Spectral applications of metric surgeries

Pierre Jammes

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Introduction and motivations

Examples of applications of metric surgeries

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Theorem

$\nu(M^n)$ is uniformly bounded on manifold of dimension n.

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Theorem (Bär, Dahl, Ammann, Humbert)

This inequality is an equality for a generic set of metrics. In particular, The Dirac operator is generically invertible if $n = 3, 5, 6, 7 \mod 8$.

Introduction and motivations

Proposition

If the scalar curvature of (M^n, g) *and* (M'^n, g') *is positive* $(n \geq 3)$, *then* $M \# M'$ *carries a metric of positive scalar curvature.*

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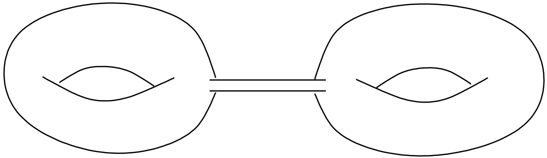
Theorem (Gromov, Lawson, '80)

Every closed simply-connected non spin manifold of dimension ≥ 5 carries a metric of positive scalar curvature.

Surgeries I : connected sum



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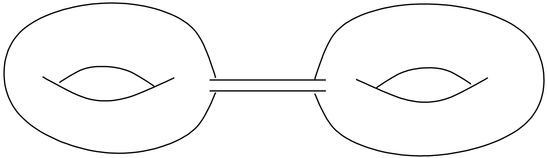

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- Attach a handle $S^{n-1} \times [0, 1]$.

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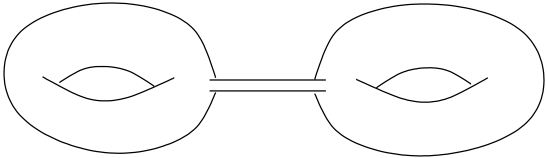

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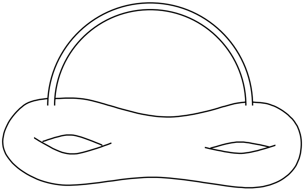

Surgeries II : definition

- Let M^n be a closed manifold, and $S^k \hookrightarrow M^n$ an embedded sphere whose tubular neighborhood is diffeomorphic to $S^k \times B^{n-k}$.
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Definition

The manifold obtained from M by a surgery along S^k (k dimensional surgery) is

$$M \backslash (S^k \times B^{n-k}) \bigcup_{S^k \times S^{n-k-1}} (B^{k+1} \times S^{n-k-1})$$

$n-k$ is the *codimension* of the surgery.

Surgeries II : definition & examples

Example I

The connected sum is a surgery along a sphere S^0.

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Example II

1-codimensional surgery

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Example III

The sphere S^3 is the union of two copies of $S^1 \times D^2$.
A surgery along a trivial knot in S^3 produces the manifold $S^1 \times S^2$.

Surgeries III : applications

Theorem (Gromov, Lawson, '80)

Let M^n be a closed riemannian manifold with positive scalar curvature. If M' is obtained from M by a surgery of codimension ≥ 3, then M' carries a metric of positive scalar curvature.

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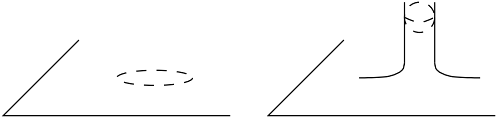

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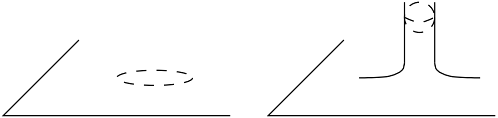

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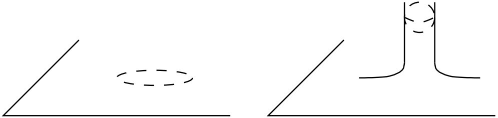

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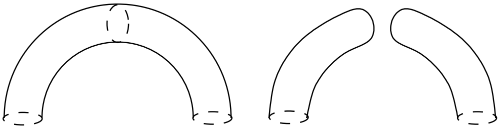
Theorem (Ammann, Dahl, Humbert, '09)

If D is invertible on M and M' is obtained from M by a surgery of codimension 2, then D is invertible on (M', g').

Surgeries IV: cancellation



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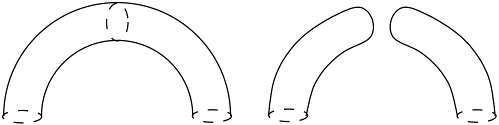

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$$S^k \times B^{n-k} \leftrightarrow B^{k+1} \times S^{n-k-1}$$

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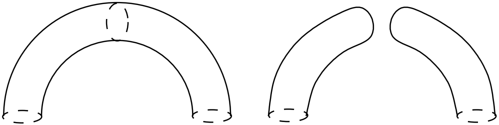

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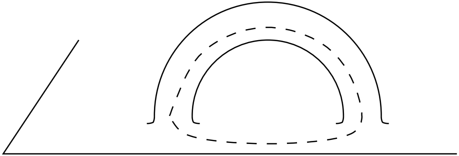

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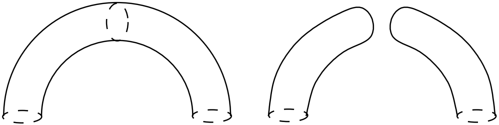

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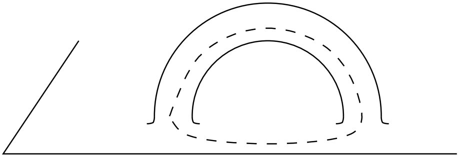

A k-surgery is cancelled by a surgery along a $(k+1)$-sphere that intersects transversally the belt sphere of the k surgery in one point (Smale's cancellation lemma).

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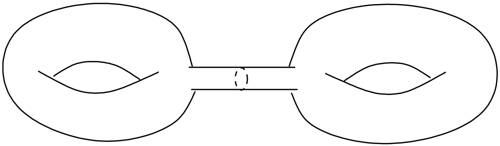

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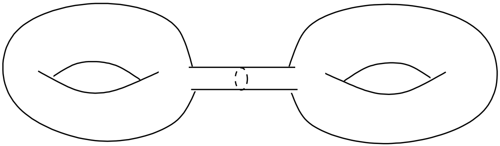
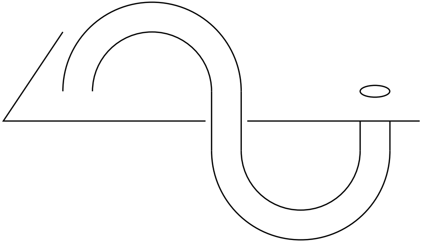


- non oriented handle



Cobordism I : definition

Definition

Let M and N be two closed n-dimensional manifolds. A cobordism between M and N is a compact $n+1$-manifold W whose boundary is $M \coprod N$. M are N are *cobordant* if such a cobordim exists.

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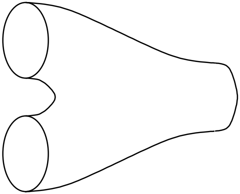
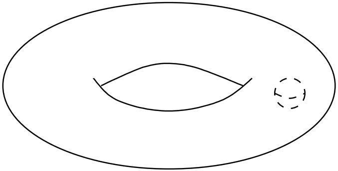


$S^1 \coprod S^1$ is cobordant to S^1

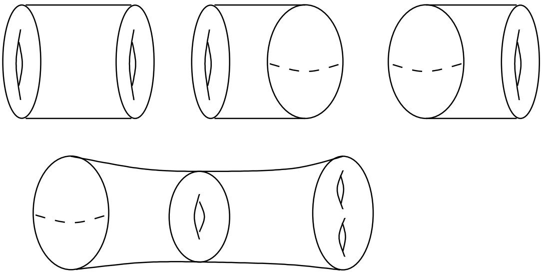


T^2 is cobordant to S^2

Cobordism I : definition

Remark

Cobordism is a equivalence relation.



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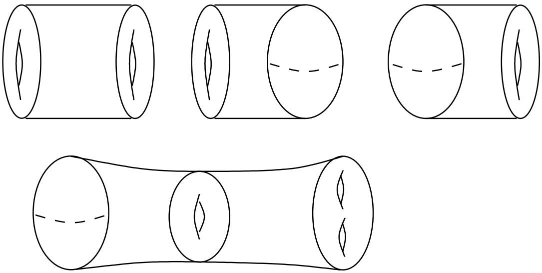

Questions

1. What can we say about the quotient set ?
2. What can we say about a given equivalence class ?

Cobordism II : cobordism ring

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- If M is the boundary of W^{n+1}, we write $[M] = [\emptyset]$.
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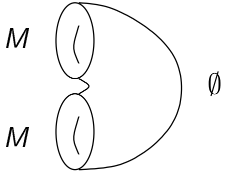
- The identity element of this group is $[\emptyset]$
- $[M] + [M] = [\emptyset]$



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- $\partial(W_1 \times N) = (M \times N) \sqcup (M' \times N) \Rightarrow [M \times N] = [M' \times N]$
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Proof : let W^{2n+1} be a cobordism between M^{2n} and N^{2n}. We obtain a closed manifold W' by gluing two copies of W along their boundaries.

$\chi(W') = 2\chi(W) - \chi(\partial W)$

$\Rightarrow \chi(\partial W) = \chi(M) + \chi(N) = 0 \mod 2.$

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- *(Dold, 1956) If* $i = 2^r(2s+1) - 1$ *is odd,* X_i *is the class of* $P(2^r - 1, s2^r)$ *where* $P(k, l) = (S^k \times P^l(\mathbb{C}))/(x, z) \sim (-x, \bar{z})$.

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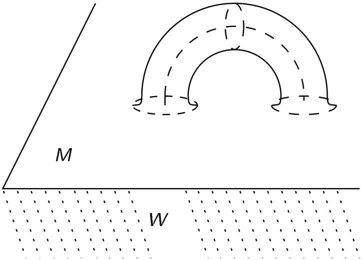
Ω_1	0	
Ω_2	$Z/2$	$P^2(\mathbb{R})$
Ω_3	0	
Ω_4	$(Z/2)^2$	$P^2(\mathbb{R}) \times P^2(\mathbb{R})$, $P^4(\mathbb{R})$
Ω_5	$Z/2$	$P(1,2)$

Cobordism III : cobordism & surgeries

Let $W^{n+1} = M \times [0,1]$ be a trivial cobordism. If $S^{k-1} \hookrightarrow M \times \{1\}$ is an embedded sphere with trivial normal bundle, we obtain a new cobordism W' by attaching a handle $B^k \times B^{n+1-k}$ along S^{k-1} :



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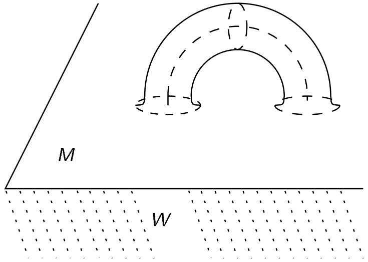

W' is called an *elementary cobordism of index* k. The new boundary is obtained from M by a surgery along S^{k-1}.

Cobordism III : cobordism & surgeries

Consequences

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- If M' is obtained from M by a finite number of surgeries, then $[M] = [M']$.

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Theorem (Smale, Wallace)

If W is a cobordism, then $W = W_1 \cup W_2 \cup \ldots \cup W_p$, where each W_i is an elementary cobordism. Moreover, we can assume that the indices of these cobordisms are increasing with i.

Cobordism III : cobordism & surgeries

Proof

Let W be a cobordism between M and N, and $f : W \to [0,1]$ a Morse function such that $f^{-1}(0) = M$ and $f^{-1}(1) = N$.

- f Morse function $\Leftrightarrow$ all critical points of f are non degererates.

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- Near a critical point, $f(x) = f(0) + x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - x_n^2$.

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  $f(x) = f(0) + x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - x_n^2$.
- W compact $\Rightarrow$ f has finitely many critical points.
- We may assume that the critical values of f are distincts, and $\neq 0, 1$.
- If there is no critical value in $[a, b]$, then $f^{-1}([a,b])$ is a trivial cobordism.

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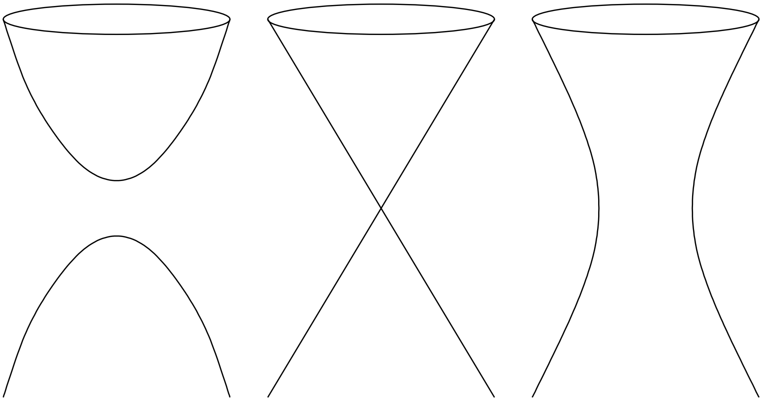

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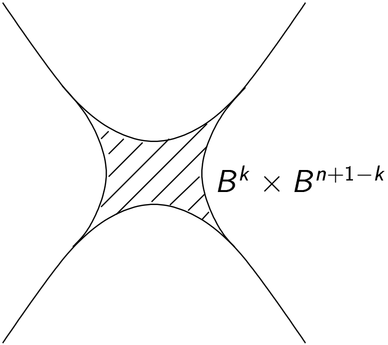

$\rightarrow$ each critical point corresponds to an elementary cobordism.

Cobordism IV : oriented cobordism

- All manifolds are supposed orientable and oriented.
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Remark

For a trivial cobordism $M \times [0,1]$, the orientation induced on $M \times \{0\}$ and $M \times \{1\}$ are opposite.
$\Rightarrow -[M] = [-M]$

Cobordism IV : oriented cobordism

Let Ω_*^{SO} be the oriented cobordism ring.

Theorem (R. Thom, 1954)

- *For each n, Ω_n^{SO} is finitely generated.*
- $\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[Y_{4i}], i \geq 1$ *with* $Y_{4i} = [P^{2i}(\mathbb{C})]$.

dimension	1	2	3	4	5	6	7	8
group	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	$\mathbb{Z}^2$

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Proof : two 0-handle + one 1-handle = one 0-handle

Handle decomposition

Exercise

Every compact surface with boundary admits a flat metric.

$$2\pi\chi(M) = \int_M K \; dA \; + \int_{\partial M} k \; dl$$

Conformal bounds for λ_1

Let (M, g) be a closed connected riemannian manifold.

$$\Delta : C^\infty(M) \to C^\infty(M)$$

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \dots$$

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Definition

The conformal volume of M is the infimum of $V_c(\varphi)$ on all conformal immersion $\varphi \to S^k$, for all k.

$$V_c(M, [g]) = \inf_{\varphi} V_c(\varphi)$$

Conformal bounds for λ_1

Theorem (Li & Yau, El Soufi & Ilias)

$$\lambda_1(M, g)\mathrm{Vol}(M)^{2/n} \leq nV_c(M, [g])^{2/n}$$

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Examples of manifold that admits a minimal immersion in the sphere : S^n, $P^n(\mathbb{R})$, $P^n(\mathbb{C})$, $P^n(\mathbb{H})$, $\ldots$

Conformal bounds for λ_1

$S^k \hookrightarrow \mathbb{R}^{k+1}$ The coordinates x_i of $\mathbb{R}^{k+1}$ satisfies

- $\sum_i x_i^2 = 1$ on S^k
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Conformal bounds for λ_1

Definition

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There is a constant $c(n) > 0$ such that for all closed manifold M^n, $V_{\mathcal{M}}(M) \leq c$.

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Principle of the proof : to study the behavior of the Möbius volume when performing surgeries.

Conformal bounds for λ_1

Proof for $n = 2$

Lemma

Let M be a compact surface. If M' is obtained by adding a handle to M, then $V_{\mathcal{M}}(M') \leq \sup\{V_{\mathcal{M}}(M), c\}$ where c doesn't depend on M.

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Let $\varphi : M \to S^k$ such that $V_{\mathcal{M}}(M) \leq V(\varphi) \leq V_{\mathcal{M}}(M) + \varepsilon$ and $V_c(\varphi) - \varepsilon \leq \mathrm{Vol}(\varphi(M)) \leq V_c(\varphi)$.

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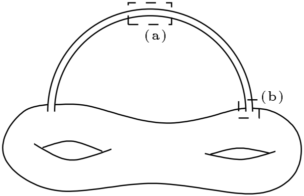
Stereographic projection : $S^k \to \mathbb{R}^k \cup \{\infty\}$

$$g_{S^k} = \frac{4}{(1 + \|x\|^2)^2} g_{\mathrm{eucl}}$$

Conformal bounds for λ_1

By attaching a thin handle to $\varphi(M)$, we obtain an immersion $\varphi' : M' \to S^k$ such that $\mathrm{Vol}(\varphi'(M')) \sim \mathrm{Vol}(\varphi(M))$



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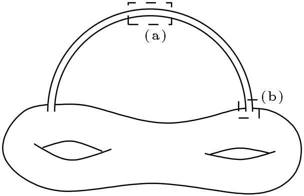

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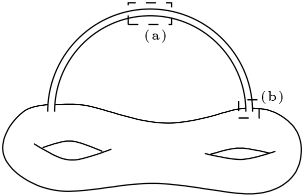

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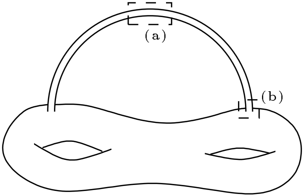

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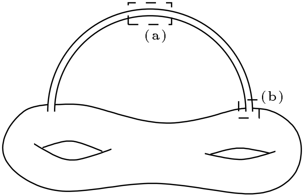

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- If γ is a translation, $\mathrm{Vol}(\gamma \circ \varphi'(M'))$ is close to or smaller than $\mathrm{Vol}(\varphi(M))$.
- Same conclusion if the factor of the homothety is small or or "not too large".

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Suppose that the factor of γ is large. The parts of M' that are close to ∞ have a small area.

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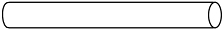
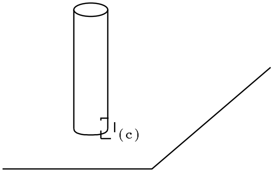
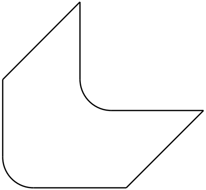
$M' \sim$  or 



Conformal bounds for λ_1

Proof for $n \geq 3$

Lemma

Let M^n be a closed manifold. If M' is obtained from M by a surgery of codimension ≥ 2, then $V_{\mathcal{M}}(M') \leq \sup\{V_{\mathcal{M}}(M), c(n)\}$.

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$$S^k \times B^{n-k}(\varepsilon) \leftrightarrow B^{k+1} \times S^{n-k-1}(\varepsilon)$$

$$\text{Codimension } \geq 2 \Leftrightarrow n - k - 1 \geq 1$$

Conformal bounds for λ_1

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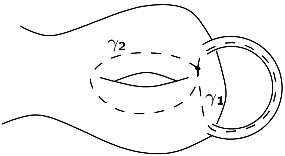

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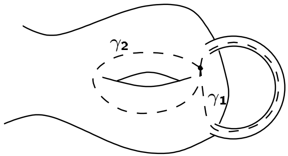

Since M' is non orientable, we can find a transversally orientable loop and apply the cancellation lemma.

Conformal bounds for λ_1

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$\Rightarrow$ the Möbius volume is bounded on all classes of oriented cobordism

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Example

Let $M^n \subset \mathbb{R}^n$ be an euclidean domain. The stereographic projection induces a conformal immersion $\varphi : M \to S^n$.
For all $\gamma \in G_n$, $\gamma \circ \varphi(M)$ is a domain of S^n, hence $\text{Vol}(\gamma \circ \varphi(M)) \leq \text{Vol}(S^n)$.
$\Rightarrow \; V_{\mathcal{M}}(M) \leq \text{Vol}(S^n)$

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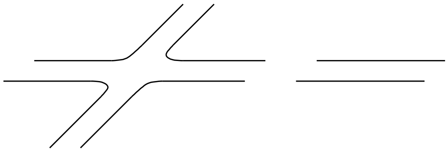
- If ε is small, $\mathrm{Vol}(\varphi(M))$ is small.
- If $\gamma \in G_3$ has not a large homothetic factor, $\mathrm{Vol}(\gamma \circ \varphi(M))$ is still small.

Conformal bounds for λ_1: manifolds with boundary

We consider $\gamma \in G_3$ with large factor.



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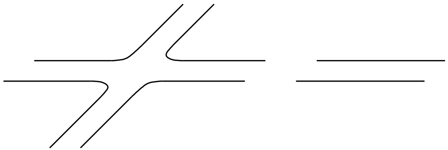

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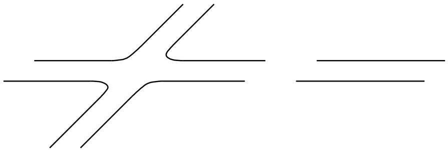

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- If we increase the ratio of γ, the volume of D' decreases as fast as $\mathrm{Vol}(S^2 \backslash D)$.

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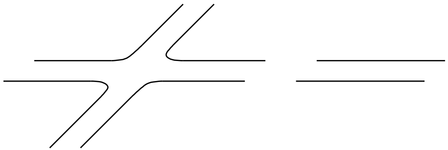

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$\Rightarrow \mathrm{Vol}(\gamma \circ \varphi(M)) < \mathrm{Vol}(S^2)$.

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- Control of the volume of $\mathrm{Vol}(\gamma \circ \varphi'(M'))$ in the same way as for dimension 2.

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