1(\sigma) = \text{ si } F^{\rm int}_{\kappa_1(\sigma),\sigma} > 0,\\ \kappa_2(\sigma) = \text{ si } F^{\rm int}_{\kappa_1(\sigma),\sigma} \leq 0. \end{array}\right.$$

Five Spots example: $\Sigma^{\rm prod}$ = one producer, $\Sigma^{\rm inj}$ = the four injectors.

Discretization of Black Oil Model: discrete system (without gravity)

Heavy HC residual

$$
\left\{
\begin{aligned}
R_{h,\kappa} = {} & \Big(\rho^o(P_\kappa, c_\kappa)\; S^o_\kappa\; (1 - c_\kappa) - m^{n-1}_{h,\kappa}\Big) \frac{|\kappa|}{\Delta t} \\
& + \sum_{\sigma \in \Sigma_{\rm int} \cap \partial\kappa} \left(1 - c_{\kappa_{\rm up}(\sigma)}\right)\; \rho^o\left(P_{\kappa_{\rm up}(\sigma)}, c_{\kappa_{\rm up}(\sigma)}\right) \frac{k_{r_o}\left(S^o_{\kappa_{\rm up}(\sigma)}\right)}{\mu^o\left(P_{\kappa_{\rm up}(\sigma)}, c_{\kappa_{\rm up}(\sigma)}\right)}\; F^{\rm int}_{\kappa,\sigma} \\
& + \sum_{\sigma_w \in \Sigma^{\rm prod} | \kappa_{\rm prod}(\sigma_w) = \kappa} \rho^o(P_\kappa, c_\kappa)\; (1 - c_\kappa)\; \frac{k_{r_o}(S^o_\kappa)}{\mu^o(P_\kappa, c_\kappa)}\; \left(F^{\rm prod}_{\sigma_w}\right)^+ = 0,
\end{aligned}
\right.
$$

Discretization of Black Oil Model: discrete system (without gravity)

Light HC residual

$$
\left\{
\begin{aligned}
R_{l,\kappa} = & \Big(\rho^o(P_\kappa, c_\kappa)\ S^o_\kappa\ c_\kappa + \rho^g(P_\kappa)\ S^g_\kappa - m^{n-1}_{l,\kappa}\Big)\frac{|\kappa|}{\Delta t} \\
& + \sum_{\sigma\in\Sigma_{\rm int}\cap\partial\kappa} c_{\kappa_{\rm up}(\sigma)}\ \rho^o(P_{\kappa_{\rm up}(\sigma)}, c_{\kappa_{\rm up}(\sigma)})\ \frac{k_{r_o}(S^o_{\kappa_{\rm up}(\sigma)})}{\mu^o(P_{\kappa_{\rm up}(\sigma)}, c_{\kappa_{\rm up}(\sigma)})}\ F^{\rm int}_{\kappa,\sigma} \\
& + \sum_{\sigma\in\Sigma_{\rm int}\cap\partial\kappa} \rho^g(P_{\kappa_{\rm up}(\sigma)})\ \frac{k_{r_g}(S^g_{\kappa_{\rm up}(\sigma)})}{\mu^g(P_{\kappa_{\rm up}(\sigma)})}\ F^{\rm int}_{\kappa,\sigma} \\
& + \sum_{\sigma_w\in\Sigma^{\rm prod}|\kappa_{\rm prod}(\sigma_w)=\kappa} \Big[c_\kappa\ \rho^o(P_\kappa, c_\kappa)\frac{k_{r_o}(S^o_\kappa)}{\mu^o(P_\kappa, c_\kappa)} + \rho^g(P_\kappa)\frac{k_{r_g}(S^g_\kappa)}{\mu^g(P_\kappa)}\Big]\Big(F^{\rm prod}_{\sigma_w}\Big)^+ = 0, \\
& + \sum_{\sigma_w\in\Sigma^{\rm inj}|\kappa_{\rm inj}(\sigma_w)=\kappa} \rho_g(P^{\rm inj}_{\sigma_w})\frac{k_{r_g}(1)}{\mu_g(P^{\rm inj}_{\sigma_w})}\ \Big(F^{\rm inj}_{\sigma_w}\Big)^- = 0,
\end{aligned}
\right.
$$

Discretization of Black Oil Model: discrete system (without gravity)

Local closure laws (volume conservation an thermodynamical equilibrium)

$$
\begin{cases}
S_\kappa^o + S_\kappa^g = 1, \\
S_\kappa^g \left(\bar{c}(P_\kappa) - c_\kappa\right) = 0, \\
S_\kappa^g \geq 0, \\
\left(\bar{c}(P_\kappa) - c_\kappa\right) \geq 0.
\end{cases}
$$

Discretization of Black Oil Model: Newton type algorithm

- The oil phase is always present in all cells ($S_{or} > 0$)
- Gas phase indicator for all cell $\kappa \in \mathcal{K}$

$$\mathcal{I}_\kappa = \begin{cases} 1 & \text{if} \quad \text{gas phase is present} \\ 0 & \text{if} \quad \text{gas phase is absent} \end{cases}$$

- Given $\mathcal{I}_\kappa$ for all cells, the closure laws are eliminated at the non linear level:

$$\begin{cases} \text{if } \mathcal{I}_\kappa = 1: \quad S^o_\kappa = 1 - S^g_\kappa, \quad c_\kappa = \bar{c}(P_\kappa), \\ \text{if } \mathcal{I}_\kappa = 0: \quad S^g_\kappa = 0, \quad S^o_\kappa = 1, \end{cases}$$

- the unknowns X_κ for the Newton linearization depend on $\mathcal{I}_\kappa$ for all cells:

$$\begin{cases} \text{if } \mathcal{I}_\kappa = 1: \quad X_\kappa = \Big(P_\kappa, S^g_\kappa\Big), \\ \text{if } \mathcal{I}_\kappa = 0: \quad X_\kappa = \Big(P_\kappa, c_\kappa\Big). \end{cases}$$

Solution of the discrete system of equations: Newton type algorithm

- The conservation equations are linearized with respect to the set of unknowns $X = \left(X_\kappa\right)_{\kappa\in\mathcal{K}}$ taking into account the previous elimination of the closure laws for a given $\mathcal{I}$:

$$R_\kappa(X) = \begin{pmatrix} R_{h,\kappa}(X) \\ R_{l,\kappa}(X) \end{pmatrix}$$

$R = \left(R_\kappa\right)_{\kappa\in\mathcal{K}}$, $dX = -\left(\frac{\partial R}{\partial X}\right)^{-1} R$

- On each cell κ update of the unknowns $P_\kappa, S_\kappa, c_\kappa$ and of the Gas phase indicator $\mathcal{I}_\kappa$ to satisfy the inequality constraints $S_\kappa^g \geq 0$ and $c \leq \bar{c}(P_\kappa)$, $\kappa \in \mathcal{K}$.

Solution of the discrete system of equations: algorithm

$P = (P_\kappa)_\mathcal{K}$, $\mathcal{I} = (\mathcal{I}_\kappa)_\mathcal{K}$, $S = (S_\kappa = S^g_\kappa)_\mathcal{K}$,$c = (c_\kappa)_\mathcal{K}$.

$R = \Big(R_{h,\kappa}, R_{l,\kappa}\Big)_{\kappa\in\mathcal{K}}$.

- Initialization of $P = P^0, \mathcal{I} = \mathcal{I}^0, S = S^0, c = c^0$ and Δt at $t = 0$.
- Time loop: while $t < t_f$: $t = t + \Delta t$
  - Compute the initial residual $R(P, S, c, P^0, S^0, c^0, \Delta t)$ and its norm $\|R\|_0 = \|R\|$
  - Newton Loop: while $\frac{\|R\|}{\|R\|_0} < \epsilon$ or $Nit_{newton} < Nit_{max}$
    - Linearization of the residuals $R(X) + \left(\frac{\partial R}{\partial X}\right) dX = 0$
    - $dX = -\left(\frac{\partial R}{\partial X}\right)^{-1} R$
    - Update of P, S, c and $\mathcal{I}$
    - Compute the residual $R(P, S, c, P^0, S^0, c^0, \Delta t)$ and its norm $\|R\|$
  - If $Nit_{newton} = Nit_{max}$ and $\frac{\|R\|}{\|R\|_0} > \epsilon$ then Restart the time step: $t = t - \Delta t$, $\Delta t = \Delta t/2$, $P = P^0, \mathcal{I} = \mathcal{I}^0, S = S^0, c = c^0$
  - Else new time step: update Δt and set $P^0 = P, \mathcal{I}^0 = \mathcal{I}, S^0 = S, c^0 = c$

Solution of the discrete system of equations: update of P, S, c and $\mathcal{I}$

- $dX = -J^{-1}\ R$
- Newton update: for $\kappa = 1, \cdots, N$:
  - $P_\kappa \leftarrow P_\kappa + dX(2\kappa - 1)$,
  - if $\mathcal{I}_\kappa = 1$ then $S_\kappa \leftarrow S_\kappa + dX(2\kappa)$, $c_\kappa = \bar{c}(P_\kappa)$
  - else $c_\kappa \leftarrow c_\kappa + dX(2\kappa)$, $S_\kappa = 0$.

- Gas phase indicator update: for $\kappa = 1, \cdots, N$:
  - If $\mathcal{I}_\kappa = 1$ then
    - if $S_\kappa < 0$ then
      $$\mathcal{I}_\kappa \leftarrow 0,\ S_\kappa \leftarrow 0,\ c_\kappa \leftarrow \bar{c}(P_\kappa)$$
    - else if $S_\kappa > 1 - S_{or}$ then
      $$S_\kappa \leftarrow 1 - S_{or},$$
  - Else if $\mathcal{I}_\kappa = 0$ then
    - if $c_\kappa > \bar{c}(P_\kappa)$ then
      $$\mathcal{I}_\kappa \leftarrow 1,\ S_\kappa \leftarrow 0,\ c_\kappa \leftarrow \bar{c}(P_\kappa)$$

Solution of the discrete system of equations: computation of the residual $R(P, S, c, P^0, S^0, c^0, \Delta t)$

- numbering of the cells: $\kappa = 1, \cdots, N$
- numbering of the equations:

$$\left\{ \begin{array}{l} R^h_\kappa \to R(2k-1),\ \kappa = 1, \cdots, N \\ R^l_\kappa \to R(2k),\ \kappa = 1, \cdots, N \end{array} \right.$$

- Loop on cells $\kappa = 1, \cdots, N$

$$\left\{ \begin{array}{rcl} \rho^o_\kappa & = & \rho^o(P_\kappa, c_\kappa), \\ \rho^g_\kappa & = & \rho^g(P_\kappa), \\ R(2\kappa-1) & \leftarrow & R(2\kappa-1) + \dfrac{\left(\rho^o_\kappa\ (1-S_\kappa)\ (1-c_\kappa) - \rho^o(P^0_\kappa, c^0_\kappa)\ (1-S^0_\kappa)\ (1-c^0_\kappa)\right)|\kappa|}{\Delta t} \\ R(2\kappa) & \leftarrow & R(2\kappa) + \dfrac{\left(\rho^o_\kappa\ (1-S_\kappa)\ c_\kappa + \rho^g_\kappa\ S_\kappa - \rho^o(P^0_\kappa, c^0_\kappa)\ (1-S^0_\kappa)\ c^0_\kappa - \rho^g(P^0_\kappa)\ S^0_\kappa\right)|\kappa|}{\Delta t} \end{array} \right.$$

Solution of the discrete system of equations: computation of the residual $R(P, S, c, P^0, S^0, c^0, \Delta t)$

- Loop on interior faces:

$$\begin{cases} F_\sigma^h = (1 - c_{\kappa_{\mathrm{up}}(\sigma)})\ \rho^o_{\kappa_{\mathrm{up}}(\sigma)}\ \frac{k_{r_o}(1-S_{\kappa_{\mathrm{up}}(\sigma)})}{\mu^o}\ F^{\mathrm{int}}_{\kappa_1(\sigma),\sigma} \\ \\ F_\sigma^l = \left[c_{\kappa_{\mathrm{up}}(\sigma)}\ \rho^o_{\kappa_{\mathrm{up}}(\sigma)}\ \frac{k_{r_o}(1-S_{\kappa_{\mathrm{up}}(\sigma)})}{\mu^o} + \rho^g_{\kappa_{\mathrm{up}}(\sigma)}\ \frac{k_{r_g}(S_{\kappa_{\mathrm{up}}(\sigma)})}{\mu^g}\right]\ F^{\mathrm{int}}_{\kappa_1(\sigma),\sigma} \\ \\ R(2\kappa_1(\sigma) - 1) \leftarrow R(2\kappa_1(\sigma) - 1) + F_\sigma^h \\ R(2\kappa_2(\sigma) - 1) \leftarrow R(2\kappa_2(\sigma) - 1) - F_\sigma^h \\ R(2\kappa_1(\sigma)) \leftarrow R(2\kappa_1(\sigma)) + F_\sigma^l \\ R(2\kappa_2(\sigma)) \leftarrow R(2\kappa_2(\sigma)) - F_\sigma^l \end{cases}$$

Solution of the discrete system of equations: computation of the residual $R(P, S, c, P^0, S^0, c^0, \Delta t)$

- Loop on injector wells σ_w:

$$R(2\kappa_{\mathrm{inj}}(\sigma_w)) \leftarrow R(2\kappa_{\mathrm{inj}}(\sigma_w)) + \rho_g(P^{\mathrm{inj}}_{\sigma_w})\frac{k_{r_g}(1)}{\mu_g} \left(F^{\mathrm{inj}}_{\sigma_w}\right)^-$$

- Producer well σ_w

$$\begin{cases} R(2\kappa_{\mathrm{prod}}(\sigma_w) - 1) \leftarrow R(2\kappa_{\mathrm{prod}}(\sigma_w) - 1) + \rho^o_\kappa \ (1 - c_\kappa) \ \frac{k_{r_o}(1-S_\kappa)}{\mu^o} \left(F^{\mathrm{prod}}_{\sigma_w}\right)^+ \\ R(2\kappa_{\mathrm{prod}}(\sigma_w)) \leftarrow R(2\kappa_{\mathrm{prod}}(\sigma_w)) + \left[c_\kappa \ \rho^o_\kappa \frac{k_{r_o}(1-S_\kappa)}{\mu^o} + \rho^g_\kappa \frac{k_{r_g}(S_\kappa)}{\mu^g}\right]\left(F^{\mathrm{prod}}_{\sigma_w}\right)^+ \end{cases}$$

Solution of the discrete system of equations: Newton linearization $R(X) + \left(\frac{\partial R}{\partial X}\right) dX = 0$

- Unknowns numbering for the Newton linearization: $\kappa = 1, \cdots, N$

$$\left\{ \begin{array}{rl} dX(2\kappa - 1) = & dP_\kappa, \\ dX(2\kappa) = & \left\{ \begin{array}{ll} dS_\kappa & \text{if } \mathcal{I}_\kappa = 1, \\ dc_\kappa & \text{if } \mathcal{I}_\kappa = 0. \end{array} \right. \end{array} \right.$$

- Notation

$$\left(X^1, X^2\right) = \left\{ \begin{array}{ll} \left(P, S\right) & \text{if } \mathcal{I} = 1 \rightarrow c = \bar{c}(P), \\ \left(P, c\right) & \text{if } \mathcal{I} = 0 \rightarrow S = 0. \end{array} \right.$$

- Derivatives: example of $\rho^o(P, c)$:

$$\frac{\partial \rho^o}{\partial X^1} = \left[\frac{\partial \rho^o}{\partial P}(P, c) + \frac{\partial \rho^o}{\partial c}(P, c)\frac{\partial \bar{c}}{\partial P}(P)\right] \mathcal{I} \; + \; \left[\frac{\partial \rho^o}{\partial P}(P, c)\right] (1 - \mathcal{I})$$

$$\frac{\partial \rho^o}{\partial X^2} = \left[\frac{\partial \rho^o}{\partial c}(P, c)\right](1 - \mathcal{I})$$

Solution of the discrete system of equations: Newton linearization $R(X) + \left(\frac{\partial R}{\partial X}\right) dX = 0$

- Jacobian: example of the accumulation term for R^h_κ:

$$\frac{\left(\rho^o_\kappa\ (1-S_\kappa)\ (1-c_\kappa) - \rho^o(P^0_\kappa, c^0_\kappa)\ (1-S^0_\kappa)\ (1-c^0_\kappa)\right)|\kappa|}{\Delta t}$$

- Loop on cells κ

$$\begin{aligned}
J(2\kappa-1, 2\kappa-1) \leftarrow{}& J(2\kappa-1, 2\kappa-1) + \mathcal{I}_\kappa\ \Big[-\rho^o_\kappa\ (1-S_\kappa)\ \tfrac{\partial \bar{c}}{\partial P}(P_\kappa) \\
&+(1-S_\kappa)\ (1-c_\kappa)\Big(\tfrac{\partial \rho^o}{\partial P}(P_\kappa, c_\kappa) + \tfrac{\partial \rho^o}{\partial c}(P_\kappa, c_\kappa)\tfrac{\partial \bar{c}}{\partial P}(P_\kappa)\Big)\Big]\tfrac{|\kappa|}{\Delta t} \\
&+(1-\mathcal{I}_\kappa)\ \Big[\tfrac{\partial \rho^o}{\partial P}(P_\kappa, c_\kappa)\ (1-S_\kappa)\ (1-c_\kappa)\Big]\tfrac{|\kappa|}{\Delta t}
\end{aligned}$$

$$\begin{aligned}
J(2\kappa-1, 2\kappa) \leftarrow{}& J(2\kappa-1, 2\kappa) + \mathcal{I}_\kappa\Big[-\rho^o_\kappa\ (1-c_\kappa)\Big]\tfrac{|\kappa|}{\Delta t} \\
&+(1-\mathcal{I}_\kappa)\ \Big[\tfrac{\partial \rho^o}{\partial c}(P_\kappa, c_\kappa)\ (1-S_\kappa)\ (1-c_\kappa) - \rho^o_\kappa\ (1-S_\kappa)\Big]\tfrac{|\kappa|}{\Delta t}
\end{aligned}$$

Solution of the discrete system of equations: Newton linearization $R(X) + \left(\frac{\partial R}{\partial X}\right) dX = 0$

- Jacobian: example of the interior face flux term:
  $F_\sigma^h = (1 - c_{\kappa_{\rm up}(\sigma)})\ \rho^o_{\kappa_{\rm up}(\sigma)}\ \frac{k_{r_o}(1-S_{\kappa_{\rm up}(\sigma)})}{\mu^o}\ F^{\rm int}_{\kappa_1(\sigma),\sigma}$

- Loop on interior faces σ: $\kappa_1 = \kappa_1(\sigma)$, $\kappa_2 = \kappa_2(\sigma)$, $\kappa_{\rm up} = \kappa_{\rm up}(\sigma)$

$$
\begin{array}{l}
G_\sigma^h = (1 - c_{\kappa_{\rm up}})\ \rho^o_{\kappa_{\rm up}}\ \frac{k_{r_o}(1-S_{\kappa_{\rm up}})}{\mu^o}\\
J(2\kappa_1 - 1, 2\kappa_1 - 1) \leftarrow J(2\kappa_1 - 1, 2\kappa_1 - 1) + G_\sigma^h\ T_\sigma^{\rm int}\\
J(2\kappa_2 - 1, 2\kappa_2 - 1) \leftarrow J(2\kappa_2 - 1, 2\kappa_2 - 1) + G_\sigma^h\ T_\sigma^{\rm int}\\
J(2\kappa_1 - 1, 2\kappa_2 - 1) \leftarrow J(2\kappa_1 - 1, 2\kappa_2 - 1) - G_\sigma^h\ T_\sigma^{\rm int}\\
J(2\kappa_2 - 1, 2\kappa_1 - 1) \leftarrow J(2\kappa_2 - 1, 2\kappa_1 - 1) - G_\sigma^h\ T_\sigma^{\rm int}
\end{array}
$$

to be continued ...

Solution of the discrete system of equations: Newton linearization $R(X) + \left(\frac{\partial R}{\partial X}\right) dX = 0$

- suite of the interior flux term F_σ^h

$$\gamma = \Big(\mathcal{I}_\kappa \Big[(1-c_{\kappa_{\rm up}})\tfrac{\partial \rho^o}{\partial P}(P_{\kappa_{\rm up}}, c_{\kappa_{\rm up}}) + (1-c_{\kappa_{\rm up}})\tfrac{\partial \rho^o}{\partial c}(P_{\kappa_{\rm up}}, c_{\kappa_{\rm up}})\tfrac{\partial \bar{c}}{\partial P}(P_{\kappa_{\rm up}}) - \tfrac{\partial \bar{c}}{\partial P}(P_{\kappa_{\rm up}})\rho^o_{\kappa_{\rm up}}\Big] + (1-\mathcal{I}_\kappa)\Big[(1-c_{\kappa_{\rm up}})\tfrac{\partial \rho^o}{\partial P}(P_{\kappa_{\rm up}}, c_{\kappa_{\rm up}})\Big]\Big)\frac{k_{r_o}(1-S_{\kappa_{\rm up}})}{\mu^o}\; F^{\rm int}_{\kappa_1,\sigma}$$

$$\delta = \mathcal{I}_\kappa \Big[-\frac{(1-c_{\kappa_{\rm up}})\rho^o_{\kappa_{\rm up}}}{\mu^o}\; \frac{\partial k_{r_o}}{\partial S}((1-S_{\kappa_{\rm up}})\Big]\; F^{\rm int}_{\kappa_1,\sigma} + (1-\mathcal{I}_\kappa)\Big[-\rho^o_{\kappa_{\rm up}} + (1-c_{\kappa_{\rm up}})\tfrac{\partial \rho^o}{\partial c}(P_{\kappa_{\rm up}}, c_{\kappa_{\rm up}})\Big]\frac{k_{r_o}(1-S_{\kappa_{\rm up}})}{\mu^o}\; F^{\rm int}_{\kappa_1,\sigma}$$

$$\begin{aligned}
J(2\kappa_1-1, 2\kappa_{\rm up}-1) &\leftarrow J(2\kappa_1-1, 2\kappa_{\rm up}-1) + \gamma \\
J(2\kappa_2-1, 2\kappa_{\rm up}-1) &\leftarrow J(2\kappa_2-1, 2\kappa_{\rm up}-1) - \gamma \\
J(2\kappa_1-1, 2\kappa_{\rm up}) &\leftarrow J(2\kappa_1-1, 2\kappa_{\rm up}) + \delta \\
J(2\kappa_2-1, 2\kappa_{\rm up}) &\leftarrow J(2\kappa_2-1, 2\kappa_{\rm up}) - \delta
\end{aligned}$$

- to be continued for all remaining terms of the Jacobian ...