HIGHER ORDER APPROXIMATION OF ANALYTIC SETS BY TOPOLOGICALLY EQUIVALENT ALGEBRAIC SETS

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Abstract. It is known that every germ of an analytic set is homeomorphic to the germ of an algebraic set. In this paper we show that the homeomorphism can be chosen in such a way that the analytic and algebraic germs are tangent with any prescribed order of tangency. Moreover, the space of arcs contained in the algebraic germ approximates the space of arcs contained in the analytic one, in the sense that they are identical up to a prescribed truncation order.

1. Introduction

The problem of approximation of analytic sets (or functions) by algebraic ones is one of the most fundamental problems in singularity theory and an old subject of investigation (see e.g. [2], [4], [7], [15], [20], [24], [29]). In general this problem has been considered by two different approaches.

Firstly one may seek to approximate the germs of analytic sets by the germs of algebraic ones so that both objects are homeomorphic. For instance, it is well-known that a germ of analytic set with an isolated singularity is analytically equivalent to a germ algebraic set (see [13] for a general account of this case). But in general an analytic set is not even locally diffeomorphically equivalent to an algebraic one (cf. [29]). Nevertheless by a result of T. Mostowski [15] every analytic set is locally topologically equivalent to an algebraic one.

Theorem 1.1 ([15], [4])**.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(X, 0) \subset (\mathbb{K}^n, 0)$ be an analytic germ. Then there is a homeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $h(X)$ is the germ of an algebraic subset of $\mathbb{K}^n$.*

Secondly one may seek to approximate analytic germs by the algebraic ones with the higher order tangency, cf. e.g. [5], [6], [9], [10], [11]. But the classical methods of such approximation do not provide the objects which are homeomorphic. In this paper we show how to construct approximations that satisfy both requirements that is approximate with homeomorphic objects and with a given order of tangency.

Theorem 1.2. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(X, 0) \subset (\mathbb{K}^n, 0)$ be an analytic germ. Then there are $C, c > 0$ and an open neighborhood U of 0 in $\mathbb{K}^n$ such that for every $m \in \mathbf{N}$ there are a subanalytic and arc-analytic homeomorphism $\varphi_m : U \to \varphi_m(U) \subset \mathbb{K}^n$*

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and an algebraic subset V_m of $\mathbb{K}^n$ with the following properties:

(a) $\varphi_m(X \cap U) = V_m \cap \varphi_m(U)$,
(b) $||\varphi_m(a) - a|| \leq C^m ||a||^m$ *for every* $a \in U$.

Moreover, there is a nowhere dense analytic subset $Z \subset U$ independent of m such that φ_m is real analytic on $U \setminus Z$ and such that the jacobian determinant of φ_m on $U \setminus Z$ satisfies

(c) $c \leq |jacdet(\varphi_m)(x)| \leq C$.

In the above statement, and throughout the paper, "analytic" means "complex analytic" if $\mathbb{K} = \mathbb{C}$ and "real analytic" if $\mathbb{K} = \mathbb{R}$, unless we say explicitly "real analytic" or "complex analytic".

The proof of Theorem 1.2 is based on Mostowski's approach, its recent refinement [4], and a new result on Zariski equisingularity given in [16]. This proof will be divided into two parts. Firstly, using Popescu Approximation Theorem and a strong version of Varchenko's Theorem given in [16], we show that there is an arc-analytic homeomorphism ψ such that $\psi(X)$ is a Nash set tangent to X at 0 with any prescribed order of tangency (cf. Proposition 3.4, Section 3). Next, using Artin-Mazur's Theorem of [3] (cf. also [7]), we prove that there is a Nash analytic diffeomorphism θ such that $\theta(\psi(X))$ is an algebraic set tangent to $\psi(X)$ (cf. Proposition 4.1, Section 4).

In Section 5 we give an application of Theorem 1.2 to the space of analytic arcs on a given germ of analytic set. Namely, we show that for the space of truncated arcs there is an algebraic germ, homeomorphic to the original one, with the identical space of truncated arcs. We shall use the following notation. For any analytic germ $(Z, 0) \subset (\mathbb{K}^n, 0)$, let $\mathcal{A}_m^{\mathbb{R}}(Z)$ denote the space of all truncations up to order m of real analytic arcs contained in $(Z, 0)$. For a complex analytic germ $(Z, 0)$, $\mathcal{A}_m^{\mathbb{C}}(Z)$ denotes the space of all truncations up to order m of complex analytic arcs contained in $(Z, 0)$.

Theorem 1.3. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(X, 0) \subset (\mathbb{K}^n, 0)$ be a germ of real (resp. complex) analytic set. Then for every $m \in \mathbf{N}$, there is an algebraic germ $(V_m, 0) \subset (\mathbb{K}^n, 0)$ with the same embedded topological type as $(X, 0)$ such that $\mathcal{A}_m^{\mathbb{R}}(X) = \mathcal{A}_m^{\mathbb{R}}(V_m)$ if $\mathbb{K} = \mathbb{R}$, and $\mathcal{A}_m^{\mathbb{C}}(X) = \mathcal{A}_m^{\mathbb{C}}(V_m)$, $\mathcal{A}_m^{\mathbb{R}}(X) = \mathcal{A}_m^{\mathbb{R}}(V_m)$ if $\mathbb{K} = \mathbb{C}$.*

We also present an example showing that it is not enough in general to choose for V_m a germ of algebraic set which has a high tangency with X.

Convention. The constants denoted C, c may change from line to line. We also use auxiliary constants $C_1, C_2, etc.$. They also are not fixed and may depend from line to line. Sometimes we write $C(m)$ to stress that this particular constant may depend on m.

2. Preliminaries

2.1. **Nash functions and sets.** Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Let Ω be an open subset of $\mathbb{K}^q$ and let f be an analytic function on Ω. We say that f is a *Nash function* at $\zeta \in \Omega$ if there exist an open neighbourhood U of ζ in Ω and a non zero polynomial $P \in \mathbb{K}[Z_1, \ldots, Z_q, W]$ such that $P(z, f(z)) = 0$ for $z \in U$. An analytic function on Ω is a Nash function if it is a Nash function at every point of Ω. An analytic mapping $\varphi : \Omega \to \mathbb{K}^N$ is a *Nash mapping* if each of its components is a Nash function on Ω.

A subset X of Ω is called a *Nash subset* of Ω if for every $\zeta \in \Omega$ there exist an open neighbourhood U of ζ in Ω and Nash functions $f_1, \ldots, f_s$ on U, such that $X \cap U = \{z \in U \ ; \ f_1(z) = \cdots = f_s(z) = 0\}$. A germ X_ζ of a set X at $\zeta \in \Omega$ is a *Nash germ* if there exists an open neighbourhood U of ζ in Ω such that $X \cap U$ is a Nash subset of U. Note that X_ζ is a Nash germ if its defining ideal can be generated by convergent power series algebraic over the ring $\mathbb{K}[Z_1, \ldots, Z_q]$. For more details on real and complex Nash functions and sets see [8], [25].

2.2. **Nested Artin-Płoski-Popescu Approximation Theorem.** We set $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$. The ring of convergent power series in $x_1, \ldots, x_n$ is denoted by $\mathbb{K}\{x\}$. If A is a commutative ring then the ring of algebraic power series with coefficients in A is denoted by $A\langle x\rangle$.

The following result of [4] is a generalization of Płoski's result [17]. It is a corollary of Theorem 11.4 [22] which itself is a corollary of the Néron-Popescu Desingularization (see [18], [22] or [23] for the proof of this desingularization theorem in whole generality or [19] for a proof in characteristic zero).

Theorem 2.1. [4] *Let $f(x, y) \in \mathbb{K}\langle x\rangle[y]^p$ and let us consider a solution $y(x) \in \mathbb{K}\{x\}^m$ of*

$$f(x, y(x)) = 0.$$

Let us assume that $y_i(x)$ depends only on $(x_1, \ldots, x_{\sigma(i)})$ where $i \longmapsto \sigma(i)$ is an increasing function. Then there exist a new set of variables $z = (z_1, \ldots, z_s)$, an increasing function τ, convergent power series $z_i(x) \in \mathbb{K}\{x\}$ vanishing at 0 such that $z_1(x), \ldots, z_{\tau(i)}(x)$ depend only on $(x_1, \ldots, x_{\sigma(i)})$, and an algebraic power series vector solution $y(x, z) \in \mathbb{K}\langle x, z\rangle^m$ of

$$f(x, y(x, z)) = 0,$$

such that for every i,

$$y_i(x, z) \in \mathbb{K}\langle x_1, \ldots, x_{\sigma(i)}, z_1, \ldots, z_{\tau(i)}\rangle \text{ and } y(x) = y(x, z(x)).$$

Moreover, given $k \in \mathbb{N}$ we may always require that $y_i(x, z) - y_i(x) \in \mathfrak{m}^k$, where $\mathfrak{m} = (x_1, ..., x_n) \subset \mathbb{K}\{x, z\}$.

Proof. Except the latter claim this theorem is proven in [4]. This latter claim can be shown by the following standard argument. We consider only the case $k = 1$. Write $y(x) = y(0) + \sum_{j=1}^n x_j w_j(x)$. Then $w(x)$ is a solution of

$$g(x, w(x)) = f(x, y(0) + \sum_{j=1}^{n} x_j w_j(x)) = 0.$$

Let $w(x, z)$ be an algebraic power series vector solution of the above equation, given by the standard version of the theorem, with convergent power series vector $z(x)$ so that $w(x) = w(x, z(x))$. Then $y(x, z) := y(0) + \sum_{j=1}^n x_j w_j(x, z)$ solves $f(x, y(x, z)) = 0$ and satisfies the required properties. □

2.3. **Arc-analytic maps and arc-wise analytic families.** Let Z, Y be analytic spaces. A map $f : Z \to Y$ is called *arc-analytic* if $f \circ \delta$ is real analytic for every real analytic arc $\delta : I \to Z$, where $I = (-1, 1) \subset \mathbb{R}$ (cf. [14]). By *an arc-analytic homeomorphism* we mean a homeomorphism φ such that both φ and φ^{-1} are arc-analytic.

Let T be a nonsingular analytic space. We say that a map $f(t, z) : T \times Z \to Y$ is *an arc-wise analytic family in* t if it is analytic in t and arc-analytic in z. This means that for every real analytic arc $z(s) : I \to Z$, the map $f(t, z(s))$ is real analytic and moreover, if $\mathbb{K} = \mathbb{C}$, complex analytic with respect to t (cf. [16]).

2.4. **Algebraic Equisingularity of Zariski.** Notation: Let $x = (x_1, \dots, x_n) \in \mathbb{K}^n$. Then we set $x^i = (x_1, \dots, x_i) \in \mathbb{K}^i$.

2.4.1. *Assumptions.* Let V be an analytic hypersurface in a neighborhood of the origin in $\mathbb{K}^l \times \mathbb{K}^n$ and let $T = V \cap (\mathbb{K}^l \times \{0\})$. Suppose there are given pseudopolynomials [1]

$$F_i(t, x^i) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(t, x^{i-1}) x_i^{p_i - j}, \qquad i = 0, ..., n,$$

$t \in \mathbb{C}^l$, $x^i \in \mathbb{C}^i$, with analytic coefficients $a_{i-1,j}$, that satisfy

1. $V = F_n^{-1}(0)$,
2. for all i, j, $a_{i,j}(t, 0) \equiv 0$,
3. F_{i-1} vanishes on the set of those (t, x^{i-1}) for which $F_i(t, x^{i-1}, x_i) = 0$, considered as an equation on x_i, has fewer complex roots than for generic (t, x^{i-1}),
4. Either $F_i(t, 0) \equiv 0$ or $F_i \equiv 1$ (and in the latter case $F_k \equiv 1$ for all $k \le i$ by convention),
5. $F_0 \equiv 1$.

For condition (3) it suffices that the discriminant of $(F_i)_{red}$, that is of F_i reduced, divides F_{i-1}. Equivalently, the first not identically equal to zero generalized discriminant of F_i divides F_{i-1}, see subsection 3.0.2.

We call a system $\{F_i(t, x^i)\}$ satisfying the above conditions *Zariski equisingular*. We also say that the family of germs $V_t := V \cap (\{t\} \times \mathbb{K}^n)$ is *Zariski equisingular* if such a system with $V = \{F_n = 0\}$ exists, maybe after a local analytic change of coordinates x. Answering a question posed by O. Zariski, A. Varchenko [26, 27, 28] showed that a Zariski equisingular family V_t is locally topologically trivial. The following stronger version of Varchenko's Theorem follows from Theorems 3.3 and 4.3 of [16].

Theorem 2.2. ([26, 27, 28], [16]) *If a system $\{F_i(t, x^i)\}$ is Zariski equisingular then the family V_t is locally topologically trivial along T, that is there exist $\varepsilon, \delta > 0$, $C, c > 0$ and a homeomorphism*

$$\Phi : B_\varepsilon \times \Omega_0 \longrightarrow \Omega,$$

[1] A pseudopolynomial is a polynomial in x_i with coefficients that are analytic in the other variables. The pseudopolynomials F_i that we consider are moreover distinguished polynomials in x, i.e. are of the form $x_i^p + \sum_{j=1}^{p} a_j(x^{i-1}) x_i^{p-j}$ with $a_j(0) = 0$ for all j. They may depend analytically on t that is considered as a parameter.

where $B_\varepsilon = \{t \in \mathbb{K}^l \ ; \ \|t\| < \varepsilon\}$, $\Omega_0 = \{x \in \mathbb{K}^n \ ; \ \|x\| < \delta\}$ *and* Ω *is a neighborhood of the origin in* $\mathbb{K}^{l+n}$*, that preserves the family* V_t*:* $\Phi^{-1}(V) = B_\varepsilon \times V_0$*. Moreover,* Φ *has a triangular form*

$$\Phi(t,x) = (t, \Psi(t,x)) = (t, \Psi_1(t,x^1), \Psi_2(t,x^2), \cdots, \Psi_n(t,x^n))$$

and satisfies the following additional properties:

1. *if* F_n *is a product of pseudopolynomials then* Φ *preserves the zero set of each factor,*
2. Φ *is subanalytic (semi-algebraic if all* F_i *are polynomials or Nash functions),*
3. Φ *is an arc-wise analytic family in* t *and* Φ^{-1} *is arc-analytic,*
4. *there is an analytic subset* $Z \subset \Omega_0$ *such that* Φ *is real analytic in the complement of* $B_\varepsilon \times Z$ *and such that the jacobian determinant of* Φ *satisfies for* $(t,x) \in B_\varepsilon \times (\Omega_0 \setminus Z)$

$$c \le |jacdet(\Phi)(t,x)| \le C.$$

Moreover, if the multiplicity of $F_i(0,x^i)$ *at* $0 \in \mathbb{K}^i$ *is equal to* p_i *for every* $2 \le i \le n$ *then for all* $(t,x) \in B_\varepsilon \times \Omega_0$

$$\left\| \frac{\partial \Psi}{\partial t}(t,x) \right\| \le C \|\Psi(t,x)\|.$$

We call the map Φ satisfying the conditions (1)-(4) *an arc-wise analytic trivialization.*

Remark 2.3. The last condition can be written equivalently, maybe for different $\varepsilon, \delta > 0$, $C, c > 0$, as

$$c\|x\| \le \|\Psi(t,x)\| \le C\|x\|,$$

see Propositions1.6 and 1.7 of [16]. That means geometrically that the trivialization Φ preserves the size of the distance to the origin.

Remark 2.4. The condition (1) of the conclusion can be given much stronger form, cf. Proposition 1.9 of [16]. For any analytic function G dividing F_n, there are constant $C, c > 0$, depending on G, such that

$$c|G(0,x)| \le |G(\Phi(t,x))| \le C|G(0,x)|.$$

3. Nash approximation of analytic sets

3.0.2. *Generalized discriminants.* (see e.g. [30] Appendix IV) Let $f(T) = T^p + \sum_{j=1}^p a_i T^{p-i} = \prod_{j=1}^p (T - T_i)$. Then the expressions

$$\sum_{r_1,\dots r_{j-1}} \prod_{k<l, k,l \neq r_1,\dots r_{j-1}} (T_k - T_l)^2$$

are symmetric in $T_1, \dots, T_p$ and hence polynomials in $a = (a_1, \dots, a_p)$. We denote these polynomials by $\Delta_j(a)$. Thus Δ_1 is the standard discriminant and f has exactly $p - j$ distinct roots if and only if $\Delta_1 = \dots = \Delta_j = 0$ and $\Delta_{j+1} \neq 0$.

3.0.3. *Construction of a normal system of equations.* We recall here the main construction of [15], [4].

Let $g_1,\dots,g_l \in \mathbb{K}\{x\}$ be a finite set of pseudopolynomials:

$$g_s(x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}(x^{n-1})x_n^{r_s-j}.$$

We arrange the coefficients $a_{n-1,s,j}$ in a row vector $a_{n-1}\in\mathbb{K}\{x^{n-1}\}^{p_n}$, $p_n=\sum_s r_s$. Let f_n be the product of the g_s's. The generalized discriminants $\Delta_{n,i}$ of f_n are polynomials in a_{n-1}. Let j_n be a positive integer such that

$$\Delta_{n,j_n}(a_{n-1})\not\equiv 0 \quad \text{and} \quad \Delta_{n,i}(a_{n-1})\equiv 0 \text{ for } i<j_n. \tag{3.1}$$

Then, after a linear change of coordinates x^{n-1}, we may write

$$\Delta_{n,j_n}(a_{n-1}) = u_{n-1}(x^{n-1})\left(x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j}\right)$$

where $u_{n-1}(0)\neq 0$ and for all j, $a_{n-2,j}(0)=0$, and

$$f_{n-1} = x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j}$$

is the Weierstrass polynomial associated to Δ_{n,j_n}. We denote by $a_{n-2}\in\mathbb{K}\{x^{n-2}\}^{p_{n-1}}$ the vector of its coefficients $a_{n-2,j}$.

Similarly we define recursively a sequence of pseudopolynomials $f_i(x^i)$, $i=1,\dots,n-1$, such that $f_i = x_i^{p_i}+\sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}$ is the Weierstrass polynomial associated to the first non identically equal to zero generalized discriminant $\Delta_{i+1,j_{i+1}}(a_i)$ of f_{i+1}, where we denote in general $a_i=(a_{i,1},\dots,a_{i,p_{i+1}})$ and

$$\Delta_{i+1,j_{i+1}}(a_i) = u_i(x^i)\left(x_i^{p_i}+\sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}\right), \quad i=0,...,n-1. \tag{3.2}$$

Thus the vector of functions a_i satisfies

$$\Delta_{i+1,k}(a_i)\equiv 0 \qquad k<j_{i+1}, \quad i=0,...,n-1. \tag{3.3}$$

This means in particular that

$$\Delta_{1,k}(a_0)\equiv 0 \quad \text{for } k<j_1 \text{ and } \Delta_{1,j_1}(a_0)\equiv u_0,$$

where u_0 is a non-zero constant.

Remark 3.1. At each step of this construction we may use a linear change of coordinates in order to assume that the multiplicity of f_i at the origin is equal to p_i, that is f_i as a power series does not contain monomials of degree smaller than p_i. Equivalently it means that for all j, $\operatorname{mult}_0 a_{i-1,j}\geq j$. We will assume in the following that this condition is satisfied for every i.

3.0.4. *Approximation by Nash functions.* If we consider (3.2) and (3.3) as a system of polynomial equations on $a_i(x^i)$, $u_i(x^i)$ then, by construction, this system admits convergent solutions. Therefore, by Theorem 2.1, there exist a new set of variables $z = (z_1, \dots, z_s)$, an increasing function τ, convergent power series $z_i(x) \in \mathbb{K}\{x\}$ vanishing at 0, algebraic power series $u_i(x^i, z) \in \mathbb{K}\langle x^i, z_1, \dots, z_{\tau(i)}\rangle$ and vectors of algebraic power series $a_i(x^i, z) \in \mathbb{K}\langle x^{(i)}, z_1, \dots, z_{\tau(i)}\rangle^{p_{i+1}}$ such that the following holds:

$$\begin{aligned}
&z_1(x), \dots, z_{\tau(i)}(x) \text{ depend only on } (x_1, \dots, x_i),\\
&a_i(x^i, z),\ u_i(x^i, z) \text{ are solutions of } (3.2), (3.3),\\
&a_i(0, z) \equiv 0,\\
&\text{and } a_i(x^i) = a_i(x^i, z(x^i)),\ u_i(x^i) = u_i(x^i, z(x^i)).
\end{aligned}$$

Then we define

$$F_n(z,x) = \prod_s G_s(z,x), \quad G_s(z,x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}\left(x^{n-1}, z(x^{n-1}) - z\right) x_n^{r_s - j},$$

$$F_i(z,x) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}\left(x^{i-1}, z^{\tau(i-1)}(x^{i-1}) - z^{\tau(i-1)}\right) x_i^{p_i - j}, \quad i = 0, \dots, n-1.$$

Finally we set $F_0 \equiv 1$.

We consider the system $\{F_i(z,x)\}$ as a system of pseudopolynomials in x parameterized by z. If we substitute $z = 0$ we obtain the original normal system of equations. The system $\{F_i(z,x)\}$ is Zariski equisingular. Indeed, $a_{i,j}(0,z) \equiv 0$ and hence the condition (2) of Assumptions 2.4.1 is satisfied. Moreover for every F_i, F_{i-1} is the first not identically equal to zero generalized discriminant, and hence the condition (3) of Assumptions 2.4.1 is satisfied. Therefore Theorem 2.2 implies the following lemma.

Lemma 3.2. *There are $\varepsilon, \delta > 0$, $C, c > 0$ and a subanalytic, arc-wise analytic as a family in z, an arc-analytic homeomorphism*

$$\Phi(z,x) = (z, \Psi(z,x)) : B_\varepsilon \times \Omega_0 \to \Omega, \tag{3.4}$$

where $B_\varepsilon = \{z \in \mathbb{K}^l\ ;\ \|z\| < \varepsilon\}$, $\Omega_0 = \{x \in \mathbb{K}^n\ ;\ \|x\| < \delta\}$, and Ω is a neighborhood of the origin in $\mathbb{K}^{l+n}$, preserving the sets $\{G_s = 0\}$ for $1 \le s \le l$, i.e. $\Phi^{-1}(\{G_s = 0\}) = B_\varepsilon \times (\{G_s = 0, z = 0\})$. There is an analytic nowhere dense subset $Z \subset \Omega_0$ such that Φ is real analytic in the complement of $B_\varepsilon \times Z$ and such that the jacobian determinant of Φ satisfies for $(z,x) \in B_\varepsilon \times (\Omega_0 \setminus Z)$

$$c \le |jacdet(\Phi)(z,x)| \le C. \tag{3.5}$$

Moreover, there exists a constant $C_1 > 1$ such that for all $(z,x) \in B_\varepsilon \times \Omega_0$

$$\left\| \frac{\partial \Psi}{\partial z}(z,x) \right\| \le C_1 \|\Psi(z,x)\| \tag{3.6}$$

$$C_1^{-1}\|x\| \le \|\Psi(z,x)\| \le C_1 \|x\|.$$

Let us consider for $m \in \mathbb{N}$

$$z(x) = z_m(x) + \tilde{z}_m(x),$$

where $z_m(x)$ is the m-th Taylor polynomial of $z(x)$. We shall use in the sequel the following bounds that hold for x sufficiently close to the origin.

$$\|\tilde{z}_m(x)\| \le C(m)\|x\|^{m+1}, \qquad \|\frac{\partial \tilde{z}_m}{\partial x}(x)\| \le C(m)\|x\|^m. \tag{3.7}$$

By classical formulae we may bound $C(m) \le C \sup_{\|x\|<(3/2)\delta'} \|z_{\mathbb{C}}(x)\| \delta'^{-(m+1)}$, where $z_{\mathbb{C}}$ is the complexification of z (if $\mathbb{K} = \mathbb{R}$), that we suppose defined on $\|x\| < 2\delta'$, $0 < \delta' < (1/2)\delta$, and C is a universal constant. Then (3.7) holds on $\|x\| \le \delta'$.

Set

$$\tilde{f}_n(t,x) = \prod_s \tilde{g}_s(t,x), \quad \tilde{g}_s(t,x) = G_s(t\tilde{z}_m(x), x)$$
$$\tilde{f}_i(t,x) = F_i(t\tilde{z}_m(x), x), \quad i = 0, \dots, n-1.$$

The system $\{\tilde{f}_i(t,x)\}$ is a deformation of the original normal system of equations, given by $t = 0$, to a Nash system of equations, given by $t = 1$. It is Zariski equisingular by construction. We show that its arc-wise analytic trivialization can be induced from Φ given by Lemma 3.2.

The construction of Φ of Theorem 2.2 given in [16] is universal, as we explain below, provided the Whitney Interpolation map, ψ in the notation of [16], is fixed. Thus the map of Lemma 3.2 is of the form

$$\Phi(z,x) = (z, \Psi(z,x)) = (z, \Psi_1(z,x^1), \Psi_2(z,x^2), \dots, \Psi_n(z,x^n))$$

and is constructed inductively as follows. Denote for short $x' = x^{n-1}$ and let

$$\Phi'(z,x') = (z, \Psi'(z,x')) = (z, \Psi_1(z,x^1), \dots, \Psi_{n-1}(z,x')).$$

Let $\eta(z,x') = (\eta_1(z,x'), ..., \eta_{p_n}(z,x'))$ be the vector of the roots of $F_n(z,x',x_n)$ (it is denoted by $a(t,x')$ in [16]). Then by (3.5) of [16], Ψ_n is defined by

$$\Psi_n(z,x',x_n) = \psi(x_n, \eta(0,x'), \eta(z, \Psi'(z,x'))), \tag{3.8}$$

where ψ is the Whitney Interpolation map, see [16] Appendix I.

In our case, the coefficients, and hence the roots, of $F_i(z,x^{i-1},x_i)$, depend only on $z^{\tau_{i-1}}, x^{i-1}$. Hence Φ is of the form

$$\Phi(z,x) = (z, \Psi(z,x)) = (z, \Psi_1(x^1), \Psi_2(z^{\tau(1)},x^2), \dots, \Psi_n(z^{\tau(n-1)},x^n)).$$

Now let

$$H(t,x) = (t, h(t,x)) = (t, h_1(x^1), h_2(t,x^2), \dots, h_n(t,x^n)).$$

be an arc-wise analytic trivialization constructed by the same recipe for the system $\{\tilde{f}_i(t,x)\}$ and let

$$H'(t,x') = (t, h'(t,x')) = (t, h_1(x^1), h_2(t,x^2), \dots, h_{n-1}(t,x^{n-1}))$$

be the trivialization of $\{\tilde{f}_i(t,x')\}_{i \le n-1}$ obtained in the inductive step. Then h can be deduced from Ψ by the following recursive formula.

Lemma 3.3.

$$h(t,x) = \Psi(t\tilde{z}_m(h'(t,x')),x). \tag{3.9}$$

Proof. The above formula means that for each $i = 1,\ldots,n$

$$h_i(t,x^i) = \Psi_i(t\tilde{z}_m^{\tau(i-1)}(h^{i-1}(t,x^{i-1})),x^i). \tag{3.10}$$

In particular, Ψ_1 is independent of z, h_1 is independent of t and $h_1(x_1) = \Psi_1(x_1)$.

We assume by induction that $h'(t,x') = \Psi'(t\tilde{z}_m^{\tau(n-2)}(h^{n-2}(t,x^{n-2})),x')$ and then show that (3.10) holds for $i = n$.

$$\begin{aligned}
&\Psi_n(t\tilde{z}_m(h'(t,x')),x',x_n)\\
&= \psi(x_n,\eta(0,x'),\eta(t\tilde{z}_m(h'(t,x')),\Psi'(t\tilde{z}_m(h'(t,x')),x')))\\
&= \psi(x_n,\eta(0,x'),\eta(t\tilde{z}_m^{\tau(n-1)}(h'(t,x')),\Psi'(t\tilde{z}_m^{\tau(n-2)}(h^{n-2}(t,x^{n-2})),x')))\\
&= \psi(x_n,\eta(0,x'),\eta(t\tilde{z}_m^{\tau(n-1)}(h'(t,x')),h'(t,x')))\\
&= \psi(x_n,\xi(0,x'),\xi(t,h'(t,x'))) = h_n(t,x),
\end{aligned}$$

where $\xi(t,h'(t,x'))$ is the vector of the roots of $\tilde{f}_n(t,h'(t,x'),x_n)$. □

In particular, by (3.6) and (3.9)

$$\|h(t,x)\| \le C_1\|x\|. \tag{3.11}$$

There is a constant $C > 0$ such that we have for x close to the origin and t from a neighborhood of $[0,1]$ in $\mathbb{R}$

$$\left\|\frac{\partial h}{\partial t}(t,x)\right\| \le CC(m)\|x\|^{m+2}, \tag{3.12}$$

where $C(m)$ is given by (3.7). Indeed, this is obvious for h_1 since h_1 is independent of t. The inductive step then follows from the identity (3.9) and the inequalities (3.6), (3.7) and (3.11):

$$\begin{aligned}
\left\|\frac{\partial h}{\partial t}(t,x)\right\| &\le \|\frac{\partial \Psi}{\partial z}(t\tilde{z}_m(h'(t,x')),x)\|\times\\
&\quad\times(\|\tilde{z}_m(h'(t,x'))\| + \|t\frac{\partial \tilde{z}_m}{\partial x}(h'(t,x'))\|\,\|\partial h'/\partial t(t,x')\|)\\
&\le C'C(m)\|\Psi(t\tilde{z}_m(h'(t,x')),x)\|\|x\|^{m+1}\\
&\le CC(m)\|x\|^{m+2}.
\end{aligned}$$

Then by integration we obtain

$$\|h(1,x) - x\| \le CC(m)\|x\|^{m+2}. \tag{3.13}$$

This allows us to give the following result:

Proposition 3.4. *Let X be an analytic subset of an open neighborhood U_0 of $0 \in \mathbb{K}^n$ with $0 \in X$. Then there are constants $C, c > 0$ and an open neighborhood $U \subset U_0$ of 0 in $\mathbb{K}^n$ such that for every $m \in \mathbf{N}$, there are a subanalytic arc-analytic homeomorphism $\varphi_m : U \to \varphi_m(U) \subset \mathbb{K}^n$ and a Nash subset V_m of $\varphi_m(U)$ with the following properties:*

(a) $\varphi_m(X \cap U) = V_m$,

(b) $\|\varphi_m(a) - a\| \leq C^m \|a\|^m$ *for every* $a \in U$.

Moreover, there is a nowhere dense analytic subset $Z \subset U$ *such that* φ_m *is real analytic in the complement of* Z *and*

$$c \leq |jacdet(\varphi_m)(x)| \leq C,$$

for $x \in U \setminus Z$ *and* $m > 1$.

Proof. We may assume that X is defined by the pseudopolynomials $g_1, \dots, g_l \in \mathbb{K}\{x\}$. We set $\hat{g}_i(x) = G_i(1, x)$ and we denote by V_m the zero locus of the $\hat{g}_i$. Then, by Lemma 3.3, $X \cap U$ is homeomorphic to $V_m \cap U$ where $U = \{x \in \mathbb{K}^n \ ; \ \|x\| < \delta'\}$. The homeomorphism is obtained by setting $t = 1$ in (3.9)

$$\varphi_m(x) = \Psi(\tilde{z}_m(x), x).$$

Condition (b) now follows from (3.13).

The last claim of the proposition follows from (3.5). Indeed, Ψ_i depends only on $z_1, ..., z_{\tau_{i-1}}$ and x^i and therefore

$$\frac{\partial}{\partial y_i} \Psi_j(\tilde{z}_m(y), y) = \frac{\partial \tilde{z}_m}{\partial y_i}(y) \frac{\partial \Psi_j}{\partial z}(\tilde{z}_m(y), y) + \frac{\partial \Psi_j}{\partial x_i}(\tilde{z}_m(y), y)$$

is non-zero only for $j \geq i$. (Formally the above formula makes sense only for $\mathbb{K} = \mathbb{R}$. In the complex case we should consider the derivatives with respect to $\mathrm{Re}\, y_i$ and $\mathrm{Im}\, y_i$.) Moreover, for $i = j$ we have $\frac{\partial}{\partial y_i} \Psi_j(\tilde{z}_m(y), y) = \frac{\partial \Psi_j}{\partial x_i}(\tilde{z}_m(y), y)$, since $z_1(x), ..., z_{\tau_{i-1}}(x)$ depend only on x^{i-1}. Therefore, taking into account that the jacobian matrix of $\Phi(z, x)$ is block triangular,

$$jacdet(\varphi_m)(y) = jacdet(\Psi(\tilde{z}_m(y), y)) = jacdet(\Phi)(\tilde{z}_m(y), y),$$

and the claim now follows from (3.5). □

4. Algebraic approximation of Nash sets

The main result of this section is the following proposition. For any $\mathbb{K}$-differentiable map $\varphi : U \subset \mathbb{K}^n \to \mathbb{K}^n$, let J_φ be the map assigning to every $a \in U$ the jacobian matrix of φ at a.

Proposition 4.1. *Let* X *be a Nash subset of an open neighborhood* U_0 *of* $0 \in \mathbb{K}^n$ *with* $0 \in X$. *Then there are a constant* $C > 0$ *and an open neighborhood* $U \subset U_0$ *of* 0 *in* $\mathbb{K}^n$ *such that for every* $m \in \mathbf{N}$ *there are a Nash diffeomorphism* $\varphi_m : U \to \varphi_m(U) \subset \mathbb{K}^n$ *and an algebraic subset* V_m *of* $\mathbb{K}^n$ *with the following properties:*

(a) $\varphi_m(X \cap U) = V_m \cap \varphi_m(U)$,
(b) $||\varphi_m(a) - a|| \leq C^m ||a||^m$ *for every* $a \in U$.

Moreover, the sequence $(J_{\varphi_m})_{m \in \mathbf{N}}$ *converges uniformly on* U *to the constant map assigning to every* $a \in U$ *the identity matrix.*

Remark 4.2. By the last assertion, for every $\varepsilon > 0$ the maps φ_m in Proposition 4.1 can be chosen in such a way that $1 - \varepsilon < |jacdet(\varphi_m)(x)| < 1 + \varepsilon$ for every $x \in U$, $m \in \mathbf{N}$.

*Proof of Proposition 4.1.*First we discuss the case $\mathbb{K} = \mathbb{C}$. We may assume that $\dim(X) = k < n$ because otherwise there is nothing to prove. By Artin-Mazur's construction [3] (cf. also [7]), there are $s \in \mathbf{N}$, an algebraic set $M \subset \mathbb{C}^n \times \mathbb{C}^s$ with

$0 \in M$ and a complex Nash submanifold N of a polydisc $E \times F \subset \mathbb{C}^n \times \mathbb{C}^s$ centered at zero such that:

(i) $\pi|_M : M \to \mathbb{C}^n$ is a proper map, where $\pi : \mathbb{C}^n \times \mathbb{C}^s \to \mathbb{C}^n$ denotes the natural projection, and $M \cap (E \times F) \subset N$,
(ii) $\pi|_N : N \to E$ is a biholomorphism and $\pi(M \cap (E \times F)) = X \cap E$.

We may assume that $s = 1$ (i.e. F is a disc in $\mathbb{C}$). Indeed, we may replace M, N by $v_L(M), v_L(N) \subset \mathbb{C}^n \times \mathbb{C}$, respectively. Here v_L is defined by the formula $v_L(x, z) = (x, L(z))$, where $L : \mathbb{C}^s \to \mathbb{C}$ is a linear form such that v_L restricted to $(\{0\}^n \times \mathbb{C}^s) \cap M$ is injective. (Note that it may be necessary to replace $E \times F$ by a smaller polydisc.)

We may also assume (applying a change of variables) that $\pi(M) \subset \mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$ has proper projection onto $\mathbb{C}^{n-1}$ and $X \cap (E' \times E'')$ has proper projection onto E', where $E = E' \times E'' \subset \mathbb{C}^{n-1} \times \mathbb{C}$.

Denote $\pi = (\pi_1, \ldots, \pi_{n-1}, \pi_n) = (\pi', \pi_n)$. Then after replacing π_n by any polynomial w we obtain a polynomial map $\rho = (\pi', w)$ such that $\rho|_M$ is proper (so $\rho(M)$ is an algebraic subset of $\mathbb{C}^n$). Now, by (ii), there is a holomorphic function $\tau : E' \times E'' \to F$ such that N is the graph of τ. We will choose $w = w_m$ such that (perhaps after shrinking E', E'', F) for $\rho_m = (\pi', w_m)$ the map $\varphi_m(x) = \rho_m((x, \tau(x)))$ satisfies (a) and (b) with $V_m = \rho_m(M)$. The size of $E' \times E'' \times F$ will be independent of m. The idea to obtain algebraic V_m (equivalent to or approximating some X) as the image of an algebraic set by a certain polynomial map ρ_m appeared before (see e.g. [7], [1], [6]). But here ρ_m must be chosen in a special way to ensure that V_m is both analytically equivalent and higher order tangent to X.

Let us verify that after shrinking E', E'', F the following hold:

(x) $(\overline{E'} \times \mathbb{C} \times \partial F) \cap M = \emptyset$,
(y) $(\overline{E'} \times \partial E'' \times \overline{F}) \cap M = \emptyset$,
(z) $(\overline{E} \times \partial F) \cap \overline{N} = \emptyset$.

First, by the fact that $N \subset E \times F$ is a graph of a holomorphic function defined on E, we know that (z) holds after replacing E by any of its relatively compact subsets. Hence, we may assume to have (z). Observe also that if (z) is true, then it remains such after shrinking F slightly with fixed E.

Next, by the facts that $\pi|_M$ is proper and $\pi(M)$ has proper projection onto $\mathbb{C}^{n-1}$, we know that $(\{0\}^{n-1} \times \mathbb{C} \times \mathbb{C}) \cap M$ is a finite set. Therefore after shrinking F slightly (in such a way that (z) remains true) we have $(\{0\}^{n-1} \times \mathbb{C} \times \partial F) \cap M = \emptyset$. Thus, in view of the properness of the projections, (x) holds for F chosen above with any sufficiently small neighborhood E' of 0. It follows that both (x) and (z) hold with any sufficiently small $E = E' \times E''$. We can pick E in such a way that $(\overline{E'} \times \partial E'') \cap \pi(M) = \emptyset$, which implies (y).

Let δ denote the radius of F. Define $w_m : E \times F \to \mathbb{C}$ by the formula $w_m(x, z) = \pi_n(x, z) + (\frac{z}{\delta})^m$ with large m. Let us check that $\varphi_m : U \to \mathbb{C}^n$ given by $\varphi_m(x) = \rho_m((x, \tau(x)))$ has all the required properties, where $U = E' \times E''$. For $z \in F$ we have $|z| < \delta$ and the graph of τ is contained in $E' \times E'' \times F$, hence in view of (z), for m large enough, φ_m is a biholomorphism onto its image. The injectivity of φ_m requires a brief explanation. First, by the definition of φ_m, we can explicitly write

$$\varphi_m(x) = \left(x_1, \cdots, x_{n-1}, x_n + \left(\frac{\tau(x)}{\delta} \right)^m \right). \tag{4.1}$$

Thus φ_m is injective on U if the map $x_n \mapsto x_n + \left(\frac{\tau(x)}{\delta}\right)^m$ is injective for every fixed $(x_1, \cdots, x_{n-1})$. The latter assertion is true for large m if $\sup_{x\in U}|\frac{\tau(x)}{\delta}| < 1$ (because then the modulus of the derivative of the map $x_n \mapsto \left(\frac{\tau(x)}{\delta}\right)^m$ is small). Let us check that $\sup_{x\in U}|\frac{\tau(x)}{\delta}| < 1$. Recall that $N = \mathrm{graph}(\tau)$. Hence, by (z), we have

$$0 < \inf\{||a-b|| \ ; \ a \in \overline{\mathrm{graph}(\tau)}, b \in \overline{E} \times \partial F\} \leq \inf\{|\tau(x) - c| \ ; \ x \in \overline{E}, c \in \partial F\}.$$

Since δ is the radius of F, and $U = E$, we obtain $\sup_{x\in U}|\tau(x)| < \delta$, as required.

Let us verify that for large m,

$$\varphi_m(X \cap (E' \times E'')) = \rho_m(M) \cap \varphi_m(E' \times E'').$$

By (i), (ii) and the definition of φ_m, we have $\varphi_m(X \cap (E' \times E'')) = \rho_m(\mathrm{graph}(\tau) \cap M)$. Moreover, $\rho_m(\mathrm{graph}(\tau)) = \varphi_m(E' \times E'')$, hence it is sufficient to show that $\rho_m(\mathrm{graph}(\tau) \cap M) = \rho_m(\mathrm{graph}(\tau)) \cap \rho_m(M)$. In fact, the inclusion "$\subset$" is trivial so we prove "$\supset$".

Fix $a \in \rho_m(\mathrm{graph}(\tau)) \cap \rho_m(M)$. Then there are $z \in \mathrm{graph}(\tau)$ and $v \in M$ such that $\rho_m(z) = \rho_m(v) = a$. Observe that $z, v \in E' \times \mathbb{C} \times \mathbb{C}$. Indeed, write $z = (z_1, \ldots, z_{n-1}, z_n, z_{n+1})$, $v = (v_1, \ldots, v_{n-1}, v_n, v_{n+1})$. Since $\mathrm{graph}(\tau) \subset E' \times E'' \times F$ we have $(z_1, \ldots, z_{n-1}) \in E'$. By the definition of ρ_m and by $\rho_m(z) = \rho_m(v)$ we have $(z_1, \ldots, z_{n-1}) = (v_1, \ldots, v_{n-1})$.

Moreover, $(E' \times \mathbb{C} \times \mathbb{C}) \cap M$ is bounded so v must belong to $E' \times \mathbb{C} \times F$ because otherwise (for large m) in view of the definition of w_m and (x), $\rho_m(v) = a$ lies outside $\rho_m(\mathrm{graph}(\tau))$, which is a contradiction. Now if $v \notin E' \times E'' \times F$, then by (y), (z), (x) (for large m) $\rho_m(v) \notin \rho_m(\mathrm{graph}(\tau))$, again a contradiction. Consequently, in view of (i), we have $v \in M \cap (E' \times E'' \times F) \subset \mathrm{graph}(\tau)$. Since $\rho_m|_{\mathrm{graph}(\tau)}$ is injective, we have $v = z$, hence $a \in \rho_m(\mathrm{graph}(\tau) \cap M)$. This shows that

$$\varphi_m(X \cap (E' \times E'')) = \rho_m(M) \cap \varphi_m(E' \times E'')$$

and completes the proof of (a).

Finally, by (4.1) we have

$$||\varphi_m(a) - a|| \leq \frac{|\tau(a)|^m}{\delta^m}$$

for every $a \in E' \times E''$. Since $\tau(0) = 0$, we immediately obtain (b).

The last assertion of the proposition follows by (b). Namely, we may assume that U is so small that $c||a|| < 1$ for every $a \in U$. Then (b) implies that $(\varphi_m)_{n\in\mathbf{N}}$ converges to id uniformly on U. In view of the Weierstrass theorem, after shrinking U slightly, we obtain that $(\frac{\partial \varphi_m}{\partial x_i})_{m\in\mathbf{N}}$ converges uniformly on U to $\frac{\partial id}{\partial x_i}$ for every $i = 1, \ldots, n$. This completes the proof in the case $\mathbb{K} = \mathbb{C}$.

As for $\mathbb{K} = \mathbb{R}$, we cannot simply repeat the procedure above because the image of a real algebraic set by a proper polynomial map need not be algebraic. Instead we proceed as follows. First for the given Nash set X let $X_{\mathbb{C}}$ denote its complexification. More precisely, $X_{\mathbb{C}}$ is a representative of the smallest complex Nash germ in $(\mathbb{C}^n, 0)$ containing the germ $(X, 0)$. Note that $X_{\mathbb{C}}$ is defined by real equations. For $X_{\mathbb{C}}$ one can repeat the construction described above obtaining M, N also defined by real equations. In particular, the Taylor expansion of φ_m around zero has real coefficients i.e. the restriction of φ_m to $\mathbb{R}^n$ is a real Nash isomorphism. It is not difficult to observe that $(\varphi_m(X),0)$ is the germ of a real algebraic set as required. $\square$

5. Proof of Theorem 1.3

Let $(X,0)\subset(\mathbb{K}^n,0)$ be a germ of analytic set. In both real and complex cases $(X,0)$ can be considered as a real analytic set germ defined by equations

$$g_1=\cdots=g_l=0$$

where the g_i are convergent power series with real coefficients. A real analytic arc on $(X,0)$ is a germ of real analytic map $(\mathbb{R},0)\longrightarrow(X,0)$. Then the space of real analytic arcs of $(X,0)$ is in bijection with set of morphisms

$$\frac{\mathbb{R}\{x_1,\ldots,x_n\}}{(g_1,\ldots,g_l)}\longrightarrow\mathbb{R}\{t\}$$

which are defined by the data of n convergent power series $x_1(t),\ldots,x_n(t)\in\mathbb{R}\{t\}$ such that

$$g_i(x_1(t),\ldots,x_n(t))=0\quad\forall i.$$

Let m be a positive integer and let φ_m be the homeomorphism given by Theorem 1.2. Let γ be an arc on $(X,0)$, i.e. γ is a real analytic map $(-\varepsilon,\varepsilon)\longrightarrow X\cap U$ for some $\varepsilon>0$. Then we define $\gamma':(-1,1)\longrightarrow X\cap U$ by $\gamma'(t)=\gamma(\varepsilon t)$ for all $t\in(-1,1)$. For such a real analytic arc γ', $\varphi_m\circ\gamma'$ is a real analytic arc on $V_m\cap\varphi_m(U)$ since φ_m is an arc-analytic map (cf. Theorem 1.2). Thus $\varphi_m\circ\gamma$ is a real analytic arc on $(V_m,0)$. On the other hand φ_m^{-1} is arc-analytic thus the same procedure applies. This shows that φ_m induces a bijection between the space of real analytic arcs on $(X,0)$ and the space of real analytic arcs on $(V_m,0)$.

Now if γ is a real analytic arc on $(X,0)$ then

$$\|\varphi_m(\gamma(t))-\gamma(t)\|\leq C^m\|\gamma(t)\|^m\leq C(m,\gamma)|t|^m$$

for all t small enough and some positive constant $C(m,\gamma)$ depending only on m and γ. This shows that $x_i\circ\varphi_m\circ\gamma(t)$ has the same Taylor expansion as $x_i\circ\gamma$ up to order $m-1$. Thus φ_m induces the identity map between the space of m-truncations of real analytic arcs on $(X,0)$ and the space of m-truncations of real analytic arcs on $(V_m,0)$.

If $(X,0)$ is complex analytic then any real analytic arc germ $\gamma:(\mathbb{R},0)\longrightarrow(X,0)$ extends uniquely to a complex analytic arc germ $\gamma_{\mathbb{C}}:(\mathbb{C},0)\longrightarrow(X,0)$ and similarly for the arc-germs in $(V_m,0)$. Moreover, any complex arc germ in $(X,0)$, resp. in $(V_m,0)$, is the complexification of a real arc germ. Thus $\mathcal{A}^{\mathbb{C}}(X)=\mathcal{A}^{\mathbb{C}}(V_m)$ follows from $\mathcal{A}^{\mathbb{R}}(X)=\mathcal{A}^{\mathbb{R}}(V_m)$.

Remark 5.1. By a result of M. Greenberg (cf. [12] or [21] for the analytic case) for a given analytic germ $(X,0)\subset(\mathbb{K}^n,0)$ there exists a constant $a=a_X>0$ such that for every integer m we have that

$$\mathcal{A}_m^{\mathbb{K}}(X)=\pi_m\left(\mathcal{B}_{am}^{\mathbb{K}}(X)\right)$$

where $\mathcal{B}_k^{\mathbb{K}}(X)$ denotes the space of k-jets on $(X,0)$, i.e. the space of $\mathbb{K}$-analytic arcs on $(\mathbb{K}^n,0)$ whose contact order with $(X,0)$ is at least $k+1$, and π_m is the truncation map at order m, i.e. the map sending an arc on $(\mathbb{K}^n,0)$ onto its truncation at order

m. If we denote by $(X',0)$ an analytic germ defined by equations that coincide with the equations defining $(X,0)$ up to order am, then we obviously have that

$$\mathcal{B}^{\mathbb{K}}_{am}(X)=\mathcal{B}^{\mathbb{K}}_{am}(X').$$

But while $\mathcal{A}^{\mathbb{K}}_m(X)$ is the truncation of $\mathcal{B}^{\mathbb{K}}_{am}(X)$, $\mathcal{A}^{\mathbb{K}}_m(X')$ has no reason in general to be equal to the truncation of $\mathcal{B}^{\mathbb{K}}_{am}(X')$ since the constant $a_{X'}$ of Greenberg's Theorem may be strictly greater than $a=a_X$. Thus we cannot prove, using Greenberg's Theorem, that $\mathcal{A}^{\mathbb{K}}_m(X)$ is equal to $\mathcal{A}^{\mathbb{K}}_m(X')$ when $(X',0)$ is an analytic germ whose equations coincide with those of $(X,0)$ up to a high order.

In fact a high order of tangency of two analytic germs does not guarantee that the spaces of truncated arcs associated with these germs are equal. We can explicitly show this on the following example. Let

$$f(x,y,z)=z^2-xy^4,\ \ f_k(x,y,z)=z^2-x(y^4+x^{2k})$$

and define $X=\{(x,y,z):f(x,y,z)=0\}, X_k=\{(x,y,z):f_k(x,y,z)=0\}$ for any positive integer k which is not divisible by 2. Then the arc γ given by $\gamma(t)=(t,0,0)$ satisfies $\gamma\in\mathcal{A}^{\mathbb{K}}_1(X)$ but in both cases real or complex $\gamma\notin\mathcal{A}^{\mathbb{K}}_1(X_k)$. Indeed, let $\tilde\gamma$ be any arc whose truncation up to order 1 equals γ. Then after substituting $\tilde\gamma$ to $x(y^4+x^{2k})$ and to z^2, we obtain power series with odd and even order of zero, respectively, so $f_k\circ\tilde\gamma\neq 0$. Thus $\mathcal{A}^{\mathbb{K}}_1(X_k)\neq\mathcal{A}^{\mathbb{K}}_1(X)$, although $2k$-truncations of f and f_k are equal and the multiplicities of X and X_k at 0 are also equal for k large enough.

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