ARC-WISE ANALYTIC STRATIFICATION, WHITNEY FIBERING CONJECTURE AND ZARISKI EQUISINGULARITY

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ABSTRACT. In this paper we show Whitney's fibering conjecture in the real and complex, local analytic and global algebraic cases.

For a given germ of complex or real analytic set, we show the existence of a stratification satisfying a strong (real arc-analytic with respect to all variables and analytic with respect to the parameter space) trivialization property along each stratum. We call such a trivialization arc-wise analytic and we show that it can be constructed under the classical Zariski algebro-geometric equisingularity assumptions. Using a slightly stronger version of the Zariski equisingularity, we show the existence of Whitney's stratified fibration, satisfying the conditions (b) of Whitney and (w) of Verdier. Our construction is based on the Puiseux with parameter theorem and a generalization of Whitney's interpolation. For algebraic sets our construction gives a global stratification.

We also present several applications of the arc-wise analytic trivialization, mainly to the stratification theory and the equisingularity of analytic set and function germs. In the real algebraic case, for an algebraic family of projective varieties, we show that the Zariski equisingularity implies local constancy of the associated weight filtration.

CONTENTS

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Introduction and statement of results.

In 1965 Whitney stated the following conjecture.

Conjecture. [Whitney fibering conjecture, [70] section 9, p.230] *Any analytic subvariety $V \subset U$ (U open in $\mathbb{C}^n$) has a stratification such that each point $p_0 \in V$ has a neighborhood U_0 with a semi-analytic fibration.*

By a semi-analytic fibration Whitney meant the following (it has nothing to do with the notion of semi-analytic set introduced about the same time by Łojasiewicz in [37]). Let p_0 belong to a stratum M and let $M_0 = M \cap U_0$. Let N be the analytic plane orthogonal to M at p_0 and let $N_0 = N \cap U_0$. Then Whitney requires that there exist a homeomorphism

$$\phi(p, q) : M_0 \times N_0 \to U_0,$$

complex analytic in p, such that $\phi(p, p_0) = p$ ($p \in M_0$) and $\phi(p_0, q) = q$ ($q \in N_0$), and preserving the strata. He also assumes that for each $q \in N_0$ fixed, $\phi(\cdot, q) : M_0 \to U_0$ is a complex analytic embedding onto an analytic submanifold $L(q)$ called the fiber (or the leaf) at q, and thus U_0 is fibered continuously into submanifolds complex analytically diffeomorphic to M_0. Note that due to the existence of continuous moduli it is in general impossible to find $\phi(p, q)$ complex analytic in both variables, see [70].

Whitney stated his conjecture in the context of his regularity conditions (a) and (b) for stratifications introduced in [69]. These conditions imply the topological triviality (equisingularity) along each stratum. This trivialization is obtained by the flow of some "controlled" vector fields and does not imply the existence of a fibration as required in Whitney's conjecture. Thus Whitney conjectured the existence of a better trivialization, given by his fibration, that should, moreover, imply the regularity conditions (a) and (b). As Whitney claims in [70] a semi-analytic (in his sense) fibration ensures the continuity of the tangent spaces to the leaves of the fibration and hence Whitney's condition (a) for the stratification. This seems not to be obvious. We recall Whitney's argument in Subsection 7.4, but to complete it we need an extra assumption. To have the condition (b), quoting Whitney, "one should probably require more than just the continuity of ϕ in the second variable".

Whitney's fibering conjecture as stated above was proven by Hardt and Sullivan in the local analytic and global projective cases, in Theorem 6.1 of [22]. But it is not clear to us whether ϕ of [22] ensures the continuity of the tangent spaces to the leaves or the condition (b). In the real algebraic case an analog of Whitney's conjecture was proven in [21]. In this case the continuity of the tangent spaces is not clear either.

Whitney's fibering conjecture has been studied in the context of abstract C^∞ stratified spaces and topological equisingularity, cf. [46], [47], [48]. Assuming that the conjecture is true, Murolo and Trotman have shown in [48] a horizontally-C^1 version of Thom's first isotopy theorem.

0.1. **Ehresmann Theorem.** Whitney's conjecture is consistent with the following holomorphic version of the Ehresmann fibration theorem, see [66]. Let $\pi : \mathcal{X} \to B$ be a proper holomorphic submersion of complex analytic manifolds. Then, for every $b_0 \in B$ there is a neighborhood B_0 of b_0 in B and a C^∞ trivialization

$$\phi(p, q) : B_0 \times X_0 \to \mathcal{X}_{B_0},$$

holomorphic in p, where $X_0 = \pi^{-1}(b_0)$, $\mathcal{X}_{B_0} = \pi^{-1}(B_0)$. Note that ϕ can be made real analytic but, in general, due to the presence of continuous moduli, not holomorphic. This version of Ehresmann's theorem is convenient to study the variation of Hodge structures in families of Kähler manifolds, see [66].

As we show in this paper there are no continuous moduli for complex analytic families of singular complex analytic germs, nor for families of algebraic varieties, provided φ is assumed complex analytic in p and real arc-analytic in q, see Theorems 7.6 and 9.3 and Lemma 7.4.

0.2. **Statement of main results.** In this paper we show Whitney's fibering conjecture in the real and complex, local analytic and global algebraic cases. For this, for a given germ of complex or real analytic set, we show the existence of a stratification that can locally be trivialized by a map $\phi(p,q)$ that is not only real/complex analytic (depending on the case) in p, continuous in both variables, but also arc-wise analytic, see Definition 1.2. In particular, both ϕ and ϕ^{-1} are analytic on real analytic arcs. Moreover, for every real analytic arc $q(s)$ in N_0, $(p,s) \to \phi(p,q(s))$ is analytic. As we show in Proposition 1.3 this ensures the continuity of tangent spaces to the fibers and hence Whitney's condition (a) on the stratification (both in the real and complex cases). Then, by additionally requiring that the trivialization preserve the size of the distance to the stratum, we show the existence of Whitney's fibration satisfying the conditions (b) of Whitney and (w) of Verdier [64]. We call such an arc-wise analytic trivialization regular along the stratum, Definition 1.5.

Theorem (Theorem 7.6)**.** *Let $\mathcal{X} = \{X_i\}$ be a finite family of analytic subsets of an open $U \subset \mathbb{K}^N$, ($\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$). Let $p_0 \in U$. Then there exist an open neighborhood U' of p_0 and an analytic stratification of U' compatible with each $U' \cap X_i$ admitting regular arc-wise analytic trivialization along each stratum.*

In Section 8 we extend these results to stratifications of analytic functions. Recall that a stratification of a $\mathbb{K}$-analytic function $f : X \to \mathbb{K}$ is a stratification of X such that the zero set $V(f)$ of f is a union of strata. Theorem 8.2 together with Proposition 1.10 implies the following result.

Theorem ($\mathbb{K} = \mathbb{C}$)**.** *If a stratification of f admits an arc-wise analytic trivialization along a stratum $S \subset V(f)$ then it satisfies the Thom condition (a_f) along this stratum. If such trivialization is, moreover, regular along S, then it satisfies the strict Thom condition (w_f) along S.*

We also give an analogous result in the real case using the notion of regularity of a function for an arc-wise analytic trivialization, defined in Subsection 1.3. Thom's regularity conditions are used to show topological triviality of functions along strata. We discuss this in detail in Section 8, where we develop three different constructions guaranteeing such triviality.

In Section 9 we treat the algebraic case. By reduction to the homogeneous analytic case we show the following results.

Theorem (Theorem 9.2)**.** *Let $\{V_i\}$ be a finite family of algebraic subsets of $\mathbb{P}^n_{\mathbb{K}}$. Then there exists an algebraic stratification of $\mathbb{P}^n_{\mathbb{K}}$ compatible with each V_i and admitting semialgebraic regular arc-wise analytic trivializations along each stratum.*

Theorem (Theorem 9.3)**.** *Let T be an algebraic variety and let $\mathcal{X} = \{X_k\}$ be a finite family of algebraic subsets $T \times \mathbb{P}^{n-1}_{\mathbb{K}}$. Then there exists an algebraic stratification $\mathcal{S}$ of T such that for every stratum S and for every $t_0 \in S$ there is a neighborhood U of t_0 in S and a semialgebraic, arc-wise analytic trivialization of π, preserving the family $\mathcal{X}$*

$$\Phi : U \times \mathbb{P}^{n-1}_{\mathbb{K}} \to \pi^{-1}(U), \tag{0.1}$$

$\Phi(t_0, x) = (t_0, x)$, *where* $\pi : T \times \mathbb{P}^{n-1}_{\mathbb{K}} \to T$ *denotes the projection.*

The arc-wise analytic triviality is particularly friendly to the curve selection lemma argument. Recall that in analytic geometry many properties can be proven by checking them along real analytic arcs. We use this argument many times in this paper. For precise statement and a proof of the curve selection lemma we refer the reader to [10], [68], [37], [25] or [43]. To prove the classical regularity conditions, (a) of Whitney, (w) of Verdier, or Thom's conditions (a_f) or (w_f), we use a wing lemma type argument originated by Whitney in [69], see Proposition 7.3. Arc-wise analytic trivializations naturally provide such wings. For instance, in Whitney's notation, if $q(s)$ is a real analytic arc in N_0 then $\phi(p, q(s))$ constitutes such an arc-wise analytic wing. Moreover, arc-wise analytic trivializations preserve the multiplicities and the singular loci of the sets they trivialize, see Propositions 1.13 and 1.14 for precise statements.

Thus this paper, in order to get local arc-wise analytic trivializations, we redefine many classical notions and reprove many classical results of stratification theory on analytic and algebraic sets. Our approach is based on the classical Puiseux with parameter theorem and the algebro-geometric equisingularity of Zariski (called also Zariski's equisingularity). Our main tool in the construction of arc-wise analytic trivializations is Theorem 3.3, which says that the Zariski equisingularity implies arc-wise analytic triviality. To show it, we use Whitney's interpolation adapted to arc-analytic geometry. This is explained in Appendix I.

Besides the proofs of the Puiseux with parameter theorem and the curve selection lemma this paper is self-contained. Our method is based on the Zariski equisingularity, hence is constructive; it involves the computation of the discriminants of subsequent linear projections.

0.3. **Zariski Equisingularity.** Let V be a real or complex analytic variety. Then there exists a stratification $\mathcal{S}$ of V such that V is equisingular along each stratum S. There are several different notions of equisingularity, the basic one is the topological one, with many possible refinements, such as stratified topological triviality. Whitney introduced in [70], [69], the regularity conditions (a) and (b) that guarantee, by the Thom-Mather first isotopy theorem, the topological equisingularity along each stratum. He showed in [69] that any complex analytic variety admits (a) and (b) regular stratifications. The real analytic case was established in [37] and the subanalytic case in [25].

Topological equisingularity can also be obtained by means of the Zariski equisingularity, as shown by Varchenko in [61, 62, 63]. Zariski's definition, see [74], is recursive and is based on the geometry of discriminants. Let $V \subset \mathbb{K}^N$ be a hypersurface. We say that V is Zariski equisingular along stratum S at $p \in S$ if, after a change of a local system of coordinates, the discriminant of a linear projection $\pi : \mathbb{K}^N \to \mathbb{K}^{N-1}$ restricted to V is equisingular along $\pi(S)$ at $\pi(p)$. The kernel of π should be transverse to S and π restricted to V should be finite at p. Stronger notions of Zariski's equisingularity are obtained if one assumes that the kernel of π is not contained in the tangent cone to V at p (transverse Zariski equisingularity) or that π is generic (generic Zariski equisingularity).

The special case, when S is of codimension one in V, was studied by Zariski in [72]. Note that in this case V can be considered as a family of plane curves parameterized by S. As Zariski shows, in this case the Zariski equisingularity is equivalent to Whitney's conditions

(a) and (b) on the pair of strata $V \setminus S, S$. Such equisingular families of plane curves admit a uniform Puiseux representation parameterized by S, this result is known in literature as the parametrized Puiseux or the Puiseux with parameter theorem. We recall it in Subsection 2.1.

In this paper we show that the Zariski equisingularity implies arc-wise analytic triviality. In the case of the Zariski transverse equisingularity we obtain an arc-wise analytic triviality that is also regular.

Theorem (see Theorems 3.3 and 4.3)**.** *If a hypersurface $V \subset \mathbb{K}^N$ is Zariski equisingular along stratum S at $p \in S$, then there is a local arc-wise analytic trivialization of $\mathbb{K}^N$ along S at p that preserves V.*

Our proof is different from that of Varchenko and is based on Whitney's interpolation that gives a precise algebraic formula for such a trivialization. The main idea is the following. Suppose V is Zariski equisingular along S and $\pi : \mathbb{K}^N \to \mathbb{K}^{N-1}$ is the projection giving this equisingularity. By the inductive assumption, there is an arc-wise analytic trivialization of $\pi(V)$ along $\pi(S)$. This trivialization is then lifted to a trivialization of V along S, and extended to a trivialization of the ambient space $\mathbb{K}^n$ along S by our version of Whitney's interpolation. Therefore the lift is continuous, subanalytic, and, by the Puiseux with parameter theorem, arc-wise analytic. This latter conclusion is obtained thanks to the arc-wise analyticity in the inductive assumption, see Remark 3.4.

For an analytic function germ F we denote by F_{red} its reduced (i.e. square free) form. Let (Y, y) be a germ of a $\mathbb{K}$-analytic space. For a monic polynomial $F \in \mathcal{O}_Y[z]$ in z we often consider the discriminant of F_{red}. If Y has arbitrary singularities then this discriminant should be replaced by an appropriate generalized discriminant that is a polynomial in the coefficients of F, see Appendix II.

Finally, we note that the Zariski equisingularity can be used to trivialize not only hypersurfaces but also analytic spaces of arbitrary embedding codimension. This follows from the fact that if a hypersurface V is Zariski equisingular along S and $V = \cup V_i$ is the decomposition of V into irreducible components, then the arc-wise analytic trivialization preserves each V_i and hence any set-theoretic combination of the V_i's.

0.4. **Proofs of the main theorems are constructive.** The main theorems, Theorem 7.6 and Theorem 9.2 can be shown in a virtually algorithmic way. For this we proceed as follows. Given an ideal $\mathcal{I}$ of $\mathbb{K}[x_1, ..., x_n]$ or $\mathbb{K}\{x_1, ..., x_n\}$ we choose a finite set of generators of $\mathcal{I}$ and consider their product $f(x_1, ..., x_n)$. Then we complete f to a system of (pseudo)polynomials $F_i(x_1, ..., x_i)$, $i = 1, ..., n$, see Definition 5.1. This process, explained in detail in subsections 7.3 and 9.1, involves a generic linear change of coordinates. That is the only point not entirely algorithmic. It follows from Theorem 3.3, see Proposition 5.2, that the canonical stratification associated to a system of (pseudo)polynomials admits locally arc-wise analytic trivializations. To get a regular arc-wise analytic stratification, and hence a Whitney stratification, we need to refine this construction and consider not only f but also its partial derivatives with respect to x_n. This way we get a system of (pseudo)polynomials F_i that we call derivation complete, as explained in Example 4.4.

0.5. **Zariski Equisingularity and regularity conditions on stratifications.** In general, Whitney's conditions and Zariski's equisingularity, do not imply one another. We recall several classical examples in Section 7.5. By Zariski [72], they coincide for a hypersurface V along a nonsingular subvariety of codimension 1 in V.

It was shown by Speder [57] that in the complex case Zariski's equisingularity obtained by taking generic projections implies the regularity conditions (a) and (b) of Whitney. As it follows from our Theorem 4.3 the assumption that the projections are transverse, in both complex and real cases, is sufficient. We also show in Proposition 3.6 that the Zariski equisingularity (arbitrary projections) implies equimultiplicity.

Whitney's stratification approach is independent of the choice of local analytic coordinates and simple to define. But the trivializations obtained by this method are not explicit and difficult to handle. These trivializations are obtained by integration of "controlled" vector fields whose existence can be theoretically established. Stronger regularity conditions, such as (w) of Verdier [64], or Lipschitz of Mostowski [45], [51], lead to easier constructions of such vector fields, but in general, even if these vector fields can be chosen subanalytic, not much can be said about their flows.

Zariski's equisingularity method is more explicit and in a way constructive. It uses the actual equations and local coordinate systems. This can be considered either as a drawback or as an advantage. Zariski's equisingularity was used, for instance, by Mostowski [44], see also [3], to show that analytic set germs are always homeomorphic to algebraic ones.

In this paper we apply the Zariski equisingularity to construct stratifications via corank one linear projections. This method was developed by Hardt and Hardt & Sullivan [19, 20, 21, 22].

In [76] Zariski has proposed a general theory of equisingularity for hypersurfaces by introducing the notion of dimensionality type of their points. The dimensionality type is defined through an inductive process, using discriminants of generic (not necessarily linear) projections. Besides the codimension one case [72], this notion has been studied in the codimension two for families of isolated surface singularities in [5] and [59].

0.6. **Applications to real algebraic geometry.** Semialgebraic arc-analytic maps are often used in real algebraic geometry. The arc-analytic maps were introduced by Kurdyka in [31]. It was shown by Bierstone and Milman in [2] (see also [53]) that semialgebraic arc-analytic maps are blow-analytic. Semialgebraic arc-analytic maps and semialgebraic arc-symmetric sets were used in [33], [54], to show that injective self-morphisms of real algebraic varieties are surjective. For more on this development we refer the reader to [34]. Let us also note that recently studied [27], [28], [29], [13] continuous rational maps are, in particular, arc-analytic and semialgebraic.

The weight filtration on real algebraic varieties, recently introduced [41, 42], is stable under semialgebraic arc-analytic homeomorphisms. By Theorem 9.3 any algebraic family of algebraic sets is generically semialgebraically arc-wise analytic trivial, and therefore we have the following result.

Theorem (see Corollary 9.6). *Let T be a real algebraic variety and let X be an algebraic subset of $T \times \mathbb{P}^{n-1}_{\mathbb{K}}$. Then there exists a finite stratification $\mathcal{S}$ of T such that for every stratum S and for every $t_0, t_1 \in S$ the fibers X_{t_0} and X_{t_1} have isomorphic the weight filtration on homology.*

0.7. **Resolution of singularities and blow-analytic equivalence.** The resolution of singularities can also be used to show topological equisingularity, though the results are partial and many questions are still open. This method works for the families of isolated singularities, cf. Kuo [30], and gives local arc-analytic trivializations. But little is known if the singularities are not isolated, see e.g. [26]. Let us explain the encountered problem on a simple example. Suppose that $Y \subset V$ is nonsingular and let $\sigma : \tilde V \to V$ be a resolution of singularities such that $\sigma^{-1}(Y)$ is a union of the components of exceptional divisors. Fix a local projection $\pi : V \to Y$. The exceptional divisor of σ as a divisor with normal crossings is naturally stratified by the intersections of its components. Let $Z \subset Y$ be the closure of the union of all critical values of $\pi \circ \sigma$ restricted to the strata. By Sard's theorem $\dim Z < \dim Y$. We say that V is *equiresoluble along* Y if $Y \cap Z = \emptyset$. Thus V is equiresoluble along $Y' = Y \setminus Z$ and $\pi \circ \sigma$ is locally topologically (and even real analytically) trivial over Y'. If σ is an isomorphism over $V \setminus Y$ (family of isolated singularities case) then this trivialization blows down to a topological trivialization of a neighborhood of Y in V. But in the non-isolated singularity case there is no clear reason why a trivialization of $\pi \circ \sigma$ comes from a topological trivialization of a neighborhood of Y in V. Thus, in general, we do not know whether equiresolubility implies topological equisingularity.

As before, one may ask how the equiresolution method is related to the other methods of establishing topological equisingularity. A non-trivial result of Villamayor [65], says that the generic Zariski equisingularity of a hypersurface implies a weak version of equiresolution, see loc. cit. for details, but the main problem remains, it does not show the existence of a topological trivialization that lifts to the resolution space.

Notation and terminology. We denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. Thus, by $\mathbb{K}$-analytic we mean either real analytic or holomorhic (complex analytic).

By an analytic space we mean one in the sense of [48]. As we work only locally in the analytic case, it suffices to consider only analytic set germs. For an analytic space X by $Sing(X)$ we denote the set of singular points of X and by $Reg(X)$ its complement, the set of regular points of X. For an analytic function germ F we denote by $V(F)$ its zero set and by F_{red} its reduced (i.e. square free) form. By a real analytic arc we mean a real analytic map $\gamma : I \to X$, where $I = (-1, 1)$ and X is a real or a complex analytic space.

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Part 1. Arc-wise analytic trivializations.

1. Definition and basic properties

Let Z, Y be $\mathbb{K}$-analytic spaces. A map $f : Z \to Y$ is called *arc-analytic* if $f \circ \delta$ is analytic for every real analytic arc $\delta : I \to Z$, where $I = (-1, 1) \subset \mathbb{R}$. The arc-analytic maps were introduced by Kurdyka in [31] and have been subsequently used intensively in real analytic and algebraic geometry, see [34]. It was shown by Bierstone and Milman in [2] (see also [53] for a different proof) that the arc-analytic maps with subanalytic graphs are continuous and that the arc-analytic maps with semi-algebraic graphs are blow-analytic, i.e. can be made real analytic after composing with blowings-up. Therefore the arc-analytic maps are closely related to the blow-analytic trivialization in the sense of Kuo [30].

In this paper we consider arc-analytic trivializations satisfying some additional properties. Below we define the notion of arc-wise analytic trivialization, that is not only arc-analytic with arc-analytic inverse, but it is also $\mathbb{K}$-analytic with respect to the parameter $t \in T$. For simplicity we assume that the parameter space T is nonsingular.

Definition 1.1. Let T, Y, Z be $\mathbb{K}$-analytic spaces, T nonsingular. We say that a map $f(t, z) : T \times Z \to Y$ is *arc-wise analytic in* t if it is $\mathbb{K}$-analytic in t and arc-analytic in z, that is if for every real analytic arc $z(s) : I \to Z$, the map $f(t, z(s))$ is real analytic, and moreover, if $\mathbb{K} = \mathbb{C}$, complex analytic with respect to t.

All arc-wise analytic maps considered in this paper are subanalytic and hence continuous.

We stress that even for complex analytic spaces we define the notion of arc-analyticity using only real analytic arcs. (A map of complex analytic spaces $f : Z \to Y$, with Z nonsingular, that is complex analytic on complex analytic arcs is, by Hartogs Theorem, complex analytic.)

Definition 1.2. Let Y, Z be $\mathbb{K}$-analytic spaces and let T be a nonsingular $\mathbb{K}$-analytic space. Let $\pi : Y \to T$ be a $\mathbb{K}$-analytic map. We say

$$\Phi(t, z) : T \times Z \to Y$$

is an *arc-wise analytic trivialization of* π if it satisfies the following properties

(1) Φ is a subanalytic homeomorphism,
(2) Φ is arc-wise analytic in t (in particular it is $\mathbb{K}$-analytic with respect to t),
(3) $\pi \circ \Phi(t, z) = t$ for every $(t, z) \in T \times Z$,
(4) the inverse of Φ is arc-analytic,
(5) there exist $\mathbb{K}$-analytic stratifications $\{Z_i\}$ of Z and $\{Y_i\}$ of Y, such that for each i, $Y_i = \Phi(T \times Z_i)$ and $\Phi_{|T \times Z_i} : T \times Z_i \to Y_i$ is a real analytic diffeomorphism.

Sometimes we say for short that such Φ is an arc-wise analytic trivialization if it is obvious from the context what the projection π is.

In the algebraic case we require Φ to be semialgebraic and that the stratifications are algebraic in the sense explained in Section 7.

If $\Phi(t, z) : T \times Z \to Y$ is an *arc-wise analytic trivialization* then, for each $z \in Z$, the map $T \ni t \to \Phi(t, z) \in Y$ is a $\mathbb{K}$-analytic embedding. We denote by L_z its image and we call it *a leaf or a fiber of* Φ. We say that Φ *preserves* $X \subset Y$ if X is a union of leaves. We denote by $T_y = T_y L_z$, $y = \Phi(t, z)$, the tangent space to the leaf through y.

1.1. **Computation in local coordinates.** Let $(t_0, z_0) \in T \times Z$, $y_0 = \Phi(t_0, z_0)$. Choosing local coordinates, we may always assume that $(T, t_0) = (\mathbb{K}^m, 0)$, (Z, z_0) is an analytic subspace of $(\mathbb{K}^n, 0)$, and (Y, y_0) is an analytic subspace of $(T \times \mathbb{K}^n, 0)$ with $\pi(t, x) = t$. Thus we may write

$$\Phi(t, z) = (t, \Psi(t, z)). \tag{1.1}$$

We also suppose that $L_0 = \Phi(T \times \{0\}) = T \times \{0\}$ as germs at the origin.

Using local coordinates, we identify T_y with an m-dimensional vector subspace of $\mathbb{K}^m \times \mathbb{K}^n$ and consider T_y as a point in the Grassmannian $G(m, m+n)$. These tangent spaces are spanned by the vector fields v_i on Y defined by

$$v_i(\Phi(t,z)) := (\partial/\partial t_i, \partial \Psi/\partial t_i) \qquad i = 1, ..., m. \tag{1.2}$$

Proposition 1.3. *Let $\Phi(t,z) : T \times Z \to Y$ be an arc-wise analytic trivialization. Then the vector fields v_i and the tangent space map $y \to T_y$ are subanalytic, arc-analytic, and continuous.*

Proof. The subanalyticity follows from the classical argument of subanalyticity of the derivative of a subanalytic map, see [32] Théorème 2.4. Let $(t(s), z(s)) : (I, 0) \to (T \times Z, (t_0, z_0))$ be a real analytic arc germ. Consider the map $\Psi : T \times I \to \mathbb{K}^n$

$$\Psi(t, z(s)) = \sum_{k \geq k_0} D_k(t) s^k. \tag{1.3}$$

The arc-analyticity of v_i on $(t(s), z(s))$ follows from the analyticity of $(t, s) \to \partial\Psi(t, z(s))/\partial t_i$. Finally, subanalytic and arc-analytic maps are continuous, cf. [2] Lemma 6.8. $\square$

Remark 1.4. For $y = \Phi(t,z)$ fixed, $\tau \to \Phi(t + \tau e_i, z)$ is an integral curve of v_i through y. Moreover, such an integral curve is unique as follows from (5) of Definition 1.2.

1.2. **Arc-wise analytic trivializations regular along a fiber.** We now define regular arc-wise analytic trivializations along a fiber that will be important for applications in stratification theory including our proof of Whitney's fibering conjecture, c.f. section 7. Regular arc-wise analytic trivializations preserve the size of the distance to a fixed fiber.

Definition 1.5. We say that an arc-wise analytic trivialization $\Phi(t,z) : T \times Z \to Y$ is *regular at* $(t_0, z_0) \in T \times Z$ if there are a neighborhood U of (t_0, z_0) and a constant $C > 0$ such that for all $(t, z) \in U$ (in local coordinates at (t_0, z_0) and $y_0 = \Phi(t_0, z_0)$)

$$C^{-1} \|\Psi(t_0, z)\| \leq \|\Psi(t, z))\| \leq C \|\Psi(t_0, z)\|, \tag{1.4}$$

where as in (1.1), $\Phi(t,z) = (t, \Psi(t,z))$, $\Psi(t, z_0) \equiv 0$. We say that Φ is *regular along* L_{z_0} if it is regular at every (t, z_0), $t \in T$.

We have the following criterion of regularity which follows from the more general Proposition 1.7 that we prove in the next subsection.

Proposition 1.6. *The arc-wise analytic trivialization $\Phi(t,z)$ is regular at $(0,0)$ if and only if for every real analytic arc germ $z(s) : (I, 0) \to (Z, 0)$, the leading coefficient of (1.3) does not vanish at $t = 0$: $D_{k_0}(0) \neq 0$.*

Moreover, if $\Phi(t,z)$ is regular at $(0,0)$, then in a neighborhood of $(0,0) \in T \times Z$

$$\|\frac{\partial \Psi}{\partial t}(t, z))\| \leq C \|\Psi(t, z)\|. \tag{1.5}$$

1.3. **Functions and maps regular along a fiber.** In this section we generalize the notion of regularity for arc-wise analytic trivializations to $\mathbb{K}$-analytic function germs $f : (Y, y_0) \to (\mathbb{K}, 0)$, see Definition 1.8. First we show the following criterion that we state for f of a slightly more general form.

Proposition 1.7. *Let $\Phi(t, z) : T \times Z \to Y$ be an arc-wise analytic trivialization and let $f : (Y, y_0) \to (\mathbb{R}^k, 0)$, $y_0 = \Phi(t_0, z_0)$, be a real analytic map germ. Then the following conditions are equivalent:*

(i) there is $C > 0$ such that for all (t, z) sufficiently close to (t_0, z_0)

$$C^{-1}\|f(\Phi(t_0, z))\| \le \|f(\Phi(t, z))\| \le C\|f(\Phi(t_0, z))\|. \tag{1.6}$$

(ii) for every real analytic arc germ $z(s) : (I, 0) \to (Z, z_0)$ the leading coefficient D_{k_0} of

$$f(\Phi(t, z(s))) = \sum_{k \ge k_0} D_k(t)s^k \tag{1.7}$$

satisfies $D_{k_0}(t_0) \neq 0$.

(iii) there is $C > 0$ such that for all (t, z) sufficiently close to (t_0, z_0)

$$\|\frac{\partial(f \circ \Phi)}{\partial t}(t, z))\| \le C\|f \circ \Phi(t, z)\|. \tag{1.8}$$

Proof. To show that (ii) implies (i) we use the curve selection lemma. If (i) fails then there is a real analytic arc germ $(t(s), z(s)) : (I, 0) \to (T \times Z, (t_0, z_0))$ along which one of the inequalities of (i) fails, that is, for instance, $\frac{\|f(\Phi(t(s),z(s)))\|}{\|f(\Phi(t_0,z(s)))\|} \to \infty$ as $s \to 0$. But this contradicts (ii). To complete this argument we note that $f(\Phi(t_0, z(s))) \equiv 0$ iff $f(\Phi(t, z(s))) \equiv 0$, that is what (ii) means in this case. Clearly (ii) follows from (i).

Similarly, it is sufficient to show (iii) on every real analytic arc and this follows immediately from (ii). Finally (i) follows from (iii). □

Definition 1.8. Let $\Phi(t, z) : T \times Z \to Y$ be an arc-wise analytic trivialization in t. We say that an analytic function germ $f : (Y, y_0) \to (\mathbb{K}, 0)$, $y_0 = \Phi(t_0, z_0)$, is Φ-*regular* (regular for short), if it satisfies one of the equivalent conditions of Proposition 1.7.

We say that f is Φ-*regular along* L_{z_0} (regular for short) if it is regular at every (t, z_0), $t \in T$.

Proposition 1.9. *Let $\Phi(t, z) : T \times Z \to Y$ be an arc-wise analytic trivialization and let $f, g : (Y, y_0) \to (\mathbb{K}, 0)$ be two analytic function germs not vanishing identically on each component of (Y, y_0). Then f and g are regular if and only if so is fg.*

Proof. It follows from (ii) of Proposition 1.7. □

In the complex case the regularity is a geometric notion as the following proposition shows.

Proposition 1.10. *Suppose $\mathbb{K} = \mathbb{C}$. Let $\Phi(t, z) : T \times Z \to Y$ be an arc-wise analytic trivialization and let $f : Y \to \mathbb{C}$ be a complex analytic function. Suppose that Φ preserves $V(f)$. Then f is Φ-regular at every point of $V(f)$.*

Proof. Suppose that this is not the case. Then there exists a real analytic arc $z(s) : (I, 0) \to (Z, z_0)$, such that in (1.7), $D_{k_0} \not\equiv 0$ and $D_{k_0}(t_0) = 0$. Clearly $f \circ \Phi(t_0, z(s)) \neq 0$ for $s \neq 0$. We show that for $s \neq 0$ there is $t(s)$, $t(s) \to t_0$ as $s \to 0$, such that $f \circ \Phi(t(s), z(s)) = 0$. This would contradict the assumption on Φ. For this, by restricting to a $\mathbb{K}$-analytic arc through t_0, we may suppose that t is a single variable $t \in (\mathbb{C}, 0)$. Let us then write $f \circ \Phi(t, z(s)) = s^{k_0} h(t, s)$, where

$$h(t, s) = D_{k_0}(t) + \sum_{k > k_0} D_k(t) s^{k-k_0}.$$

Since 0 is an isolated root of $h(t, 0) = 0$, Rouché's Theorem implies that $h(t, s) = 0$ has roots also for $s \neq 0$. □

Definition 1.11. We say that an ideal $\mathcal{I}$ of $\mathcal{O}_{Y,y_0}$ is Φ-*regular* (regular for short) if, for one or equivalently for every finite system of generators $f_1, ..., f_k$ of $\mathcal{I}$, $f = (f_1, \ldots, f_k)$ satisfies the equivalent conditions of Proposition 1.7.

This definition generalizes both the notion of regularity of a function and of a fiber. It follows that Φ-regularity of an ideal implies that its zero set $V(\mathcal{I})$ is preserved by the trivialization,. But except the complex function case, Proposition 1.10, this is a strictly stronger condition. We will need the following lemma for the proof of Proposition 7.3.

Lemma 1.12. *Suppose $\mathbb{K} = \mathbb{C}$. Let $\mathcal{I}$ be an ideal of $\mathcal{O}_{Y,y_0}$ and let $\Phi(t, z) : T \times Z \to Y$ be an arc-wise analytic trivialization. Let $z(s) : (I, 0) \to (Z, z_0)$ be real analytic and denote by $\varphi(t, s) : (T \times \mathbb{C}, t_0 \times 0) \to (Y, y_0)$ the complexification of $\Phi(t, x(s))$. Suppose that $\varphi(t, s) \notin V(\mathcal{I})$ for $s \in \mathbb{R}, s > 0$. Then $\varphi(t, s) \notin V(\mathcal{I})$ for $s \in \mathbb{C} \setminus \{0\}$.*

Proof. As in the proof of Proposition 1.10, we may assume T one dimensional. Consider the ideal $\varphi^*(\mathcal{I})$ in $\mathbb{C}\{t, s\}$. The only interesting case is if $V(\mathcal{I})$ is one-dimensional, that is a complex curve germ. Write its defining function in the form $s^{k_0} h(t, s)$ with h not vanishing on the t-axis. (We do not claim here that $\varphi^*(\mathcal{I})$ is principal. It may have embedded components at the origin.) If $h(0, 0) \neq 0$ we are done. Otherwise, 0 is an isolated root of $h(t, 0) = 0$ and then, by Rouché's Theorem, $h(t, s) = 0$ has roots for all $s \neq 0$, that contradicts the assumption that there are no such roots for $s \in \mathbb{R}, s > 0$. □

1.4. **Preservation of multiplicity and singular locus.** In this subsection we suppose that T, Z, and Y are open subsets of $\mathbb{K}^m$, $\mathbb{K}^n$, and $\mathbb{K}^{n+m}$, respectively, and that $\Phi : T \times Z \to Y$ is an arc-wise analytic trivialization of the standard projection $\pi : \mathbb{K}^{n+m} \to \mathbb{K}^m$.

Under these assumptions we first show the preservation of multiplicities of Φ-regular functions. Let us denote $Y_t = Y \cap \pi^{-1}(t)$ and for a function $f : Y \to \mathbb{K}$, by f_t, the restriction of f to Y_t. In the following lemma we compare the multiplicities of f at $(t, z) \in Y$ and the multiplicities of the restrictions f_t at $z \in Y_t$.

Proposition 1.13. *If $f : (Y, y_0) \to (\mathbb{K}, 0)$, $y_0 = \Phi(t_0, z_0)$, is Φ-regular then for t close to t_0 the following multiplicities are equal*

$$\operatorname{mult}_{y_0} f = \operatorname{mult}_{\Phi(t,z_0)} f = \operatorname{mult}_{y_0} f_{t_0} = \operatorname{mult}_{\Phi(t,z_0)} f_t. \tag{1.9}$$

Proof. We use the argument of Fukui's proof of invariance of multiplicity by blow-analytic homeomorphisms, cf. [14]. It is based on the observation that, on a smooth space, $\operatorname{mult}_{y_0} f = \min_{y(s)} \operatorname{ord}_0 f(y(s))$, where the minimum is taken over all real analytic arcs $y(s) : (I, 0) \to (Y, y_0)$. Thus since Φ and Φ^{-1} are arc-analytic

$$\operatorname{mult}_{\Phi(t,z_0)} f_t = \min_{z(s)\to z_0} \operatorname{ord}_0 f(\Phi(t, z(s))).$$

For Φ-regular f such orders are preserved by Φ and this shows the last equality in (1.9). The other ones follow from (ii) of Proposition 1.7. $\square$

Consider an ideal $\mathcal{I} = (f_1, ..., f_k)$ of $\mathcal{O}_Y$ and denote by $X = V(\mathcal{I})$ its zero set and by X_t the set $X \cap \pi^{-1}(t)$. Recall that for $y \in X \subset \mathbb{K}^{n+m}$ the Zariski tangent space T_yX is the kernel of the differential $D_y(f_1, ..., f_k)$.

Proposition 1.14. *Let $f_i \in \mathcal{O}_{Y,y_0}, i = 1, ..., k$, be Φ-regular and let $X = V(\mathcal{I})$. Then, for every y close to y_0, $y = \Phi(t, z)$, $T_yX_t = \pi^{-1}(t) \cap T_yX$ and $\dim_{\mathbb{K}} T_{\Phi(t,z)}X_t$ is independent of t. In particular, $SingX_t = \pi^{-1}(t) \cap SingX$ and Φ preserves $SingX$.*

Proof. The equality $T_yX_t = \pi^{-1}(t)\cap T_yX$ follows from the fact that the tangent space to the leaf through y satisfies $T_yL_z \subset T_yX$ and is transverse to the fibers of π.

The differential of f at y vanishes if and only if

$$\min_{y(s)} \operatorname{ord}_0 f(y(s)) > 1$$

where the minimum is taken over all real analytic arc germs $y(s) : (I, 0) \to (Y, y)$. Similarly, the differentials of $f_1, ..., f_l$ at y are linearly independent if and only if for every $i = 1, ..., l$ there is a real analytic arc $y(s) : I \to (Y, y)$ such that

$$\operatorname{ord}_0 f_i(y(s)) = 1 \text{ and } \operatorname{ord}_0 f_j(y(s)) > 1 \text{ for all } j = 1, ..., \hat{\imath}, ..., l.$$

This condition is preserved by Φ. $\square$

2. Construction of arc-wise analytic trivializations

In this section we use the Whitney Interpolation and the Puiseux with parameter theorem to construct arc-wise analytic trivializations of equisingular (in the sense of Zariski) families of plane curve singularities. In Part 2 we will extend this construction to the Zariski equisingular families of hypersurface singularities in an arbitrary number of variables.

Let

$$F(t, x, z) = z^N + \sum_{i=1}^{N} c_i(t, x)z^{N-i} \tag{2.1}$$

be a unitary polynomial in $z \in \mathbb{K}$ with $\mathbb{K}$-analytic coefficients $c_i(t, x)$ defined on $U_{\varepsilon,r} = U_\varepsilon \times U_r$, where $U_\varepsilon = \{t \in \mathbb{K}^m; \|t\| < \varepsilon\}$, $U_r = \{x \in \mathbb{K}; |x| < r\}$. Here t is considered as a parameter. Suppose that the discriminant $\Delta_{F_{red}}(t, x)$ of F_{red} is of the form

$$\Delta_{_{red}F_{red}}(t, x) = x^M u(t, x), \quad u \neq 0 \text{ on } U_{\varepsilon,r}. \tag{2.2}$$

For F reduced, if $M = 0$ then by the Implicit Function Theorem the complex roots of F, denoted later by $a_1(t, x), ..., a_N(t, x)$, are distinct $\mathbb{K}$-analytic functions of (t, x). In general, by the Puiseux with parameter theorem they become analytic in (t, y) after a ramification $x = y^d$. By Corollary 2.3, for x fixed, an ordering of the roots at $(0, x)$, $a_1(0, x), ..., a_N(0, x)$, gives by continuity an ordering of the roots at (t, x), $a_1(t, x), \ldots, a_N(t, x)$. Denote by $a(t, x) = (a_1(t, x), ..., a_N(t, x))$ the vector of these roots and consider the self-map $\Phi : U_{\varepsilon,r} \times \mathbb{C} \to U_{\varepsilon,r} \times \mathbb{C}$

$$\Phi(t, x, z) = (t, x, \psi(z, a(0, x), a(t, x))), \tag{2.3}$$

where $\psi(z, a, b)$ is the Whitney interpolation map given by (I.13).

Theorem 2.1. *For $\varepsilon > 0$ sufficiently small, the map Φ defined in (2.3) is an arc-wise analytic trivialization of the projection $U_{\varepsilon,r} \times \mathbb{C} \to U_\varepsilon$. It preserves the zero set $V(F)$ of F and, moreover, F is Φ-regular along $U_\varepsilon \times \{(0, 0)\}$.*

If $\mathbb{K} = \mathbb{R}$ then Φ is conjugation invariant in z.

Theorem 2.1 is shown in Subsection 2.2.

2.1. **Puiseux with parameter.**

We recall the classical Puiseux with parameter theorem, see [72] Thm. 7 and [73] Thm. 4.4, also [56]. The Puiseux with parameter theorem is a special case of the Abhyankar-Jung Theorem, see [1], [55].

Theorem 2.2. (Puiseux with parameter)
Let $F(t, x, z) \in \mathbb{C}\{t, x\}[z]$ be as in (2.1). Suppose that the discriminant of F reduced is of the form $\Delta_{F_{red}}(t, x) = x^M u(t, x)$ with $u(0, 0) \neq 0$. Then there is a positive integer d and $\tilde a_i(t, y) \in \mathbb{C}\{t, y\}$ such that

$$F(t, y^d, z) = \prod_{i=1}^{N} (z - \tilde a_i(t, y)).$$

Let θ be a dth root of unity. Then for each i there is j such that $\tilde a_i(t, \theta y) = \tilde a_j(t, y)$.

If $F(t, x, z) \in \mathbb{R}\{t, x\}[z]$ then the family $\tilde a_i(t, y)$ is conjugation invariant.

Corollary 2.3. *For x fixed, the roots of F, $a_1(t, x), \ldots, a_N(t, x)$, can be chosen complex analytic in t. Moreover, if $a_i(0, x) = a_j(0, x)$ then $a_i(t, x) \equiv a_j(t, x)$. Thus the multiplicity of each $a_i(t, x)$ as a root of F is independent of t.*

Proof. It suffices to show it for F reduced. Then for $x \neq 0$ it follows from the IFT. Let us show it for $x = 0$. The family $a_1(t, 0), \ldots, a_N(t, 0)$ coincides with $\tilde a_1(t, 0), \ldots, \tilde a_N(t, 0)$. If $\tilde a_i(0, 0) = \tilde a_j(0, 0)$ then $\tilde a_i(t, y) - \tilde a_j(t, y)$ divides y^{dM} and hence equals a power of y times a unit. □

The following corollary is well-known.

Corollary 2.4. *The Puiseux pairs of $a_i(t, x)$ and the contact exponents between different branches of $V(F)$ are independent of t.*

The next corollary is essential for the proof of Theorem 2.1. It allows us to use the bi-Lipschitz property given by Proposition I.3. Define

$$\gamma(t,x) = \max_{a_i(0,x)\neq a_j(0,x)} \frac{|(a_i(t,x)-a_i(0,x))-(a_j(t,x)-a_j(0,x))|}{|a_i(0,x)-a_j(0,x)|} \tag{2.4}$$
$$= \max_{a_i(0,x)\neq a_j(0,x)} |\frac{a_i(t,x)-a_j(t,x)}{a_i(0,x)-a_j(0,x)}-1|.$$

Corollary 2.5. *There are a positive integer r and positive real constants ε, δ, C such that for all $|x| \le \delta$ and $\|t\| \le \varepsilon$*

$$\gamma(t,x) \le C\|t\|^r.$$

Proof. We may replace x by y^d and the family $a_i(t,x)$ by complex analytic functions $\tilde a_i(t,y)$. Suppose that $\tilde a_i(t,y)-\tilde a_j(t,y)$ is not identically equal to zero. Then, since $\tilde a_i(t,y)-\tilde a_j(t,y)$ divides the discriminant of F_{red}, $\tilde a_i(t,y)-\tilde a_j(t,y) = y^{m_{ij}}u_{ij}(t,y)$ with $u_{i,j}(0,0)\neq 0$. Therefore $u_{ij}(t,y)-u_i(0,y)$ belongs to the ideal $(t_1,\dots,t_m)\mathbb{C}\{t,y\}$. Consequently there are a positive integer r_{ij} and a constant C_{ij} such that

$$\frac{|(\tilde a_i(t,y)-\tilde a_j(t,y))-(\tilde a_i(0,y)-\tilde a_j(0,y))|}{|(\tilde a_i(0,y)-\tilde a_j(0,y))|} = \frac{|u_{ij}(t,y)-u_{ij(}0,y)|}{|u_{ij}(0,y)|} \le C_{ij}\|t\|^{r_{ij}}$$

in a neighborhood of the origin. It suffices to take $C=\max C_{ij}$ and $r=\min r_{ij}$. $\square$

2.2. **Proof of Theorem 2.1.** Φ is continuous by Proposition I.4 and Remark I.2. By Proposition I.3 and Corollary 2.5, if ε is sufficiently small, then for t and x fixed, $\psi_{a(0,x),a(t,x)}: \mathbb{C}\to\mathbb{C}$ is bi-Lipschitz. Therefore Φ is bijective and the continuity of Φ^{-1} follows from the invariance of domain.

Lemma 2.6. *For any $r'<r$ there is $C>0$ such that the restriction $\Phi: U_{\varepsilon,r'}\times\mathbb{C}\to U_{\varepsilon,r'}\times\mathbb{C}$ satisfies*

$$C^{-1}|F(0,x,z)| \le |F(\Phi(t,x,z))| \le C|F(0,x,z)|. \tag{2.5}$$

Proof. The Lipschitz constants of $\psi_{a(0,x),a(t,x)}: \mathbb{C}\to\mathbb{C}$ and of its inverse can be chosen independent of $(t,x)\in U_{\varepsilon,r'}$. Let L be a common upper bound for these constants. Then, because $\psi_{a(0,x),a(t,x)}(a_i(0,x)) = a_i(t,x)$,

$$L^{-1}|z-a_i(0,x)| \le |\psi_{a(0,x),a(t,x)}(z)-a_i(t,x)| \le L|z-a_i(0,x)| \tag{2.6}$$

Because $F(\Phi(t,x,z)) = \prod_i(\psi_{a(0,x),a(t,x)}(z)-a_i(t,x))$, we obtain (2.5) with $C=L^N$ by taking the product of (2.6) over i. $\square$

Let us write the formula for $\psi(z,a,b)$ of (I.13), as

$$\psi(z,a,b) = z + \frac{Q(z,a)\overline{Q(z,a)}\big(\sum_j\sum_k\ Q_{k,j}(z,a,b)\overline{Q_k(z,a)}(b_j-a_j)\big)}{N!Q(z,a)\overline{Q(z,a)}(\sum_k Q_k(z,a)\overline{Q_k(z,a)})}, \tag{2.7}$$

where

$$Q_k(z,a) = P_k((z-a_1)^{-1},...,(z-a_N)^{-1}),$$
$$Q_{k,j}(z,a) = (z-a_j)^{-1}\frac{\partial P_k}{\partial \xi_j}((z-a_1)^{-1},...,(z-a_N)^{-1})$$
$$Q(z,a) = \prod_{s=1}^{N}(z-a_s)^{N!},$$

and the polynomials P_k are defined in Example I.7.

The numerator of the fraction in (2.7) is a polynomial in the real and imaginary parts of z, a, and a polynomial in b. The denominator of this fraction is the a's non-negative real valued polynomial in $\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} a, \operatorname{Im} a$. By Proposition I.4 this quotient is continuous on the set $\Xi = \{(z,a,b);\ \text{if } a_i = a_j \text{ then } b_i = b_j\}$. We show that ψ is real analytic on the strata of a natural stratification of Ξ.

The space $\mathbb{C}^N \ni a$ can be stratified by the type of a, that is by the number of distinct a_i and by the multiplicities m_i they appear in the vector a. We encode such a type by the multiplicity vector $\mathbf{m} = (m_1,...,m_d)$, $\sum_{s=1}^d m_i = N$. We denote by $S_{\mathbf{m}} \subset \mathbb{C}^N$ the set of the vectors a with the multiplicity vector $\mathbf{m}$. Each stratum, that is each connected component of such $S_{\mathbf{m}}$, is given by $S_W = \{a \in S_{\mathbf{m}}; a_i = a_j \text{ if } \exists s, \text{ s.t. } i,j \in W_s\}$, where $W = \{W_s\}$ is a partition $\{1,...,N\} = \sqcup_s W_s$ with $|W_s| = m_s$. We denote by S_W the stratum given by partition W.

Lemma 2.7. *The restriction of $\psi(z,a,b)$ of* (I.13) *to each $\mathbb{C}\times S_W \times S_W$ is real analytic.*

Proof. Choose the representatives $i_1,...,i_d$ so that $i_s \in W_s$. If we replace in (2.7), Q by $Q_W(z,a) = \prod_{s=1}^d (z-a_{i_s})^{N!}$ then the denominator of the fraction in (2.7) does not vanish. Indeed, first note that for all k, $Q_W(z,a)Q_k(z,a)$ is a polynomial on $\mathbb{C}\times S$. By property (5) of Appendix I, it may vanish only for z equal to one the a_i, say a_{i_1} for instance. Note that

$$Q_W(z,a)Q_{m_1}(z,a) = \prod_{s=2}^{d}(z-a_{i_s})^{N!} + (z-a_{i_1})R(z,a),$$

where R is a polynomial. Therefore $Q_W(a_{i_1},a)Q_{m_1}(a_{i_1},a) \neq 0$ which suffices to show the claim. □

For an integer d, $1 \le d \le N$, we consider

$$D_d(a) = \sum_{r_1<\cdots<r_d}\ \prod_{k<l;\, k,l\in\{r_1,...,r_d\}} (a_k-a_l)^2.$$

Lemma 2.8. *Let the germ $a(t,s) : (\mathbb{K}^m \times \mathbb{R}, (0,0)) \to \mathbb{C}^N$ be such that for every symmetric polynomial G in b, $G(a(t,s))$ is analytic in (t,s) (it equals to a power series in $(t,s) \in \mathbb{K}^m\times\mathbb{R}$). We also assume that for $s \neq 0$, $a(t,s)$ has exactly d distinct components and that $D_d(a(t,s))$ equals $s^M u(t,s)$ with $u(0,0)\neq 0$. Let $z(s) : (\mathbb{R},0)\to\mathbb{C}$ be a real analytic germ and set $a(s) = a(0,s)$. Then $\psi(z(s),a(s),a(t,s))$, where ψ is given by* (I.13)*, is analytic in (t,s).*

Proof. By subtracting $z(s)$ from every component of $a(t,s)$ we may assume that $z(s) \equiv 0$. We consider $a_i(t,s)$ as the roots of a polynomial

$$G(z,t,s) = z^N + \sum_{i=1}^{N} c_i(t,s) z^{N-i} \tag{2.8}$$

with coefficients analytic in t, s. By Lemma II.1 the discriminant of G_{red} equals a non-zero constant times $D_d(a(t,s))$. We may consider $c_i(t,s)$ as complex analytic germs of $(t,s) \in (\mathbb{C}^m \times \mathbb{C}, (0,0))$ and apply to G_{red} the Puiseux with parameter theorem, Theorem 2.2. In particular, for a fixed s, an ordering of the roots $a_1(s), ..., a_N(s)$ of $G(z,0,s)$ gives, by continuity, an ordering of the roots $a_1(t,s), ..., a_N(t,s)$ of G. Fix such an ordering and define

$$\varphi(t,s) = \psi(0, a(s), a(t,s)),$$

where ψ is given by 2.7. Thus defined φ is independent of the choice of an ordering (since passing from one ordering to another is given by the action of the same permutation on a and b). Since $P_k(a)$ is symmetric in a, $Q(a)$ and the product $Q(a)Q_k(a)$ are polynomials in the coefficients $c_i(0,s)$ of G. Hence $Q(a(s))$ and $Q(a(s))Q_k(a(s))$ are complex analytic in $s \in \mathbb{C}$.

As follows from the next lemma, for a fixed k, $Q(a(s))(\sum_{j=1}^{N} Q_{k,j}(a(s))(a_j(t,s) - a_j(s))) \in \mathbb{C}\{t,s\}$.

Lemma 2.9. *Let $P(a,b) \in \mathbb{C}[a,b]$ be a polynomial invariant under the action of the permutation group: $P(\sigma(a), \sigma(b)) = P(a,b)$ for all $\sigma \in S_N$. Then $P(a(s), a(t,s)) \in \mathbb{C}\{t,s\}$.*

Proof. We may assume that $P(a(s), a(t,s))$ is well-defined for $(t,s) \in B \times D$, where B is a neighborhood of the origin in $\mathbb{C}^m$ and D is a small disc centered at the origin in $\mathbb{C}$. By the assumption $P(a(s), a(t,s))$ is bounded and complex analytic on $B \times (D \setminus \{0\})$. Therefore it is complex analytic on $B \times D$. $\square$

In particular, by Lemma 2.9, the numerator of the fraction in (2.7), evaluated on $a(t,s), a(s)$ is analytic in $(t,s) \in \mathbb{K}^m \times \mathbb{R}$. As we have shown before its denominator is analytic in (one variable) $s \in \mathbb{R}$. Therefore, $\varphi(t,s)$ is of the form s^{-k} times a power series in (t,s). Since, moreover, $\varphi(t,s)$ is bounded in a neighborhood of the origin it has to be analytic. $\square$

It follows from Lemma 2.7 that $\psi(z, a(0,x), a(t,x))$ of (2.3) is real analytic on $x \neq 0$ and on $x = 0$ and from Lemma 2.8 that it is arc-wise analytic. The next lemma shows that the inverse of Φ is arc-analytic and completes the proof of Theorem 2.1.

Lemma 2.10. *If $(t(s), x(s), z(s))$ is a real analytic arc, then there is a real analytic $\tilde{z}(s)$ such that $(t(s), x(s), z(s)) = \Phi(t(s), x(s), \tilde{z}(s))$.*

Proof. Since Φ^{-1} is subanalytic such $\tilde{z}(s)$ exists continuous and subanalytic. Thus there is a positive integer q such that for $s \geq 0$, $\tilde{z}(s)$ is a convergent power series in $s^{1/q}$. We show that all exponents of $\tilde{z}(s)$, $s \geq 0$, are integers. Suppose that this is not the case. Then

$$\tilde{z}(s) = \sum_{i=1}^{n} v_i s^i + v_{p/q} s^{p/q} + \sum_{k>p} v_{k/q} s^{k/q},$$

with $p/q > n$ and $p/q \notin \mathbb{N}$. Denote $\tilde z_{an}(s) = \sum_{i=1}^n v_i s^i$. Then $\psi(\tilde z_{an}(s), a(0,x(s)), a(t(s),x(s)))$ is real analytic and by the bi-Lipschitz property, Proposition I.3,

$$|\psi(\tilde z_{an}(s), a(0,x(s), a(t(s),x(s))) - \psi(\tilde z(s), a(0,x(s)), a(t(s),x(s)))| \sim |\tilde z_{an}(s) - \tilde z(s)| \sim s^{p/q},$$

that is impossible since $\psi(z_{an}(s), a(0,x(s)), a(t(s),x(s)))$ and $\psi(z(s), a(0,x(s)), a(t(s),x(s)))$ are real analytic in s.

This shows that $(t(s),x(s),z(s))$, $\Phi(t(s),x(s),\tilde z(s))$ are two real analytic arcs that coincide for $s \ge 0$ and therefore also for $s \le 0$. □

2.3. **Preservation of multiplicities of roots.** Corollary 2.3 admits a multidimensional generalization, see Zariski [75]. In the sequel we will need the following result that is a consequence of [75] and Proposition 1.13. We include its proof for the reader's convenience.

Lemma 2.11 (Preservation of multiplicities of roots)**.** *Let $\Phi : T \times Z \to Y$ be an arc-wise analytic trivialization, $y_0 = \Phi(t_0, z_0)$, and let A_i, $i = 1, ..., N$, be $\mathbb{K}$-analytic functions defined in a neighborhood of y_0. Let*

$$f(y,w) = w^N + \sum_i A_i(y) w^{N-i}$$

and suppose that the discriminant $\Delta(f_{red})$ is Φ-regular. Then, for t in a neighborhood of t_0, the roots of f at $\Phi(t, z_0)$,

$$a_1(\Phi(t,z_0)), \ldots, a_N(\Phi(t,z_0)),$$

can be chosen complex analytic in t. (Moreover, any continuous choice is complex analytic.) For such a choice, if $a_i(\Phi(t_0,z_0)) = a_j(\Phi(t_0,z_0))$ then $a_i(\Phi(t,z_0)) = a_j(\Phi(t,z_0))$ for all t. In particular, the multiplicity of each $a_i(\Phi(t,z_0))$ as a root of f is independent of t.

Proof. Choose a real analytic arc germ $z(s) : I \to Z$, $z(0) = z_0$, so that $\Delta(f_{red})$ is not identically zero on $\Phi(t, z(s))$. By Corollary 2.3 it suffices to show that $F(t,s,w) = f_{red}(\Phi(t,z(s)), w)$ satisfies the assumptions of the Puiseux with parameter theorem. To show it we first note that the discriminant of F equals to $\Delta(f_{red})(\Phi(t,z(s)))$. Secondly, we observe that, by regularity of $\Delta(f_{red})$ on $z(s)$ in the form (1.6), $\Delta(f_{red})(\Phi(t,z(s)))$ equals s^k times an analytic unit. □

Part 2. Zariski Equisingularity.

3. Zariski Equisingularity implies arc-wise analytic triviality.

In this section we generalize Theorem 2.1 to an arbitrary number of variables hypersurface case.

Definition 3.1. By a *local system of pseudopolynomials in $x = (x_1, ..., x_n) \in \mathbb{K}^n$ at $(0,0) \in \mathbb{K}^m \times \mathbb{K}^n$, with parameter $t \in U \subset \mathbb{K}^m$*, we mean a family of analytic functions

$$F_i(t, x_1, \ldots, x_i) = x_i^{d_i} + \sum_{j=1}^{d_i} A_{i,j}(t, x_1, \ldots, x_{i-1}) x_i^{d_i - j}, \quad i = 0, \ldots, n, \tag{3.1}$$

defined on $U \times U_i$, where U is a neighborhood of the origin in $\mathbb{K}^m$, U_i is a neighborhood of the origin in $\mathbb{K}^i$, with the coefficients $A_{i,j}$ vanishing identically on $T = U \times \{0\}$. Thus F_0 is an analytic function depending only on t. We also assume that, for each $i = 1, \ldots, n$, the discriminant of $F_{i,red}$ divides F_{i-1}. For $d_i = 0$, by (3.1) we mean that $F_i \equiv 1$, and in this case by convention we define all $F_j, j < i$, as identically equal to 1.

We call this system *Zariski equisingular* if $F_0(0) \neq 0$. As Varchenko showed in [61], answering a question posed by Zariski in [74], for a Zariski equisingular system, the family of analytic set germs $X_t = \{F_n(t, x) = 0\} \subset (\mathbb{K}^n, 0)$ is topologically equisingular for t close to the origin. In this section we show that this equisingularity can be obtained by an arc-wise analytic trivialization.

Remark 3.2. The above definition is slightly more general than that of [74] or [61] where it is assumed that F_{i-1} is the Weierstrass polynomial associated to the discriminant of $F_{i,red}$. Our less restrictive assumption is sufficient for the proof of Theorem 3.3. In fact, in the inductive step we only need that the discriminant of $F_{i,red}$ is Φ_{i-1}-regular for the trivialization Φ_{i-1}. By Proposition 1.7 this is the case if this discriminant divides F_{i-1} and F_{i-1} is Φ_{i-1}-regular.

Theorem 3.3. *If $F_i(t, x)$, $i = 0, \ldots, n$, is a Zariski equisingular local system of pseudopolynomials, then there exist $\varepsilon > 0$ and a homeomorphism*

$$\Phi : B_\varepsilon \times \Omega_0 \to \Omega, \tag{3.2}$$

where $B_\varepsilon = \{t \in \mathbb{K}^m; \|t\| < \varepsilon\}$, Ω_0 and Ω are neighborhoods of the origin in $\mathbb{K}^n$ and $\mathbb{K}^{m+n}$ resp., such that

(Z1) *$\Phi(t, 0) = (t, 0)$, $\Phi(0, x_1, \ldots, x_n) = (0, x_1, \ldots, x_n)$;*
(Z2) *Φ has a triangular form*
$$\Phi(t, x_1, \ldots, x_n) = (t, \Psi_1(t, x_1), \ldots, \Psi_{n-1}(t, x_1, \ldots, x_{n-1}), \Psi_n(t, x_1, \ldots, x_n));$$
(Z3) *For $(t, x_1, \ldots, x_{i-1})$ fixed, $\Psi_i(t, x_1, \ldots, x_{i-1}, \cdot) : \mathbb{K} \to \mathbb{K}$ is bi-Lipschitz and the Lipschitz constants of Ψ_i and Ψ_i^{-1} can be chosen independent of $(t, x_1, \ldots, x_{i-1})$;*
(Z4) *Φ is an arc-wise analytic trivialization of the standard projection $\Omega \to B_\varepsilon$;*
(Z5) *F_n is regular along $B_\varepsilon \times \{0\}$.*

Recall after Proposition 1.9 that (Z5) implies that for any analytic G dividing a power of F_n, there is $C > 0$ such that

$$C^{-1}|G(0, x)| \leq |G(\Phi(t, x))| \leq C|G(0, x)|. \tag{3.3}$$

In particular Φ preserves the zero level of G.

Remark 3.4. Strategy of proof.
The functions Ψ_i will be constructed inductively so that every

$$\Phi_i(t, x_1, \ldots, x_i) = (t, \Psi_1(t, x_1), \ldots, \Psi_i(t, x_1, \ldots, x_i)) \tag{3.4}$$

satisfies the above properties (Z1)-(Z4) and (Z5) for F_i. Given $\Phi_{n-1} : B_{\varepsilon'} \times \Omega'_0 \to \Omega'$. We first lift it (by continuity) to all complex roots of F_n, then we extend it to $B_{\varepsilon'} \times \Omega'_0 \times \mathbb{C}$ by the Whitney Interpolation Formula. The fact that the trivialization $\Phi(t, x)$ obtained in this way is arc-wise analytic is proven by a reduction to the Puiseux with parameter case as follows.

Let $x(s) = (x'(s), x_n(s))$ be a real analytic arc. By the inductive assumption $\Phi_{n-1}(t, x'(s))$ is analytic in t, s. We show that $f(t,s,z) = (F_n(\Phi_{n-1}(t,x'(s)), z))_{red}$ satisfies the assumptions of the Puiseux with parameter theorem, and then we conclude by Theorem 2.1. We first consider $x'(s)$ sufficiently generic, so that the discriminant of $F_{n,red}(\Phi_{n-1}(t,x'(s)), z)$ does not vanish identically, and use this case to show that the number and the multiplicities of the roots of F_n are constant over each leaf of Φ_{n-1}. This will imply the case of arbitrary arcs $x(s)$.

The fact that Φ satisfies the property (5) of Definition 1.2 will be shown later in Section 5 where the appropriate stratification is introduced. In the argument below we do not use this property in the inductive step.

Proof. The proof is by induction on n. Thus suppose that $\Psi_1, \dots, \Psi_{n-1}$ are already constructed and that for $i < n$ the homeomorphisms (3.4) satisfy the properties (Z1)-(Z5). To simplify the notation we write $(x_1, \dots, x_n) = (x', x_n)$. By the inductive assumption F_{n-1}, and hence by Proposition 1.9 the discriminant of $F_{n,red}$, is regular for $\Phi_{n-1} : B_{\varepsilon'} \times \Omega'_0 \to \Omega'$. Therefore, by the preservation of multiplicities of roots principle, Lemma 2.11, for any $x' \in \Omega'_0$, the complex roots of F_n

$$a_1(\Phi_{n-1}(t,x')), \dots, a_{d_n}(\Phi_{n-1}(t,x'))$$

can be chosen $\mathbb{K}$-analytic in t. Moreover, $a_i(0,x') = a_j(0,x')$ if and only if $a_i(\Phi_{n-1}(t,x')) = a_j(\Phi_{n-1}(t,x'))$ for all $t \in B_{\varepsilon'}$. Denote by $a(\Phi_{n-1}(t,x')) = (a_1(\Phi_{n-1}(t,x')), \dots, a_{d_n}(\Phi_{n-1}(t,x')))$ the vector of such roots and set

$$\begin{aligned} \Psi_n(t,x) &:= \psi(x_n, a(0,x'), a(\Phi_{n-1}(t,x'))) \\ &= x_n + \frac{\sum_{j=1}^N \mu_j(x_n, a(0,x'))(a_j(\Phi_{n-1}(t,x')) - a_j(0,x'))}{\sum_{j=1}^N \mu_j(x_n, a(0,x'))}, \end{aligned} \tag{3.5}$$

where ψ is given by (I.13), and then define Φ by (Z2).

Thus constructed Φ satisfies (Z1) and (Z2) by its definition. We show that Φ is a homeomorphism that satisfies (Z3)-(Z5). This we check on every real analytic arc applying the Puiseux with parameter theorem.

Lemma 3.5. *Let $K \Subset \Omega'_0$. Then*

$$\sup_{x' \in K} \max_{a_i(0,x') \neq a_j(0,x')} \frac{|a_i(\Phi_{n-1}(t,x')) - a_j(\Phi_{n-1}(t,x'))|}{|a_i(0,x') - a_j(0,x')|} \to 1 \quad as \quad t \to 0 \tag{3.6}$$

Proof. Denote

$$\gamma(t,x') = \max_{a_i(0,x') \neq a_j(0,x')} \frac{|(a_i(\Phi_{n-1}(t,x')) - a_j(\Phi_{n-1}(t,x'))) - (a_i(0,x') - a_j(0,x'))|}{|a_i(0,x') - a_j(0,x')|}.$$

We show that γ is bounded on $B_{\varepsilon'} \times K$, after replacing ε' by a smaller positive number if necessary, and converges to 0 as t goes to 0. Let $x'(s)$ be a real analytic arc such that $(0, x'(s))$ is not entirely included in the zero set of F_{n-1}. By Corollary 2.5, γ is bounded on $(t, x'(s))$ and converges to 0 as t goes to 0. Thus, by the curve selection lemma, the claim holds on $\{(t,x'); F_{n-1}(t,x') \neq 0\}$. We extend it on the zero set of F_n by the lower semi-continuity of γ, Remark I.5. $\square$

Thus, taking ε' smaller if necessary, we see by Proposition I.3 that Ψ_n of (3.5) is well-defined, continuous by Proposition I.4, and satisfies (Z3).

Choose a neighborhood $\tilde{\Omega}'_0$ of the origin in $\mathbb{K}^{n-1}$, $\varepsilon \le \varepsilon'$ and $r > 0$ so that $\tilde{\Omega}'_0 \Subset \Omega'_0$ and F_n does not vanish on $B_\varepsilon \times \tilde{\Omega}'_0 \times \partial D$, where $\partial D = \{x_n \in \mathbb{K}; \|x_n\| = r\}$. Then we set $\Omega_0 = \tilde{\Omega}'_0 \times D$, where $D = \{x_n \in \mathbb{K}; \|x_n\| < r\}$, and $\Omega = \Phi(B_\varepsilon \times \Omega_0)$.

Now we show (Z4) (except the property (5) of Definition 1.2 that will be shown in Section 5). Let $x(s) : I \to \Omega_0$ be a real analytic arc. We show that $\Phi(t, x(s))$ is analytic in t and s. If $(0, x'(s))$ is not entirely included in the zero set of F_{n-1} then it follows from Theorem 2.1 (we argue as in the proof of Lemma 2.11). Thus, suppose $F_{n-1}(0, x'(s)) \equiv 0$. Consider

$$f(t, s, z) = (F_n(\Phi_{n-1}(t, x'(s)), z))_{red}. \tag{3.7}$$

By (3.6) the size of the discriminant $\Delta_f(t, s)$ of f is independent of t, that is there are constants $C, c > 0$ such that

$$c|\Delta_f(0, s) \le |\Delta_f(t, s)| \le C|\Delta_f(0, s).$$

Write Δ_f in the form $s^M h(t, s)$, where h does not vanish identically on $s = 0$. By the above inequality we conclude that $h(0, 0) \neq 0$. Hence $f(t, s, z)$ satisfies the assumption of Theorem 2.1 that implies that $\Phi(t, x(s))$ is analytic in t and s.

To show that the inverse of Φ is arc-analytic we use the inductive assumption, i.e. the assumption that the inverse of Φ_{n-1}, is arc-analytic. Then, for a real analytic arc $x'(s)$ fixed, over its flow $(t, s) \to \Phi_{n-1}(t, x'(s))$, we use Lemma 2.10.

The proof of (Z5) is similar to that of (Z4). First, by Proposition 1.7, it suffices to show it over the flow of any real analytic arc $x'(s)$, that is for $(t, s) \to \Phi_{n-1}(t, x'(s))$. If $(0, x'(s))$ is not entirely included in the zero set of F_{n-1}, then it follows directly from the proof of Lemma 2.11 and Theorem 2.1. If $F_{n-1}(0, x'(s)) \equiv 0$ then we consider (3.7) and conclude again by Theorem 2.1. $\square$

3.1. **Geometric properties.** In this subsection we summarize some geometric properties of the arc-wise analytic trivialization Φ constructed in the proof of Theorem 3.3. Firstly, Φ preserves the multiplicities and the singular loci of the Φ-regular functions.

The preservation of multiplicity follows by induction from Zariski [75], or, independently from Proposition 1.13.

Proposition 3.6 (Zariski equisingularity implies equimultiplicity)**.** *Let F_i, $i = 0, \ldots, n$, be a Zariski equisingular local system of pseudopolynomials at the origin in $\mathbb{K}^m \times \mathbb{K}^n$. Then for any $\mathbb{K}$-analytic function G dividing F_n, the multiplicities*

$$\operatorname{mult}_{(t,0)} G = \operatorname{mult}_0 G_t, \tag{3.8}$$

where $G_t(x) = G(t, x)$, are independent of t. $\square$

Note that, by construction $\Phi(t, x) = (t, \Psi(t, x))$ is real analytic in the complement of $B_\varepsilon \times Z$, where Z is a nowhere dense $\mathbb{K}$-analytic subset of Ω_0. Let us, for t fixed, denote $x \to \Psi(t, x)$ by Ψ_t. It follows from (Z2) and (Z3) that the jacobian determinant of Ψ_t, that is well-defined in the complement of $B_\varepsilon \times Z$, is bounded from zero and infinity in a neighborhood of the origin, that is there exists $C, c > 0$ such that

$$c \le |jac \det(\Psi_t)(t,x)| \le C. \tag{3.9}$$

Consider an analytic set $X = \{f_1(t,x) = ... = f_k(t,z) = 0\} \subset \Omega$ defined by $\mathbb{K}$-analytic Φ-regular functions $f_1(t,x), ..., f_k(t,z)$. Denote $X_t = X \cap \pi^{-1}(t)$. Then, as follows from Proposition 1.14, $SingX_t = \pi^{-1}(t) \cap SingX$ and Φ preserves $SingX$ and $RegX$.

3.2. **Generalizations.** The following generalization can be used to show the topological equisingularity of analytic function germs, see [3] and Subsection 8.3 below.

Proposition 3.7. *Theorem 3.3 holds if in the definition of a local system of pseudopolynomials the assumption*

(i) The discriminant of $F_{i,red}$ divides F_{i-1}.

is replaced by

(ii) There are $q_i \in \mathbb{N}$ such that $F_i = x_1^{q_i} \tilde{F}_i$, where $\tilde{F}_i(x_1, ..., x_i)$ is a monic Weierstrass polynomial in x_i, and for $i = 1, ..., n$, the discriminant of $\tilde{F}_{i,red}$ divides F_{i-1}.

Moreover, in the conclusion we may require that $\Psi_1(t, x_1) \equiv x_1$.

Proof. We can always require $\Psi_1(t, x_1) \equiv x_1$ in the first step of construction. Then, in the inductive step, we assume that x_1 and $\tilde{F}_{n-1}$ are Φ_{n-1}-regular. Hence, by Proposition 1.9, so is the discriminant of $\tilde{F}_{i,red}$. This allows us to proceed with the construction of Φ. Since x_1 is constant on the fibers of Φ, it is Φ-regular and $\tilde{F}_n$ is Φ-regular by the proof of Theorem 3.3. $\square$

4. Zariski Equisingularity with transverse projections.

Definition 4.1. We say that a local system of pseudopolynomials $F_i(t,x)$, $i = 1, \ldots, n$, is *transverse* at the origin in $\mathbb{K}^m \times \mathbb{K}^n$, if for every $i = 2, \ldots, n$, the multiplicity $\operatorname{mult}_0 F_i(0,x)$ of $F_i(0,x)$ at $0 \in \mathbb{K}^i$ is equal to d_i.

We always have the upper semi-continuity condition. If we denote $F_t(x) = F(t,x)$, then $\operatorname{mult}_0 F_t \le \operatorname{mult}_0 F_0$ for t close to 0. Since $\operatorname{mult}_0 F_t \le d_i$, the transversality is a closed condition (in the Euclidean or analytic Zariski topology) in parameter t.

If the system $\{F_i\}$ is Zariski equisingular then, by Proposition 3.6, the transversality is also an open condition. Thus in this case the system is transverse at any $(t, 0) \in U$, keeping the notation from Definition 3.1. Therefore, writing F instead of F_n, we have $d_n = \operatorname{mult}_0 F_t = \operatorname{mult}_0 F_0$ and also $d_n = \operatorname{mult}_{(0,0)} F = \operatorname{mult}_{(t,0)} F$.

Denote $X = F^{-1}(0)$, $X_t = X \cap \{t\} \times \mathbb{K}^n$. Geometrically the assumption $\operatorname{mult}_0 F(0,x) = d_n$ means that the kernel of the standard projection $\mathbb{K}^n \to \mathbb{K}^{n-1}$ is transverse to the tangent cone of X_0 at the origin, i.e. the vertical line $\{0\} \times \mathbb{K} \subset \mathbb{K}^{n-1} \times \mathbb{K}$ is not entirely included in this tangent cone. If this is the case then, in the Zariski equisingular case, by Proposition 3.6, the kernel of the standard projection $\pi : \mathbb{K}^m \times \mathbb{K}^n \to \mathbb{K}^m \times \mathbb{K}^{n-1}$ is transverse to the tangent cone of X at the origin.

Definition 4.2. We say that a local system of pseudopolynomials $F_i(t,x)$, $i = 1, \ldots, n$, is *partially transverse* if each F_i with $d_i > 0$ has a factor G_i of degree $d_i' > 0$ in x_i such that $\operatorname{mult}_0 G_i(0,x) = d_i'$.

It is clear from the definitions that a transverse system is partially transverse.

Theorem 4.3. *Let $F_i(t,x)$, $i=0,\dots,n$, be a Zariski equisingular local system of pseudopolynomials partially transverse at the origin in $\mathbb{K}^m \times \mathbb{K}^n$. Let $\Phi(t,x) = (t, \Psi(t,x)) : B_\varepsilon \times \Omega_0 \to \Omega$ be the homeomorphism constructed in the proof of Theorem 3.3. Then*

(Z6) Φ *is an arc-wise analytic trivialization regular along $B_\varepsilon \times \{0\}$.*

Proof. We have to show, see Subsection 1.2, that, after shrinking the neighborhood Ω if necessary, there is a constant $C>1$ such that for all $(t,x) \in B_\varepsilon \times \Omega_0$,

$$C^{-1}\|x\| \le \|\Psi(t,x))\| \le C\|x\|. \tag{4.1}$$

This will be shown by induction on n. Let us write for short $x=(x',x_n)$ and $\Phi(t,x) = (t,\Psi(t,x)) = (t,\Psi'(t,x'),\Psi_n(t,x))$. By the inductive assumption

$$C_1^{-1}\|x'\| \le \|\Psi'(t,x'))\| \le C_1\|x'\|. \tag{4.2}$$

Let $a_1(t,x'),\dots,a_{d'_n}(t,x'))$ denote the complex roots of $G_n = x_n^{d'_n} + \sum A'_j(t,x')x_n^{d'_n-j}$, where G_n is given by Definition 4.2. By the assumption on G_n, $|A'_j(t,x')| \le C_2\|x'\|^j$, for all $j=0,...,d'_n-1$, and hence these roots satisfy $|a_i(t,x')| \le C_3\|x'\|$. The latter bound, by the inductive assumption, is equivalent to

$$|a_i(t,\Psi'(t,x'))| \le C_4\|x'\|. \tag{4.3}$$

By formula (3.5), $\Psi_n(t,x) := \psi(x_n, a(0,x'), a(\Phi_{n-1}(t,x')))$ and $\psi(a_i(0,x'), a(0,x'), a(\Phi_{n-1}(t,x'))) = a_i(t,\Psi'(t,x'))$ and therefore by the Lipschitz property of Whitney Interpolation, Proposition I.3, we get

$$C_5^{-1}|x_n - a_i(0,x')| \le |\Psi_n(t,x) - a_i(t,\Psi'(t,x'))| \le C_5|x_n - a_i(0,x')|. \tag{4.4}$$

By (4.3) and (4.4)

$$|\Psi_n(t,x)| \le C_6(|x_n - a_i(0,x')| + |a_i(t,\Psi'(t,x'))|) \le C_7\|(x',x_n)\|$$

that shows the second inequality in (4.1). The proof of the first one is similar.

This ends the proof of Theorem 4.3. □

Example 4.4. Let $\mathcal{G}_n = \{G_{n,i}(t,x)\}$ be a finite family of monic pseudopolynomials in x_n. We say that $\mathcal{G}$ *is stable by derivation* if for every $G \in \mathcal{G}_n$, either $\partial G/\partial x_n \equiv 0$ or $\partial G/\partial x_n \in \mathcal{G}_n$ (after multiplication by a non-zero constant). We say that a pseudopolynomial F_n is *derivation complete* if it is the product of a stable by derivation family. We call a system of pseudopolynomials $\{F_i\}$ *derivation complete* if so is every $\{F_i\}$.

Suppose now that the system $\{F_i\}$ is derivation complete and let F_i be the product of a stable by derivation family $\mathcal{G}_i = \{G_{i,j}\}$. If $F_i(0,0)=0$ then there is $G \in \mathcal{G}_i$, such that $G(0,0)=0$, $\partial G/\partial x_i(0,0) \neq 0$. The Weierstrass polynomial associated to G is of degree 1 in x_i and divides F_i. Hence the family $\{F_i\}$ is partially transverse.

5. Canonical stratification associated to a system of pseudopolynomials.

In this section we extend the results of the last two sections to a more global situation.

Definition 5.1. By a *system of pseudopolynomials in* $x = (x_1, ..., x_n) \in \mathbb{K}^n$ we mean a family

$$F_i(x_1, \ldots, x_i) = x_i^{d_i} + \sum_{j=1}^{d_i} A_{i,j}(x_1, \ldots, x_{i-1}) x_i^{d_i - j}, \quad i = 1, \ldots, n, \tag{5.1}$$

with $\mathbb{K}$-analytic coefficients $A_{i,j}$, satisfying

(1) there are $\varepsilon_j \in (0, \infty]$, $j = 1, \ldots, n$, such that every F_i are defined on $U_i = \prod_{j=1}^{i} D_j$, where $D_j = \{|x_j| < \varepsilon_j\}$.
(2) if $\varepsilon_i < \infty$ then F_i does not vanish on $U_{i-1} \times \partial D_i$, where $\partial D_i = \{|x_i| = \varepsilon_i\}$.
(3) for every i, the discriminant of $F_{i,red}$ divides F_{i-1}.

It may happen that $d_i = 0$. Then $F_i \equiv 1$ and we set by convention $F_j \equiv 1$ for $j < i$.

We say that $\{F_i\}$ is *a system of polynomials* if every F_i is a polynomial.

For $i < k$ we denote by $\pi_{k,i} : U_k \to U_i$ the standard projection. For each i we define a filtration

$$U_i = X_i^i \supset X_{i-1}^i \supset \cdots \supset X_0^i, \tag{5.2}$$

where

(1) $X_0^1 = V(F_1)$. It may be empty.
(2) $X_j^i = (\pi_{i,i-1}^{-1}(X_j^{i-1}) \cap V(F_i)) \cup \pi_{i,i-1}^{-1}(X_{j-1}^{i-1})$ for $1 \le j < i$.

As we show below every connected component S of $X_j^i \setminus X_{j-1}^i$ is a locally closed j-dimensional $\mathbb{K}$-analytic submanifold of U_i and hence (5.2) defines an analytic stratification $\mathcal{S}_i$ of U_i, see Section 7 for the definition. We call $\mathcal{S} = \mathcal{S}_n$ *the canonical stratification associated to a system of pseudopolynomials.*

Proposition 5.2. *For all $j \le i \le n$ every connected component S of $X_j^i \setminus X_{j-1}^i$ is a locally closed j-dimensional $\mathbb{K}$-analytic submanifold of U_i of one of the following two types:*

(I) *$S \subset V(F_i)$ and there is a connected component S' of $X_j^{i-1} \setminus X_{j-1}^{i-1}$ such that $\pi_{i,i-1}$ induces a finite $\mathbb{K}$-analytic covering $S \to S'$.*
(II) *There is a connected component S'' of $X_{j-1}^{i-1} \setminus X_{j-2}^{i-1}$ such that S is a connected component of $\pi_{i,i-1}^{-1}(S'') \setminus V(F_i)$.*

Moreover, for every $p \in S$ there are a local system of coordinates at p in which $(S, p) = (\mathbb{K}^j, 0)$, neighborhoods B, Ω_0 and Ω of p in $\mathbb{K}^j$, $\mathbb{K}^{i-j}$, and $\mathbb{K}^i$ resp., and an arc-wise analytic trivialization

$$\Phi : B \times \Omega_0 \to \Omega$$

preserving the strata of stratification $\mathcal{S}_i$ and such that F_i is Φ-regular. If the system $\{F_i\}$ is derivation complete in the sense of Example 4.4 then the trivialization Φ can be chosen regular along $B \times \{p\}$.

Proof. Induction on n. Let S' be a stratum of $\mathcal{S}_{n-1}$ of dimension j and let $p' \in S'$. By the inductive assumption there are a local system of coordinates $y_1, ..., y_{n-1}$ at p' in which $(S', p') = (\mathbb{K}^j, 0)$, neighborhoods B', Ω'_0 and Ω' of p' in $\mathbb{K}^j$, $\mathbb{K}^{n-1-j}$, and $\mathbb{K}^{n-1}$ resp., and an arc-wise analytic trivialization

$$\Phi' : B' \times \Omega'_0 \to \Omega'$$

preserving $\mathcal{S}_{n-1}$, such that F_{n-1} is Φ'-regular. Since the discriminant of $F_{n,red}$ divides F_{n-1} it is also Φ'-regular. Therefore, by Lemma 2.11, the restriction of projection $\pi_{n,n-1}$

$$\pi^{-1}_{n,n-1}(B') \cap V(F_n) \to B'$$

is a finite analytic covering. This shows that the connected components of $\pi^{-1}_{n,n-1}(S') \cap V(F_n)$ and of $\pi^{-1}_{n,n-1}(S') \setminus V(F_n)$ are locally closed submanifolds of Ω_n of type (I) or (II).

Let S be a connected component of $\pi^{-1}_{n,n-1}(S') \setminus V(F_n)$ and let $p \in S'$ be such that $p' = \pi_{n,n-1}(p)$. Then $\mathcal{S}_n$ near p is the product of $\mathcal{S}_{n-1} \times \mathbb{K}$. Therefore the conclusion follows from the inductive assumption and the fact that $F_n(p) \neq 0$.

If S is a connected component of $\pi^{-1}_{n,n-1}(S') \cap V(F_n)$ we show that Φ' can be lifted to an arc-wise analytic trivialization

$$\Phi : B \times \Omega'_0 \times \mathbb{K} \to \Omega' \times \mathbb{K},$$

so that Φ preserves $\mathcal{S}_n$ and F_n is Φ-regular. This can be done exactly as in the proof of Theorem 3.3 as follows. Denote by $a(y) = (a_1(y), ..., a_{d_n}(y))$ the vector of complex roots of F_n and set

$$\Phi(y, x_n) = (\Phi'(y), \psi(x_n, a(0, y_{j+1}, ..., y_{n-1}), a(\Phi'(y))),$$

where ψ is given by the Whitney interpolation formula (I.13). The last claim of Proposition follows from Example 4.4 and Theorem 4.3. □

5.1. **Φ of the proof of Theorem 3.3 satisfies condition (5) of Definition 1.2.** Let $\mathcal{S}, \mathcal{S}_0$ be the canonical stratifications associated to the families $\{F_i(t,x)\}$ and $\{F_i(0,x)\}$ respectively. We show that Φ induces a real analytic diffeomorphism between the strata of $B_\varepsilon \times \mathcal{S}_0$ and $\mathcal{S}$. By induction on n we may suppose that the corresponding property holds for Φ_{n-1}. Let S be a stratum of $\mathcal{S}$ of type (I), that is a covering space over a stratum S'. Denote $S_0 = S \cap \{t = 0\}$, $S'_0 = S' \cap \{t = 0\}$. By construction Φ restricted to $B_\varepsilon \times V(F_n(0, x))$ is a lift of Φ_{n-1}. Therefore, if $\Phi_{n-1} : B_\varepsilon \times S'_0 \to S'$ is an analytic diffeomorphism, consequently so is its lift $\Phi : B_\varepsilon \times S_0 \to S$.

Now suppose that S is of type (II). By assumption, $a_i(t, x')$ of (3.5) are real analytic on $B_\varepsilon \times S''_0$ and hence, by the Whitney interpolation formula (I.13), so is Ψ_n on $B_\varepsilon \times S'_0$. This shows the claim. □

Remark 5.3. In general for an (arc-a) or (arc-w) stratification, we have to substratify to obtain the condition (5) of Definition 1.2. For the canonical stratification associated to a system of pseudopolynomials the arc-wise analytic trivializations constructed in the proof of Proposition 5.2 are real analytic on its strata.

Part 3. Applications.

6. Generic arc-wise analytic equisingularity

We use the Zariski equisingularity to show that an analytic family of analytic set germs $\mathcal{X} = \{X_t\}$, $t \in T$, is "generically" equisingular. That is, locally on the parameter space T, this family is equisingular in the complement of an analytic subset $Z \subset T$, $\dim Z < \dim T$. In this section the parameter space T may be singular.

Definition 6.1. Let T be a $\mathbb{K}$-analytic space, $U \subset \mathbb{K}^n$ an open neighborhood of the origin, $\pi : T \times U \to T$ the standard projection, and let $\mathcal{X} = \{X_k\}$ be a finite family of analytic subsets of $T \times U$. We say that $\mathcal{X}$ is *arc-wise analytically equisingular along* $T \times \{0\}$ *at* $t \in Reg(T)$, if there are neighborhoods B of t in $Reg(T)$ and Ω of $(t, 0)$ in $T \times \mathbb{K}^n$, and an arc-wise analytic trivialization $\Phi : B \times \Omega_t \to \Omega$, where $\Omega_t = \Omega \cap \pi^{-1}(t)$, such that $\Phi(B \times \{0\}) = B \times \{0\}$ and for every k, $\Phi(T \times X_{k,t}) = X_k$, where $X_{k,t} = X_k \cap \pi^{-1}(t)$.

We say that $\mathcal{X}$ is *regularly arc-wise analytically equisingular along* $T \times \{0\}$ *at* $t \in T$ if, moreover, Φ is regular at $(t, 0)$.

Theorem 6.2. *Let $\mathcal{X} = \{X_k\}$ be a finite family of analytic subsets of a neighborhood of $T \times U$ and let $t_0 \in T$. Then there exist an open neighborhood T' of t_0 in T and a proper $\mathbb{K}$-analytic subset $Z \subset T'$, containing $Sing(T')$, such that for every $t \in T' \setminus Z$, $\mathcal{X}$ is regularly arc-wise analytically equisingular along $T \times \{0\}$ at t.*

Moreover, there is an analytic stratification of an open neighborhood of t_0 in T such that for every stratum S and every $t \in S$, $\mathcal{X}$ is regularly arc-wise analytic equisingular along $S \times \{0\}$ at t.

Proof. For each X_k fix a finite system of generators $F_{k,i} \in \mathcal{O}_{T,t_0}$ of the ideal defining it. The first claim follows from Lemma 6.3 applied to the product of all $F_{k,i}$. The second claim follows by induction on $\dim T$.

Lemma 6.3. *Let T be a $\mathbb{K}$-analytic space, $t_0 \in T$. Let F be a $\mathbb{K}$-analytic function defined in a neighborhood of $(t_0, 0) \in T \times \mathbb{K}^n$. Then there exist a neighborhood T' of t_0 in T and a proper $\mathbb{K}$-analytic subset $Z \subset T'$, $\dim Z < \dim T$, $Sing(T) \subset Z$, such that, after a linear change of coordinates in $\mathbb{K}^n$, the following holds. For every $t \in T' \setminus Z$ there is a Zariski equisingular local transverse system of pseudopolynomials F_i, $i = 0, ..., n$, at $(t, 0)$, with F_n being the Weierstrass polynomial associated to F at $(t, 0)$.*

Proof. We may suppose that T is a subspace of $\mathbb{K}^m$, $t_0 = 0$, and $(T, 0)$ is irreducible.

We construct a new system of coordinates $x_1, ..., x_n$ on $\mathbb{K}^n$, analytic subspaces $(Z_i, 0) \subset (T, 0)$ and analytic function germs $G_i(t, x_1, ..., x_i)$, $i = n, n-1, ..., 0$, such that for every $t \in T \setminus Z$, $Z = Sing(T) \cup \bigcup Z_i$, the following condition is satisfied. Let F_i be the Weierstrass polynomial in x_i associated to the germ of G_i at $(t, 0)$. Then the discriminant of $F_{i,red}$ divides F_{i-1}.

The G_i are constructed by descending induction. First we set $G_n = F$. Then we construct G_{n-1} in three steps.

Step 1. Write

$$G_n(t,x) = \sum_{|\alpha|\geq m_0} A_\alpha(t)x^\alpha,$$

where m_0 is the minimal integer $|\alpha|$ for which $A_\alpha \not\equiv 0$. We may assume $m_0 > 0$ otherwise we simply take $Z = Sing(T)$. After a linear change of x-coordinates, we may assume $A_{(0,\dots,0,m_0)}(t) \not\equiv 0$. Denote $A(t) = A_{(0,\dots,0,m_0)}$.

Step 2. We define $A(t) * x := (A(t)^2x_1, \dots., A(t)^2x_{n-1}, A(t)x_n)$ and set

$$\tilde G_n(t,x) = (A(t))^{-(m_0+1)}G_n(t, A(t)*x) = \sum_{|\alpha|\geq m_0} \tilde A_\alpha(t)x^\alpha.$$

Then $\tilde A_{(0,\dots,0,m_0)} \equiv 1$ and $\tilde G_n$ is regular in x_n.

Step 3. Denote by H_n the Weierstrass polynomial in x_n associated to $\tilde G_n$. It is of degree m_0 in x_n. Let $\mathcal{K}$ be the field of fractions of $\mathcal{O}_{T\times\mathbb{K}^{n-1},0}$ and consider H_n as a polynomial of $\mathcal{K}[x_n]$. Let d be the degree of $H_{n,red}$. We define G_{n-1} as the dth generalized discriminant of H_n, see Appendix II, and set $Z_n = A^{-1}(0)$.

Then we repeat these steps for G_{n-1} and so on.

To see that the sequence G_i satisfies the required properties we note that if F_n denotes the Weierstrass polynomial at $(t,0) \in T\setminus (Z_n \cup Sing(T))$ associated to G_n, then, as a germ at $(t,0)$, the discriminant of $F_{n,red}$ divides G_{n-1}.

This ends the proof of Lemma 6.3 and Theorem 6.2. □

7. Stratifications and Whitney Fibering Conjecture.

Let X be a $\mathbb{K}$-analytic space of dimension n. By *an analytic stratification of* X we mean a filtration of X by analytic subspaces

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0$$

such that each $X_j \setminus X_{j-1}$ is a nonsingular (locally closed) analytic subspace of pure dimension j, or is empty. This filtration induces a decomposition $X = \sqcup S_i$, where the S_i are connected components of all $X_j \setminus X_{j-1}$. The analytic locally closed submanifolds S_i of X are called *strata* and their collection $\mathcal{S} = \{S_i\}$ is usually called a stratification of X. In what follows we simply say that $\mathcal{S} = \{S_i\}$ is an analytic stratification of X, meaning that it comes from an analytic filtration. Similarly we define an algebraic stratification of an algebraic variety.

Stratifications are often considered with extra regularity conditions such as Whitney's conditions (a) and (b) or the (w) condition of Verdier. For more details and insight we refer the reader to [69], [70], [67], [15], [64], [17], [16] and the references therein. Recall that for a real analytic stratification the (w) condition implies the conditions (a) and (b), see [64]. For a complex analytic stratification the conditions (w) and (b) are equivalent [59].

We say that a stratification $\mathcal{S} = \{S_i\}$ is *compatible with* $Y \subset X$ if Y is a union of strata.

7.1. **(Arc-a) and (arc-w) stratifications.** Let X be a $\mathbb{K}$-analytic space and let $\mathcal{S}$ be an analytic stratification of X. Let p be a point of a stratum $S \in \mathcal{S}$. We say that $\mathcal{S}$ is *arc-wise analytically trivial at* p, or *satisfies the condition (arc-a) at* p, if the following holds. There are a neighborhood Ω of p, $\mathbb{K}$-analytic coordinates on Ω such that $B = S \cap \Omega$ is a neighborhood of the origin in $\mathbb{K}^m \times \{0\}$, and an arc-wise analytic trivialization of the projection π on the first m coordinates

$$\Phi(t, x) : B \times \Omega_0 \to \Omega, \tag{7.1}$$

where $\Omega_0 = \Omega \cap \pi^{-1}(0)$, such that $\Phi(B \times \{0\}) = B$ and Φ preserves the stratification. By the last condition we mean that each stratum of $\mathcal{S}$ is the union of leaves of Φ, see Section 1. We say, moreover, that $\mathcal{S}$ is *regularly arc-wise analytically trivial at* p, or *satisfies the condition (arc-w) at* p, if Φ of (7.1) is regular along $B \times \{0\}$ in the sense of Definition 1.5.

We say that $\mathcal{S}$ is *arc-wise analytically trivial*, or *satisfies the condition (arc-a)*, if it does it at every point of X. Similarly we define *regularly arc-wise analytically trivial stratifications*. If $\mathcal{S}$ is regularly arc-wise analytically trivial then, for short, we say that $\mathcal{S}$ *satisfies the condition (arc-w)*.

We say that *the condition (arc-a), resp. (arc-w), is satisfied along a stratum* S if it is satisfied at every $p \in S$. Similarly we say that *the condition (a) or (w) is satisfied along a stratum* S if for every other stratum S', $S \subset \overline{S}'$, the pair S', S satisfies the respective condition.

Theorem 7.1. *If a stratification $\mathcal{S}$ satisfies the condition (arc-a), resp. the condition (arc-w), along a stratum S then it satisfies the condition (a) of Whitney, resp. the condition (w) of Verdier along S.*

Proof. The first claim follows from the continuity of the tangent spaces to the leaves of an arc-wise analytic trivialization, see Proposition 1.3.

We show the second claim. Fix two strata $S_0 \subset \overline{S}_1$. If the condition (w) fails for the pair S_1, S_0 at $p_0 \in S_0$, then, by the curve selection lemma, it fails along a real analytic arc $p(s) : [0, \varepsilon) \to \overline{S}_1$ with $p_0 = p(0) \in S_0$ and $p(s) \in S_1$ for $s > 0$. We show that there is a C^1 submanifold $(M, \partial M) \subset (\overline{S}_1, S_0)$, $\partial M = S_0$ near p_0, $p(s) \in M$, such that $M \setminus \partial M, \partial M$ satisfies the condition (w). It then follows, see for instance [11], that the condition (w) is satisfied along $p(s)$ for the pair S_1, S_0, which contradicts the choice of $p(s)$ and hence completes the proof.

We define M using the trivialization $\Phi(t, x) = (t, \Psi(x))$ of (7.1). By the arc-analyticity of Φ^{-1} there is a real analytic arc $(t(s), x(s))$ such that $p(s) = \Phi(t(s), x(s))$. Then we set

$$M = \{\Phi(t, x(s)); t \in B, s \geq 0\}. \tag{7.2}$$

It follows from Proposition 1.6 that M is a C^1-manifold with boundary and that $M \setminus \partial M, \partial M$ satisfies the condition (w). □

Corollary 7.2 ([57])**.** *If a complex analytic hypersurface X is generic Zariski equisingular along a nonsingular subspace $Y \subset Sing(X)$ then the pair $Reg(X), Y$ satisfies Whitney's conditions (a) and (b).*

In the above proof of Theorem 7.1 we use the Wing Lemma argument, the manifold M being the wing. This method was introduced by Whitney in [69] and then was used by many authors to show the existence of stratifications satisfying various regularity conditions, see for instance [67], [11], [4]. For a wing $(M, \partial M)$ the condition of being a C^1 submanifold is not sufficient to guarantee the condition (w) for the pair $M \setminus \partial M, \partial M$, see [9] for examples. Thus it is essential that the wing M admit a parameterization (7.2) satisfying (1.5) of Proposition 1.6. Moreover, we show below the existence of a wing that admits a $\mathbb{K}$-analytic parameterization and contains a given real analytic arc.

Proposition 7.3 (Wing Lemma). *Let $\mathcal{S}$ be an (arc-a) stratification of a $\mathbb{K}$-analytic space X. Let $p(s) : [0, \varepsilon) \to X$ be a real analytic arc such that $p_0 = p(0) \in S_0$ and $p(s) \in S_1$ for $s > 0$ and a pair of strata S_0, S_1. Then, there are a local system of coordinates at p_0, $(X, p_0) \subset (\mathbb{K}^N, 0)$, an open neighborhood Ω of p_0 in $\mathbb{K}^N$ such that $B = S_0 \cap \Omega$ is a neighborhood of p_0 in $\mathbb{K}^m \times \{0\}$, a neighborhood D of 0 in $\mathbb{K}$, and $\mathbb{K}$-analytic maps*

$$t(s) : D \to B, \quad \varphi : B \times D \to X, \quad \varphi(t, s) = (t, \psi(t, s)),$$

such that $\varphi(t, 0) = (t, 0) \in S_0$, $p(s) = \varphi(t(s), s)$ for $s > 0$, and

- *$\varphi(t, s) \in S_1$ for $s > 0$ if $\mathbb{K} = \mathbb{R}$*
- *$\varphi(t, s) \in S_1$ for $s \neq 0$ if $\mathbb{K} = \mathbb{C}$.*

Moreover if $\mathcal{S}$ is an (arc-w) stratification and if we write $\psi(t, s) = \sum_{k \geq k_0} D_k(t) s^k$, then we may require that $D_{k_0}(0) \neq 0$.

Proof. The real case follows from the definition of an arc-wise analytic trivialization, and in the regular case from Proposition 1.6. In the complex case we may construct a complex wing as follows. Let $\Phi(t, x)$ be the arc-wise analytic trivialization given in (7.1) and let $p(s) = \Phi(t(s), x(s)) : (I, 0) \to (\overline{S}_1, p_0)$. Then $\Phi(t, x(s))$ as a power series defines a complex analytic map $\varphi : (T \times \mathbb{C}, 0) \to (\mathbb{C}^N, 0)$. Thus $\varphi(t, s) = \Phi(t, x(s))$ for s real, but not necessarily for $s \in \mathbb{C} \setminus \mathbb{R}$. Because Φ preserves the strata, the stratum S_1 contains the image of φ for $s > 0$. Since $\overline{S}_1$ is a complex analytic set, it contains the entire image of φ. By assumption $\varphi(t, s) \notin S_0$ for $s \in \mathbb{R}, s > 0$. Therefore, by Lemma 1.12, $\varphi(t, s) \notin S_0$ for $s \in \mathbb{C} \setminus \{0\}$ as claimed. This ends the proof. $\square$

7.2. **Local Isotopy Lemma.** Let X be a Whitney stratified space, $p \in X$, and let S be the stratum containing p. Then, as follows from Thom's first isotopy lemma, [39], [60], [15], any local submersion onto S, restricted to X, can be trivialized over a neighborhood of p in S. As it follows from Proposition 7.4 below an analogous property holds for (arc-a) and (arc-w) stratifications.

Suppose that X is a $\mathbb{K}$-analytic subspace of a neighborhood of the origin in $\mathbb{K}^N$, Ω a neighborhood of the origin in X, and let $B = \Omega \cap (\mathbb{K}^m \times \{0\})$. Let f be a $\mathbb{K}$-analytic function on X and let $\mathcal{X} = \{X_k\}$ be a finite family of analytic subsets of X. Let $\pi : \Omega \to B$ denote the standard projection onto the first m coordinates.

Proposition 7.4. *Let Φ be an arc-wise analytic trivialization of π that preserves B and a family of analytic subsets $\mathcal{X} = \{X_k\}$ of X and let f be a Φ-regular function germ. Let $\tilde{\pi}$ be another analytic submersion $\Omega \to B$. Then, after restricting to a smaller neighborhood of the origin, there is an arc-wise analytic trivialization $\tilde{\Phi}$ of $\tilde{\pi}$, preserving B and the family*

$\mathcal{X}$, and such that f is $\tilde\Phi$-regular. Moreover, if Φ is regular along B then $\tilde\Phi$ can be chosen regular along B.

Proof. Let $H : B \times \Omega_0 \to B \times \Omega_0$ be given by

$$H(t, x) = (h(t, x), x) = (\tilde\pi(\Phi(t, x)), x).$$

We show that H is a local homeomorphism, arc-wise analytic in t, such that H^{-1} is also arc-wise analytic in t. Then $\tilde\Phi = \Phi \circ H^{-1}$ satisfies the claim.

Firstly H is a local homomorphism by the implicit function theorem, Theorem 2.5 Ch. I of [23]. Let $\gamma := x(s)$ be a real analytic arc. Consider

$$H_\gamma(t, s) = (h(t, x(s)), s) : (B \times I, 0) \to (B \times I, 0).$$

H_γ is clearly $\mathbb{K}$-analytic in t and real analytic in s. Since $h(t, x(s)) = t + \varphi(t, s)$ with $\varphi(t, s) \in \mathfrak{m}_{t,s}$, H_γ is a local analytic diffeomorphism and its inverse is $\mathbb{K}$-analytic in t and real analytic in s. $\square$

Let $\mathcal{S}$ be an analytic stratification of X satisfying Whitney's condition (a). One says after Définition 4.1.1 of [6], that $\mathcal{S}$ satisfies *the stratified local triviality condition*, the (TLS) condition for short, if any local submersion onto a stratum is locally topologically trivial by a strata preserving trivialization. Thus Lemma 7.4 gives the following result.

Corollary 7.5. *A stratification satisfying the condition (arc-a) also satisfies the condition (TLS) of* [6].

Proof. If we assume in Proposition 7.4 the $\tilde\pi$ is only a C^1-submersion then, by Theorem 2.5 Ch. I of [23], H constructed in the proof is a homeomorphism and so is $\tilde\Phi$. $\square$

7.3. **Proof of Whitney fibering conjecture.** We show below that every $\mathbb{K}$-analytic space admits locally an (arc-w) stratification. In the algebraic case such stratification exists globally. Since an (arc-w) stratification satisfies all the properties required by Whitney, it shows Whitney's fibering conjecture in the algebraic and local analytic cases.

Theorem 7.6. *Let $\mathcal{V} = \{V_i\}$ be a finite family of analytic subsets of an open $U \subset \mathbb{K}^N$. Let $p_0 \in U$. Then there exist an open neighborhood U' of p_0 and an analytic stratification of U' compatible with each $U' \cap V_i$ and satisfying the condition (arc-w).*

First proof of Theorem 7.6. We construct a system of pseudopolynomials $F_i(x_1, \ldots, x_n)$, see Section 5, in a system of local coordinates at p_0, so that the canonical stratification associated to $\{F_i\}$ is compatible with $\mathcal{V}$. Since, by construction, this system will be derivation complete, the theorem follows from Proposition 5.2.

First for every analytic space V_i choose a finite system of generators of its ideal $I(V_i) = (g_{i,j})_{j=1,\ldots n_i}$ in the local ring $\mathcal{O}_{p_0}$, and let $f_n = \prod_{i,j} g_{i,j}$. After changing linearly the system of coordinates, if necessary, we may assume that f_n is regular in x_n and then we replace it by the associated Weierstrass polynomial. Let F_n be the product of all (non-zero) partial derivatives $\partial/\partial x_n$ of f_n. After a multiplication by a non-zero constant, we may assume that F_n is monic in x_n.

Then define $f_{n-1}(x_1, \ldots, x_{n-1})$ as the discriminant of $f_{n,red}$ (or an appropriate higher order discriminant, see Appendix II). After a linear change of coordinates $x_1, ..., x_{n-1}$ we may assume that f_{n-1} is regular in x_{n-1} and then replace it by the associated Weierstrass polynomials. Let F_{n-1} be the product of all (non-zero) partial derivatives $\partial/\partial x_{n-1}$ of f_{n-1}. We continue this construction and thus define the system F_i. If it happens that f_i is a non-zero constant we define F_i and all $F_j, j < i$, as identically equal to one. Then, if ε_i are chosen so that $0 < \varepsilon_1 \ll \cdots \ll \varepsilon_n \ll 1$, this system satisfies the requirements of Definition 5.1. □

We now give a second proof of Theorem 7.6. It is less algorithmic but provides a stratification with local transverse Zariski equisingularitiy. This proof is based on the following lemma.

Lemma 7.7. *Let F be a $\mathbb{K}$-analytic function defined in a neighborhood of $0 \in \mathbb{K}^N$ and let Y be a $\mathbb{K}$-analytic subset of a neighborhood of $0 \in \mathbb{K}^N$, $\dim Y = m$. Then there exist a neighborhood U of $0 \in \mathbb{K}^N$ and a $\mathbb{K}$-analytic subset $Z \subset Y \cap U$, $\dim Z < m$, $Sing(Y) \subset Z$, such that for every $p \in Y \cap U \setminus Z$, there are a local system of coordinates at p in which $(Y, p) = (\mathbb{K}^m \times \{0\}, 0)$ and a Zariski equisingular transverse system pseudopolynomials F_i, $i = 0, ..., n = N - m$, at p, such that F_n is the Weierstrass polynomial associated to F.*

Proof. Choose a local system of coordinates at p such that the projection on the first m coordinates restricted to Y is finite. Let

$$\varphi : Y \times \mathbb{K}^n \to \mathbb{K}^N, \qquad \varphi(y, x) = y + (0, x).$$

Then apply Lemma 6.3 to $T = Y$ and $F(\varphi(t, x))$. □

Second proof of Theorem 7.6. We construct a sequence of analytic set germs at p_0

$$U = X_n \supset X_{n-1} \supset \cdots \supset X_0$$

whose representatives in a sufficiently small neighborhood U' of p_0 define a stratification satisfying the statement. For simplicity of notation we assume p_0 to be the origin.

First for each analytic space V_i choose a finite system of generators of its ideal $I(V_i) = (g_{i,j})_{j=1,...n_i}$ in the local ring $\mathcal{O}_0$, and let f_n be the product of all of them: $f_n = \prod_{i,j} g_{i,j}$. In the first step we apply Lemma 7.7 to $F = f_n$ and $Y = U$ and we set $X_{n-1} = Z$. If $\dim(X_{n-1}, 0) < n - 1$ then we set $X_{n-2} = X_{n-1}$. Otherwise we again apply Lemma 7.7 to $F = f_n$ and $Y = X_{n-1}$ and we set X_{n-2} equal to the obtained Z.

If $\dim(X_{n-2}, 0) < n - 2$ then we set $X_{n-3} = X_{n-2}$. Otherwise choose a finite system of generators $I(X_{n-1}) = (h_{n-1,j})$ and let $f_{n-1} = f_n \prod_j h_{n-1,j}$. Next apply Lemma 7.7 to $F = f_{n-1}$ and $Y = X_{n-2}$ and we set X_{n-3} equal to the obtained Z.

The inductive step is then the following. Given $U = X_n \supset X_{n-1} \supset \cdots \supset X_i$ and a function f_{i+1} that is the product of f_{i+2} and a finite set of generators of $I(X_{i+1})$ in $\mathcal{O}_0$. If $\dim(X_i, 0) < i$ then we set $X_{i-1} = X_i$. Otherwise we apply Lemma 7.7 to $F = f_{i+1}$ and $Y = X_i$ and we set X_{i-1} equal to the obtained Z.

Let $p \in X_k \setminus X_{k-1}$. Then by construction there is a local system of coordinates at p in which $(X_k, p) = (\mathbb{K}^k, 0)$ and an arc-wise analytic trivialization Φ of the coordinate projection on $\mathbb{K}^k$, preserving X_k and such that f_{k+1} is Φ-regular. Therefore, Φ preserves the zero set of every factor of f_{k+1} and hence every V_i and every X_j for $j > k$. This ends the proof. □

7.4. **Remark on Whitney fibering conjecture in the complex case.** Let U be a neighborhood of $0 \in \mathbb{C}^m \times \mathbb{C}^n$. Set $M = U \cap (\mathbb{C}^m \times \{0\})$ and $N = \{0\} \times \mathbb{C}^n$. Suppose, following Whitney, that there exists a homeomorphism

$$\phi(p,q) : M \times N \to U,$$

complex analytic in p, such that $\phi(p,0) = p$ and $\phi(0,q) = q$, and that for each $q \in N$ fixed, $\phi(\cdot, q) : M \times \{q\} \to U$ is a complex analytic embedding onto an analytic submanifold $L(q)$. Now we make an additional assumption:

(A) *for all $q \in N$, $L(q)$ is transverse to N.*

By continuity of $\phi(p,q)$ we may assume that the projection of $L(q)$ onto M is proper. Therefore, by (A) and the assumption $\phi(0,q) = q$, is has to be of degree 1. Therefore $L(q)$ is the graph of a complex analytic function $f_q : M \to \mathbb{C}^n$. If $q \to 0$ then the values of f_q go to 0 and hence, by Cauchy integral formula, the partial derivatives of f_q go to 0 on relatively compact subsets of M. This ensures the continuity of the tangent spaces to the leaves $L(q)$ as $q \to 0$.

7.5. **Examples.** There are several classical examples describing the relation between the Zariski equisingularity and Whitney's conditions that we recall below. The general set-up for these examples is the following. Consider a complex algebraic hypersurface $X \subset \mathbb{C}^4$ defined by a polynomial $F(x,y,z,t) = 0$ such that $Sing X = T$, where T is the t-axis. Let $\pi : \mathbb{C}^4 \to T$ be the standard projection. In all these examples $X_t = \pi^{-1}(t)$, $t \in T$, is a family of isolated singularities, topologically trivial along T. These examples relate the following conditions :

(1) X is Zariski equisingular along T, i.e. there is a local system of coordinates in which F can be completed to a Zariski equisingular system of polynomials, see Definition 3.1.
(2) X is Zariski equisingular along T for a transverse coordinate system, i.e. there is a local system of coordinates in which F can be completed to a Zariski equisingular transverse system of polynomials, Section 4.1.
(3) X is Zariski equisingular along T for a generic system of coordinates, i.e. for generic system of local coordinates, F can be completed to a Zariski equisingular system of polynomials
(4) The pair $(X \setminus T, T)$ satisfies Whitney's conditions (a) and (b).

Clearly (3)⇒(2)⇒(1). Speder showed (3)⇒(4) in [57] and (2)⇒(4) for families of complex analytic hypersurfaces with isolated singularities in $\mathbb{C}^3$ in his thesis [58] (not published). Theorem 7.1 gives (2)⇒(4) in the general case. As the examples below show, all the other implications are false.

Example 7.8 ([7]).

$$F(x,y,z,t) = z^5 + ty^6z + y^7x + x^{15} \tag{7.3}$$

This example satisfies (1) for the projections $(x,y,z) \to (y,z) \to x$ but (4) fails. As follows from Theorem 7.1, (2) fails as well.

Example 7.9 ([8]).

$$F(x, y, z, t) = z^3 + tx^4 z + y^6 + x^6 \tag{7.4}$$

In this example (4) is satisfied and (3) fails. This example satisfies (1) for the projections $(x, y, z) \to (x, z) \to x$.

Example 7.10 ([38]).

$$F(x, y, z, t) = z^{16} + tyz^3 x^7 + y^6 z^4 + y^{10} + x^{10} \tag{7.5}$$

In this example (2) is satisfied and (3) fails.

Example 7.11 ([50]).

$$F(x, y, z, t) = x^9 + y^{12} + z^{15} + tx^3 y^4 z^5 \tag{7.6}$$

In this example (4) is satisfied and (1) fails. This shows also that (4) does not imply (2).

8. Equisingularity of functions

In this section we show how to use Zariski's equisingularity to obtain local topological triviality of analytic function germs. We develop several different approaches.

Firstly we show that the assumptions of Theorem 3.3 gives not only the topological equisingularity of sets, but also of the function F_n and of any analytic function dividing F_n. To prove it we modify the vector fields defined by the arc-wise analytic trivialization Φ, so that their flows trivialize F_n. Note that this new trivialization is no longer arc-wise analytic.

Then, for an analytic function f, we introduce new stratifying conditions (arc-a$_f$) and (arc-w$_f$), analogs of conditions (arc-a) and (arc-w), and show that they imply the classical stratifying conditions (a$_f$) and (w$_f$) respectively.

Finally we show how to adapt the Zariski equisingularity to the graph of a function f in order to obtain an arc-wise analytic triviality of f.

8.1. Zariski equisingularity implies topological triviality of the defining function.

We show that the assumptions of Theorem 3.3 give not only the topological triviality of the zero set of F_n but also of F_n as a function.

Theorem 8.1. *Let B, Ω_0 and Ω be neighborhoods of the origin in $\mathbb{K}^m$, $\mathbb{K}^n$, and $\mathbb{K}^{m+n}$ respectively, and let $\Phi : B \times \Omega_0 \to \Omega$ be an arc-wise analytic trivialization satisfying the condition (Z1) of Theorem 3.3. Let $f(t, x)$ be a $\mathbb{K}$-analytic Φ-regular function. Then f is topologically trivial along $B \times \{0\}$ at the origin, i.e. there are smaller neighborhoods B', Ω'_0 and Ω' and a homeomorphism*

$$h : B' \times \Omega'_0 \to \Omega'$$

such that $h(t, 0) = (t, 0)$, $h(0, x) = (0, x)$, and $f(h(t, x)) = f(0, x)$.

Proof. The trivialization h is obtained by integrating the vector fields $w_i(t, x)$ defined below. Let v_i be the vector fields on Ω given by (1.2). The regularity condition (1.8) gives

$$|\partial f / \partial v_i| \leq C|f|. \tag{8.1}$$

Note that locally on $\Omega \setminus V(f)$ we may approximate v_i by a C^∞ vector field satisfying (8.1). Using partition of unity, we may glue such local approximations to a smooth vector field that satisfies (8.1) and extends continuously to $V(f)$ by $v_i|_{V(f)}$. In what follows we replace v_i by such approximation.

Next we consider on $\Omega \setminus V(f)$ the orthogonal projection of $v_i(t,x)$ on the levels of f

$$w_i(t,x) = v_i(t,x) - \frac{\partial f/\partial v_i}{\| \operatorname{grad} f\|^2} \operatorname{grad} f.$$

(in the complex case $\operatorname{grad} f := \overline{(\partial f/\partial z_1, ..., \partial f/\partial z_{m+n})}$ so that $\partial f/\partial v = \langle v, \operatorname{grad} f\rangle$). Then we extend w_i by $v_i(t,x)$ onto Ω. Clearly $\partial f/\partial w_i = 0$ and w_i are continuous by (8.1) and Łojasiewicz Gradient Inequality, [37], that says that there are constants $C > 0$, $\theta < 1$, such that

$$\| \operatorname{grad} f\| \ge C|f|^\theta.$$

in a neighborhood of the origin. By Remark 1.4 the integral curves of $w_i|_{V(f)}$ are unique and hence they are unique on Ω. Therefore, by Theorem 2.1 of [23], for each i the flow h_i of w_i is continuous. Then we trivialize f by composing these flows:

$$h(t_1, ..., t_m, x) = h_1(t_1, h_2(t_2, h_3(...(t_{m-1}, h_m(t_m, x))...))).$$

$\square$

8.2. **Conditions (arc-a$_f$) and (arc-w$_f$).** Let $f : X \to \mathbb{K}$ be a $\mathbb{K}$-analytic function defined on a $\mathbb{K}$-analytic space X. By *a stratification of f* we mean an analytic stratification $\mathcal{S}$ of X such that $V(f)$ is a union of strata. We also assume that for any stratum $S \subset X \setminus V(f)$, $f|_S$ has no critical points. A stratification $\mathcal{S}$ of f is called *a Thom stratification of f* if it is a Whitney stratification of X that for each pair of strata satisfies Thom's condition (a_f). For a definition of condition (a_f) we refer the reader to [60], [39], [15], [35], [18]. For the strict Thom condition (w_f) see [35] and [24].

We say that a stratification $\mathcal{S}$ of f *satisfies the condition (arc-a_f) at $p \in V(f)$* if there exists a local arc-wise analytic trivialization (7.1) at p preserving the strata of $\mathcal{S}$ and such that f is Φ-regular at p. If, moreover, Φ is regular at p then we say that $\mathcal{S}$ *satisfies the condition (arc-w_f) at p.*

We say that *the condition (arc-a_f), resp. (arc-w_f), is satisfied along a stratum S* if it is satisfied at every $p \in S$. Similarly we say that *the condition (a_f) or (w_f) is satisfied along S* if for every other stratum S', $S \subset \overline{S}'$, the pair S', S satisfies the respective condition.

We note that by the assumption that $f|_S$ has no critical points on stratum $S \subset X \setminus V(f)$, the levels of f are transverse to S. Therefore, if moreover $\mathcal{S}$ satisfies Whitney's condition (a), the conditions (a_f) and (w_f) are automatically satisfied along such S.

Theorem 8.2. *If a stratification of f satisfies the condition (arc-a_f), resp. (arc-w_f), along a stratum $S \subset V(f)$ then it satisfies the Thom condition (a_f), resp. (w_f), along S.*

Proof. Similarly to the proof of theorem 7.1 it suffices we check the conditions along a real analytic curve by considering a wing containing the curve.

Thus fix two strata $S_0 \subset \overline{S}_1$, $S_0 \subset V(f)$, $S_1 \cap V(f) = \emptyset$, and a real analytic curve $\gamma : p(s) = \Phi(t(s), x(s)) : [0, \varepsilon) \to \overline{S}_1$ with $p_0 = p(0) \in S_0$ and $p(s) \in S_1$ for $s > 0$. First we consider the case $\mathbb{K} = \mathbb{R}$ and the wing

$$M = \{\Phi(t, x(s)); t \in B, s \ge 0\}.$$

By the regularity of f for Φ, Proposition 1.7, we may reparametrize $\Phi_\gamma(t,s) = \Phi(t,x(s))$ by replacing $s = s(t,\tilde s)$ so that $f(\Phi_\gamma(t,\tilde s)) = \tilde s^{k_0}$. If we write $\Phi_\gamma(t,\tilde s)) = (t, \Psi_\gamma(t,\tilde s)))$ then the tangent space to the levels of $f_{|M}$ is generated by $D\Phi_\gamma(\partial/\partial t_i, \partial\Psi_\gamma/\partial t_i)$, that tends to $(\partial/\partial t_i, 0)$ as $s \to 0$, $i = 1, ..., m$. This shows (a_f). If moreover Φ is regular then the condition (w_f) follows form Proposition 1.6.

If $\mathbb{K} = \mathbb{C}$, then we use the complex wing of Proposition 7.3. $\square$

Corollary 8.3. *Let $f : X \to \mathbb{K}$ be $\mathbb{K}$-analytic and let $\mathcal{S}$ be a Whitney stratification of f satisfying the condition (arc-a$_f$). Then f is topologically trivial along each stratum $S \subset V(f)$.*

In the complex analytic case it is shown in [6] that any stratification of f satisfying the conditions (a) and (TLS) also satisfies the condition (a$_f$). Similarly, after [6] and [52] any Whitney stratification of f satisfies the strong Thom condition (w_f). Analogous results are false in the real analytic case. Thus in the complex case Theorem 8.2 (for the stratification and not for a single stratum) follows from Theorem 7.1, Proposition 1.10, and Corollary 7.5.

Thom's condition (a$_f$) implies the topological triviality of f along the strata of a Whitney stratification. But the condition (a$_f$) alone does not imply Whitney's condition (b) and therefore it may not imply topological triviality of f along the strata. Similarly, the condition (arc-a$_f$) alone may not itself imply topological triviality of f along the strata. Nevertheless, in some special cases, the topological triviality can be obtained by adapting the proof of Theorem 8.1.

Corollary 8.4. *Let $f : X \to \mathbb{K}$ be $\mathbb{K}$-analytic and let $\mathcal{S}$ be a stratification of f such that $X \setminus V(f)$ is a stratum of $\mathcal{S}$. Suppose that $\mathcal{S}$ satisfies the condition (arc-a$_f$) along a stratum $S \subset V(f)$. Then f is topologically trivial along S.*

8.3. **Arc-wise analytic triviality of function germs.** Consider a family of function germs $f_t(y) = f(t,y) : T \times (\mathbb{K}^{n-1}, 0) \to \mathbb{K}$, parametrized by an open $T \subset \mathbb{K}^m$. We say that f_t is *arc-wise analytically trivial along* T if there are neighborhoods Λ of $T \times \{0\}$ in $\mathbb{K}^m \times \mathbb{K}^{n-1}$ and Λ_0 of $\{0\}$ in $\mathbb{K}^{n-1}$, $f_0 : \Lambda_0 \to \mathbb{K}$, and an arc-wise analytic analytic trivialization

$$\sigma : T \times \Lambda_0 \to \Lambda, \text{ such that } f(\sigma(t,y)) = f_0(y).$$

Using the method developed in [3] we have the following result.

Theorem 8.5. *Let $f_t(y) = f(t,y) : T \times (\mathbb{K}^{n-1}, 0) \to \mathbb{K}$ be a $\mathbb{K}$-analytic family of $\mathbb{K}$-analytic function germs and let $t_0 \in T$. Then, there exist a neighborhood U of t_0 in T and a $\mathbb{K}$-analytic subset $Z \subset U$, $\dim Z < \dim T$, such that f is arc-wise analytically trivial along $U \setminus Z$.*

Proof. Set $x = (x_1, ..., x_n) = (x_1, y)$, where $y = (y_1, ..., y_{n-1})$, and $F(t, x_1, y) = x_1 - f(t,y)$. By Lemma 6.3 and Theorem 3.3, together with Proposition 3.7, there is an arc-wise analytic trivialization Φ of the zero set of F that preserves the levels of x_1. Then

$$\sigma(t,y) = \pi(\Phi(t, f_0(y), y)),$$

where π is the projection $\pi(t, x_1, y) = (t, y)$, gives an arc-wise analytic trivialization of f. $\square$

9. Algebraic case

9.1. **Construction of arc-wise analytically trivial stratifications.** Given a polynomial $F \in \mathbb{K}[x_1, ..., x_n]$ we may construct a system of polynomials $F_i \in \mathbb{K}[x_1, \ldots, x_i]$, $i = 1, ..., n$, as follows. First we set $F_n = F$ that after a linear change of coordinates we may assume monic in x_n. Then let F_{n-1} be the discriminant of $F_{n,red}$ or an appropriate higher order discriminant, see Appendix II. We again make a linear change of coordinates $x_1, ..., x_{n-1}$ so that we may assume F_{n-1} monic in x_{n-1} and we continue until we get F_j a non-zero constant. This construction is algorithmic except taking generic system of coordinates. For such a system of polynomials $F_i \in \mathbb{K}[x_1, \ldots, x_i]$, $i = 1, ..., n$, we may consider the canonical stratification defined in Section 5. Moreover, we may refine this construction to obtain a derivation complete system of polynomials. Then Proposition 5.2 gives the following.

Theorem 9.1. *Given $F \in \mathbb{K}[x_1, \ldots, x_n]$. There exists a linear system of coordinates $x_1, ..., x_n$ on $\mathbb{K}^n$ and a derivation complete system of polynomials on $\{F_i(x_1, ..., x_i)\}$ such that F divides F_n. In particular the associated canonical stratification to this system satisfies the condition (arc-w) and the condition (arc-w$_f$) for any factor of F.*

Theorem 9.1 gives Whitney's Fibering Conjecture in the affine algebraic case. Since the above constructions preserves the family of homogeneous polynomials we obtain as well an algorithmic proof of the following projective algebraic version of Whitney's Fibering Conjecture.

Theorem 9.2. *Let $\mathcal{V} = \{V_i\}$ be a finite family of algebraic subsets of $\mathbb{P}^n_{\mathbb{K}}$. Then there exists an algebraic stratification of $\mathbb{P}^n_{\mathbb{K}}$ compatible with each V_i and satisfying the condition (arc-w). Moreover, the local arc-wise analytic trivializations can be chosen semi-algebraic.*

Different proof of Theorems 9.1, 9.2, that gives local arc-wise analytic trivialization by Zariski equisingular local transverse system of polynomials follows from Lemma 9.5.

9.2. **Generic arc-wise analytic equisingularity in the algebraic case.** In the algebraic case we have a global version of Theorem 6.2. Here by a *real algebraic variety* we mean an affine real algebraic variety in the sense of Bochnak-Coste-Roy [4]: a topological space with a sheaf of real-valued functions isomorphic to a real algebraic set $X \subset \mathbb{R}^N$ with the Zariski topology and the structure sheaf of regular rational functions. For instance, the set of real points of a reduced projective scheme over $\mathbb{R}$, with the sheaf of regular functions, is a real algebraic variety in this sense.

Theorem 9.3. *Let T be an algebraic variety (over $\mathbb{K}$) and let $\mathcal{X} = \{X_k\}$ be a finite family of algebraic subsets $T \times \mathbb{P}^{n-1}_{\mathbb{K}}$. Then there exists an algebraic stratification $\mathcal{S}$ of T such that for every stratum S and for every $t_0 \in S$ there is a neighborhood U of t_0 in S and a semialgebraic arc-wise analytic trivialization of π, preserving the family $\mathcal{X}$,*

$$\Phi : U \times \mathbb{P}^{n-1}_{\mathbb{K}} \to \pi^{-1}(U), \tag{9.1}$$

$\Phi(t_0, x) = (t_0, x)$, where $\pi : T \times \mathbb{P}^{n-1}_{\mathbb{K}} \to T$ denotes the projection.

Proof. We may assume that T is affine irreducible. Let $G_i(t, x)$, $t \in T$, $x = (x_1, \ldots, x_n)$, be a finite family of polynomials, homogeneous in x, defining the sets X_k and let $F_n(t, x)$ be the product of all G_i. We consider F_n as a homogeneous polynomial over $\mathcal{K} = \mathbb{K}(T)$ and let

$$F_n(x) = \sum_{|\alpha|=d_n} A_\alpha x^\alpha, \quad A_\alpha \in \mathcal{K}.$$

After a linear change of coordinates x, we may suppose $A_n = A_{(0,...,0,d_n)} \neq 0$. Then we define $F_{n-1}(x_1, ..., x_{n-1})$ as the discriminant of $F_{n,red}$, and proceed inductively by constructing the system of homogeneous polynomials $F_j \in \mathcal{K}[x_1, ..., x_i]$ until $F_i \in \mathcal{K}$ is a non-zero constant. Then we take as $Z \subset T$ the union of zero sets of the denominators of the coefficients of all F_j and the numerators of the leading coefficients of all F_j. We show below that the statement of theorem holds for $T \setminus Z$ as an open stratum. Then the stratification $\mathcal{S}$ can be constructed by induction on $\dim T$.

By Theorem 3.3, $V(F_n)$ is arc-wise analytically equisingular along $(T \setminus Z) \times \{0\}$. By construction (3.5), the trivialisation $\Phi(t, x) = (t, \Psi(t, x))$ is semi-algebraic and $\mathbb{K}^*$-equivariant in the variable x, as follows from the interpolation formula, see Remark I.2. Moreover, by construction, it is regular along $U \times \{0\}$, U being a neighborhood of t_0 in $T \setminus Z$. Then the trivialization $U \times \mathbb{P}^{n-1}_{\mathbb{K}} \to \pi^{-1}(U)$ induced by Φ, is arc-wise analytic. □

We have the following versions of Lemmas 6.3 and 7.7.

Lemma 9.4. *Let T be a $\mathbb{K}$ algebraic variety and let $F \in \mathbb{K}[T \times \mathbb{K}^n]$, $F \not\equiv 0$. Then there exists a subvariety $Z \subset T$, $\dim Z < \dim T$, such that, after a linear change of coordinates in $\mathbb{K}^n$, F can be completed to a system of polynomials $\{F_i\}$, $F_n = F$, such that for every $t \in T \setminus Z$ the system $\{F_i\}$ is transverse and Zariski equisingular at $(t, 0)$.* □

Lemma 9.5. *Let $F \in \mathbb{K}[X_1, ..., X_N]$, $F \not\equiv 0$, and let $Y \subset \mathbb{K}^N$ be an algebraic subset. Then there exist an algebraic $Z \subset Y$, $\dim Z < \dim Y$, and polynomials $\{F_i\}$, $F_n = F$, such that the following holds. For every $p \in Y \setminus Z$ there is a local system of coordinates at p in which $(Y, p) = (\mathbb{K}^m \times \{0\}, 0)$, such that the germs of $\{F_i\}$ at p form a transverse and Zariski equisingular system of polynomials.* □

9.3. **Applications to real algebraic geometry.** Let X be a compact (projective or affine) real algebraic variety in the sense of [4]. A functorial filtration on the semi-algebraic chains $C_*(X; \mathbb{Z}_2)$ was introduced in [41]. This filtration, called *the Nash filtrtation*, defines a spectral sequence, *the weight spectral sequence of* X, that, in turn, defines *the weight filtration* on the homology $H_*(X; \mathbb{Z}_2)$. This construction can be extended to non-compact real algebraic varieties and the Borel-Moore homology. For a real algebraic variety X its virtual Poincaré polynomial $\beta(X) \in \mathbb{Z}[t]$, introduced in [40], is a multiplicative and additive invariant, an analog of the Hodge-Deligne polynomial. As shown in [41], the virtual Poincaré polynomial can be computed from the weight spectral sequence. For the cohomological counterpart of this theory see [36].

The Nash filtration is functorial not only for regular morphisms but also for the $\mathcal{AS}$-maps that can be defined as follows. Let X, Y be compact real algebraic varieties. A continuous map $f : X \to Y$ is *an $\mathcal{AS}$-map* if its graph Γ_f is a semialgebraic and arc-symmetric subset of $X \times Y$. For instance a map that is semialgebraic and arc-analytic is $\mathcal{AS}$. For more on $\mathcal{AS}$ maps see [54], [34].

Let Φ be a semialgebraic arc-wise analytic trivialization (9.1) preserving real algebraic $X \subset T \times \mathbb{P}^{n-1}_{\mathbb{K}}$ and let $X_t = \pi^{-1}(t)$. Then for each $t \in U$, Φ induces a semialgebraic and arc-analytic homeomorphism

$$\varphi_{t_0,t} : X_{t_0} \to X_t,$$

with an arc-analytic inverse. In particular, each $\varphi_{t_0,t}$ is $\mathcal{AS}$. Thus Theorem 9.3 gives the following.

Corollary 9.6. *Let T be a real algebraic variety and let X be an algebraic subset of $T \times \mathbb{P}^{n-1}_{\mathbb{K}}$. Then there exists an algebraic stratification $\mathcal{S}$ of T such that for every stratum S and for every $t_0, t_1 \in S$ the fibres X_{t_0} and X_{t_1} are $\mathcal{AS}$-homeomorphic and hence have isomorphic weight spectral sequences and weight filtration on the homology with $\mathbb{Z}_2$ coefficients.* □

Corollary 9.7. *Let T be a real algebraic variety and let X be an algebraic subset of $T \times \mathbb{P}^{n-1}_{\mathbb{K}}$. Then there exists an algebraic stratification $\mathcal{S}$ of T such that for every stratum S the virtual Poincaré polynomial $\beta(X_t)$ is independent of $t \in S$.* □

The latter result was also shown in [12] by means of the resolution of singularities.

Appendix I. Whitney Interpolation.

We generalize the classical Whitney Interpolation formula [70], [22].

Fix positive integers d, N and consider a family of functions $f_i : \mathbb{C}^N \to \mathbb{C}$, $i = 1, 2, ..., N$. We assume that, for a constant $C > 1$, this family satisfies the following properties

(1) f_i are continuous, differentiable on $(\mathbb{C}^*)^N$, and satisfies $f_i(\lambda\xi) = |\lambda|^d f_i(\xi)$ for all $\lambda \in \mathbb{C}$.
(2) for every permutation $\sigma \in S_N$: $f_i(\xi_{\sigma(1)}, ..., \xi_{\sigma(N)}) = f_{\sigma(i)}(\xi_1, ..., \xi_N)$.
(3) $|f_j(\xi_1, \dots, \xi_N)| \le C|\xi_j|(max_i|\xi_i|)^{d-1}$.
(4) for all k, j, $|\xi_k^2|(|\partial f_j/\partial \xi_k| + |\partial f_j/\partial \bar{\xi_k}|) \le C|\xi_j|(max_i|\xi_i|)^d$.
(5) $f = \sum_i f_i$ is real valued and satisfies $C^{-1}(max_i|\xi_i|)^d \le f(\xi_1, \dots, \xi_N) \le C(max_i|\xi_i|)^d$.

For examples see Examples I.6 and I.7.

Given two subsets $\{a_1, \dots, a_N\} \subset \mathbb{C}$, $\{b_1, \dots, b_N\} \subset \mathbb{C}$, of cardinality N such that if $a_i = a_j$ then $b_i = b_j$. Define $D_i = b_i - a_i$ and set

$$\gamma = \max_{a_i \ne a_j} \frac{|D_i - D_j|}{|a_i - a_j|}. \tag{I.1}$$

Then

$$|D_i - D_j| \le \gamma |a_i - a_j|. \tag{I.2}$$

Let

$$\mu_i(z) := f_i((z-a_1)^{-1}, \dots, (z-a_N)^{-1}), \quad \mu(z) := f((z-a_1)^{-1}, \dots, (z-a_N)^{-1}).$$

Define the interpolation map $\psi : \mathbb{C} \to \mathbb{C}$ by

$$\psi(z) = z + \frac{\sum_{i=1}^N \mu_i(z) D_i}{\mu(z)}, \tag{I.3}$$

if $z \notin \{a_1, \ldots, a_N\}$ and $\psi(a_i) = b_i$. Then ψ is continuous as follows from the following lemma.

Lemma I.1.

$$\lim_{z \to a_j} \psi(z) = b_j.$$

Proof. Let $I_j = \{i; a_i = a_j\}$. We rewrite the interpolation formula (I.3) as

$$\psi(z) = z + D_j + \frac{\sum_{i \notin I_j} \mu_i(z)(D_i - D_j)}{\mu(z)}. \tag{I.4}$$

By the properties (3) and (5), for $i \notin I_j$, $\frac{\mu_i(z)}{\mu(z)} \to 0$ as $z \to a_j$. □

Remark I.2. *Symmetries.*
The map ψ is also invariant under permutations $\sigma \in S_N$, $\sigma(a) = (a_{\sigma(1)}, \ldots, a_{\sigma(N)})$

$$\psi(z, \sigma(a), \sigma(b)) = \psi(z, a, b).$$

Let $\tau : \mathbb{C} \to \mathbb{C}$ be complex affine, $\tau(z) = \alpha z + \beta$. Then

$$\psi(\tau(z), \tau(a), \tau(b)) = \tau(\psi(z, a, b)).$$

Proposition I.3. *The map $\psi : \mathbb{C} \to \mathbb{C}$ is Lipschitz with Lipschitz constant $4N^3C^4\gamma + 1$. If $\gamma < (4N^3C^4)^{-1}$ then ψ is a bi-Lipschitz homeomorphism, with $(1 - 4N^3C^4\gamma)^{-1}$ a Lipschitz constant of ψ^{-1}.*

Proof. It suffices to show that for $z \notin \{a_1, \ldots, a_N\}$ and for every unit vector $v \in \mathbb{C}$

$$|(\psi(z) - z)'| \le 4N^3C^4\gamma, \tag{I.5}$$

where by "prime" we denote any directional derivative $\frac{\partial}{\partial v}$, $v \in \mathbb{C}$, $|v| = 1$. Indeed, if (I.5) holds then clearly ψ is Lipschitz. Moreover, if $\gamma < (4N^3C^4)^{-1}$ then for any $p \in \mathbb{C}$, $z \to p + z - \psi(z)$ is a contraction and hence admits a unique fixed point z_p, that is a unique z_p such that $\psi(z_p) = p$. Hence ψ is a homeomorphism by the invariance of domain. By (I.5) for any $p, q \in \mathbb{C}$, $|(\psi(p) - p) - (\psi(q) - q)| \le 4N^3C^4\gamma|p - q|$, that gives

$$|p - q| \le (1 - 4N^3C^4\gamma)^{-1}|\psi(p) - (\psi(q)| \tag{I.6}$$

if $\gamma < (4N^3C^4)^{-1}$.

To show (I.5) we use the following bounds that follow from the conditions (3)-(5).

$$\begin{aligned} |\mu_i(z)| &\le C^{2-\frac{1}{d}}|z - a_i|^{-1}\mu(z)^{1-\frac{1}{d}}, \\ |\mu_i'(z)| &\le NC^2|z - a_i|^{-1}\mu(z), \\ |\mu'(z)| &\le N^2C^{2+\frac{1}{d}}\mu(z)^{\frac{d+1}{d}}, \end{aligned} \tag{I.7}$$

By (3) we have $|\mu_i(z)| \le C|\xi_i|(max_j|\xi_j|)^{d-1})$, thus by 5) we have $|\mu_i(z)| \le C^{1+\frac{d-1}{d}}|\xi_i|\mu(z)^{\frac{d-1}{d}}$ i.e. the first inequality. We present now a detailed proof of the second inequality. By the chain rule

$$|\frac{\partial \mu_i}{\partial z}| + |\frac{\partial \mu_i}{\partial \bar z}| \le \sum_k (|\frac{\partial f_i}{\partial \xi_k}\frac{\partial \xi_k}{\partial z}| + |\frac{\partial f_i}{\partial \bar\xi_k}\frac{\partial \bar\xi_k}{\partial \bar z}|) = \sum_k (|\partial f_i/\partial \xi_k| + |\partial f_i/\partial \bar\xi_k|)|\xi_k|^2.$$

Therefore by (4) and (5)
$$|\frac{\partial \mu_i}{\partial z}| + |\frac{\partial \mu_i}{\partial \bar z}| \le C \sum |\xi_i| (max_j |\xi_j|)^d \le C^2 N |\xi_i| \mu.$$
Now for a unit vector $v = a + bi \in \mathbb{C}$, $a^2 + b^2 = 1$,
$$|a\frac{\partial \mu_i}{\partial x} + ib\frac{\partial \mu_i}{\partial y}| = |(a+ib)\frac{\partial \mu_i}{\partial z} + (a-ib)\frac{\partial \mu_i}{\partial \bar z}| \le C^2 N |\xi_i| \mu.$$
as required. Using this inequality we get
$$|\mu'(z)| \le \sum_i |\mu_i'(z)| \le NC^2 \mu \sum_i |\xi_i| \le N^2 C^2 \mu (max_j |\xi_j|)$$
which by (5) gives $|\mu'(z)| \le N^2 C^{2+\frac{1}{d}} \mu^{1+\frac{1}{d}}$.

Given $z \in \mathbb{C}$, choose j such that $|z - a_j| = \min_i |z - a_i|$. Then, for all i,
$$|a_i - a_j| \le 2|z - a_i|. \tag{I.8}$$
By differentiating (I.4)
$$|(\psi(z) - z)'| \le \frac{\sum_{i \notin I_j} |\mu_i'(z)(D_i - D_j)|}{\mu(z)} + \frac{(\sum_{i \notin I_j} |\mu_i(z)(D_i - D_j)|)|\mu'(z)|}{(\mu(z))^2}. \tag{I.9}$$
By (I.2) and (I.7)
$$|\mu_i'(z)(D_i - D_j)| \le 2NC^2 \gamma \mu(z)$$
and
$$|\mu_i(z)(D_i - D_j)||\mu'(z)| \le 2N^2 C^4 \gamma (\mu(z))^2.$$
This shows (I.5) and hence ends the proof of Proposition I.3. □

Consider ψ as a function defined for $(z, a, b) \in \mathbb{C} \times \Sigma$, where $\Sigma = \{(a, b) \in \mathbb{C}^N \times \mathbb{C}^N;$ such that if $a_i = a_j$ then $b_i = b_j\}$. Thus
$$\psi(z, a, b) = \psi_{a,b}(z) = z + \frac{\sum_{j=1}^N \mu_j(z, a)(b_j - a_j)}{\mu(z, a)}, \tag{I.10}$$
where $\mu_i(z, a) = f_i((z - a_1)^{-1}, \dots, (z - a_N)^{-1})$, $\mu(z, a) = \sum_i \mu_i(z, a)$, and $\psi_{a,b}(a_i) = b_i$. We may also consider $\psi(z, a, b)$ as a family of functions $\psi_{a,b} : \mathbb{C} \to \mathbb{C}$, parameterized by a, b.

Proposition I.4. *Let $a(x) : X \to \mathbb{C}^N, b(x) : X \to \mathbb{C}^N$ be continuous functions defined on a topological space X such that for every $x \in X$ and i, j, if $a_i(x) = a_j(x)$ then $b_i(x) = b_j(x)$. Then $\psi(z, a(x), b(x))$ is continuous as a function of (x, z).*

Proof. Let $(z, a, b) \to (z_0, a_0, b_0)$. Clearly $\psi(z, a, b) \to \psi(z_0, a_0, b_0)$ if $z_0 \notin \{a_{01}, \dots, a_{0N}\}$. Thus suppose $z_0 = a_{01}$ and then $\psi(z_0, a_0, b_0) = b_{01}$. Let $J = \{j \in \{1, ..., n\}; a_{0j} = a_{01}\}$. Then
$$\begin{aligned}\psi(z, a, b) - \psi(z_0, a_0, b_0) = (z - z_0) &+ \frac{\sum_{i \in J} \mu_i(z, a)((b_i - b_{01}) - (a_i - a_{01}))}{\mu(z, a)} \\ &+ \frac{\sum_{i \notin J} \mu_i(z, a)((b_i - b_{01}) - (a_i - a_{01}))}{\mu(z, a)}.\end{aligned}$$

We show that the last two summands converge to 0 as $(z,a,b) \to (z_0,a_0,b_0)$. Note that $b_{01} = b_{0i}, a_{01} = a_{0i}$ if $i \in J$. Therefore

$$\frac{\sum_{i\in J} \mu_i(z,a)((b_i - b_{01}) - (a_i - a_{01}))}{\mu(z,a)} = \sum_{i\in J} \frac{\mu_i(z,a)}{\mu(z,a)}((b_i - b_{0i}) - (a_i - a_{0i})).$$

By (3) and (5) we always have that $|\frac{\mu_i}{\mu}| \le C^2$, and using the fact that $((b_i - b_{0i}) - (a_i - a_{0i})) \to 0$ we get the second summand goes to zero. So does the third one because

$$\frac{\mu_i(z,a)}{\mu(z,a)} \to 0$$

if $i \notin J$. To show this last property we note that $\mu(z,a) \to \infty$, the limit of $z - a_i$ is nonzero if $i \notin J$, and use the first inequality of (I.7). □

Remark I.5. If $(a,b) \to (a_0,b_0)$ then $\gamma(a_0,b_0) \le \liminf \gamma(a,b)$, thus γ is lower semi-continuous, where formally we put $\gamma(a,b) = 0$ if $a_1 = \cdots = a_N$, $b_1 = \cdots = b_N$.

Example I.6. In the original Whitney interpolation $f_i(\xi) = |\xi_i|$, cf. [70], see also [22].

Example I.7. In this paper we use the following family. For $\xi_1, \ldots, \xi_N \in \mathbb{C}$ we denote by $\sigma_i = \sigma_i(\xi_1, \ldots, \xi_N)$ the elementary symmetric functions of $\xi_1, \ldots, \xi_N$. Let $P_k = \sigma_k^{\alpha_k}$, where $\alpha_k = (N!)/k$. Define

$$f_j(\xi) = \frac{1}{N!} \sum_k \xi_j \frac{\partial P_k}{\partial \xi_j} \bar{P}_k. \tag{I.11}$$

and therefore it follows that

$$f(\xi) = \sum f_j(\xi) = \sum_k P_k(\xi)\bar{P}_k(\xi). \tag{I.12}$$

Then ψ equals

$$\psi(z,a,b) = z + \frac{\sum_k \left(\sum_{j=1}^N \ \xi_j \frac{\partial P_k}{\partial \xi_j}(\xi)(b_j - a_j)\right) \bar{P}_k(\xi)}{N!(\sum_k P_k \bar{P}_k(\xi))}, \tag{I.13}$$

where $\xi = ((z-a_1)^{-1}, \ldots, (z-a_N)^{-1})$.

Appendix II. Generalized discriminants

We recall below the classical generalized discriminants, see e.g. [71] Appendix IV. Let $\mathcal{K}$ be a field of characteristic zero and let

$$F(Z) = Z^p + \sum_{j=1}^p A_i Z^{p-i} = \prod_{j=1}^p (Z - \xi_i) \in \mathcal{K}[Z], \tag{II.1}$$

with the roots $\xi_i \in \overline{\mathcal{K}}$. Then the expressions

$$D_j = \sum_{r_1 < \cdots < r_j} \ \prod_{k<l;\, k,l \in \{r_1,\ldots,r_j\}} (\xi_k - \xi_l)^2$$

are symmetric in $\xi_1, \dots, \xi_p$ and hence polynomials in $A_1, \dots, A_p$. Thus D_p is the standard discriminant and F has exactly d distinct roots if and only if $D_{d+1} = \cdots = D_p = 0$ and $D_d \neq 0$. The following lemma is obvious.

Lemma II.1. *Let $F \in \mathcal{K}[Z]$ be a monic polynomial of degree p that has exactly d distinct roots in $\xi_i \in \overline{\mathcal{K}}$ of multiplicities $\mathbf{m} = (m_1, ..., m_d)$. Then there is a positive constant $C = C_{p,\mathbf{m}}$ such that the generalized discriminant $D_{d,F}$ of F and the standard discriminant $\Delta_{F_{red}}$ of F_{red} are related by*

$$D_{d,F} = C\Delta_{F_{red}}.$$

We often use the following consequence of the Implicit Function Theorem.

Lemma II.2. *Let $F \in \mathbb{K}\{x_1, ..., x_n\}[Z]$ be a monic polynomial in Z such that the discriminant $\Delta_{F_{red}}$ does not vanish at the origin. Then, on a neighborhood U of $0 \in \mathbb{K}^n$, the complex roots $\xi_i(x_1, ..., x_n)$ of F are $\mathbb{K}$-analytic, distinct, and of constant multiplicities.*

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