

# MULTICOMMODITY FLOWS ON ROAD NETWORKS

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**Abstract.** In this paper, we discuss the multicommodity flow for vehicular traffic on road networks. To model the traffic, we use the “Aw-Rasclé” multiclass macroscopic model [3]. We describe a solution to the Riemann problem at junctions with a criterion of maximization of the total flux, taking into account the destination path of the vehicles. At such a junction, the *actual* distribution depends on the demands and the supplies on the incoming and outgoing roads, respectively. Furthermore, this new distribution scheme captures efficiently key merging characteristics of the traffic and contrary to [15] leads to an easy computational model to solve approximately the homogenization problem described in [15], [16]. Furthermore, we deduce the equivalent distribution scheme for the LWR multiclass model in [9] and we compare the results with those obtained with the “Aw-Rasclé” multiclass model for the same initial conditions.

**Key words.** Aw–Rasclé model, multicommodity flow models, traffic networks

**subject classifications.** 35LXX, 35L6

**1. Introduction** A typical vehicular traffic system consists of the vehicle-driver pairs and the infrastructures, i.e., a collection of highways systems and all their operational elements. In a vehicular traffic system, a number of trips - defined by their destination path, the travel route, etc. - interact on the road network and generate various dynamics and phenomena. To study these traffic phenomena and the corresponding applications, many traffic models [8, 9, 14, 18, 20, 21, 24, 29] and simulation packages [19] have been suggested in the literature. However, most of the models of multicommodity traffic systems have been made under the framework of the LWR model [25, 9]. In this study, we are particularly interested in the impact of the destination path of the vehicles on traffic dynamics at roads intersections when modelling with a class of “second order” models of traffic flow. In the present work, we consider a multiclass macroscopic model [3] derived from the “Aw-Rasclé” (AR) second order model of traffic flow and we propose a definition of the solution to the associated Riemann problem at a junction. This solution is based on the maximization of the mass flux at the junction and the conservation of the pseudo-momentum (see below). The vehicles with the same destination are considered to belong to the same commodity. Each commodity is attached to each vehicle and is therefore a Lagrangian variable that we must keep track of in the whole flow and in particular through junctions (see also [9] for another definition of commodity). Given some initial boundary conditions, we make use of the supply and demand methods [6, 23] to compute the fluxes through a junction and then we solve the associated Riemann problem. We then use homogenization techniques [16, 15] to obtain coupling conditions for the second momentum of the Aw–Rasclé equation. Other approaches conserving the second moment are recently introduced in [12, 27].

## 2. Preliminaries

We model a road network as a finite, directed graph  $G = (\mathcal{E}, \mathcal{N})$  where we denote by  $\mathcal{E}$  and  $\mathcal{N}$  the set of arcs and vertices of the graph  $G$ , respectively. We label each arc in  $\mathcal{E}$  by  $e = 1, \dots, \mathbf{E}$  and assume that each arc corresponds to a road. Similarly, each vertex in  $\mathcal{N}$  labeled  $n = 1, \dots, \mathbf{N}$  corresponds to junction. For an arbitrary given

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junction  $n \in \mathcal{N}$  we denote by  $\delta_n^-$  and  $\delta_n^+$  the set of all incoming and outgoing roads to  $n$ , respectively. In the sequel, we will use the indices  $k$  for the elements of  $\delta_n^-$  and  $j$  for the elements of  $\delta_n^+$ . Moreover, each road  $e$  is modelled by an interval  $I_e := [y_e^l, y_e^r]$ .

### 2.1. The model

We consider a class of “second order” models for traffic flow on road networks given by the “Aw-Rascle” equations [3]. These consist of the conservation of mass:

$$\partial_t \rho_e + \partial_x(\rho_e v_e) = 0, \quad (2.1)$$

where the function  $\rho_e$  describes the car density on road  $e$  and  $v_e$  the velocity. Furthermore, some additional quantities appear in the model. First,  $w_e$  is considered as a car-specific property which influences the velocity and is assumed here to satisfy the relation

$$w_e = v_e + p_e(\rho_e) \quad (2.2)$$

for some known function  $p_e(\rho)$  (“traffic pressure”) with the properties

$$\text{for all } \rho \geq 0: \rho p_e''(\rho) + 2p_e'(\rho) > 0 \text{ and } p_e(\rho) \approx \rho^\gamma, \gamma > 0, \rho \rightarrow 0. \quad (2.3)$$

In contrast with  $w_e$ , on each road  $e$ , we call  $a_e^i$  the local proportion of cars with the commodity  $i$ . Hence, the density of cars with commodity  $i$  is given by  $\rho_e^i := a_e^i \rho_e$ . We assume that we have  $i = 1, \dots, m$  commodities in the network.

Since we implicitly assume that each road is a single lane road, then there is no passing between cars: for all  $i$ ,  $a_e^i$  is thus a Lagrangian variable on each road  $e$ , but in principle has no influence on the velocity  $v_e$ . Let set  $\vec{a}_e := (a_e^i)_{1 \leq i \leq m}$ , then we have

$$\partial_t w_e + v_e \partial_x w_e = 0, \quad \partial_t \vec{a}_e + v_e \partial_x \vec{a}_e = \vec{0}. \quad (2.4)$$

Let us repeat that only  $w_e$  has an influence on  $v_e$  through (2.2), whereas  $\vec{a}_e$  is passively advected on each road and of course modified at each junction. We have the following interpretation in mind: The value of the property  $w_e$  may distinguish cars and trucks having different acceleration and deceleration behavior. On the other hand, the function  $a_e^i$  describes the proportion of  $\rho_e$  at  $(x, t)$  with destination  $i$ . For the latter interpretation we add the constraint

$$\sum_i a_e^i = 1, \quad \forall e \in \delta^\pm. \quad (2.5)$$

For a mathematical analysis we reformulate (2.1), (2.2), (2.4) and (2.5) in conservative form

$$\partial_t \rho_e + \partial_x(\rho_e v_e) = 0, \quad (2.6a)$$

$$\partial_t(\rho_e w_e) + \partial_x(\rho_e v_e w_e) = 0, \quad (2.6b)$$

$$\partial_t(\rho_e \vec{a}_e) + \partial_x(\rho_e v_e \vec{a}_e) = \vec{0}, \quad (2.6c)$$

$$w_e = v_e + p_e(\rho_e), \quad \sum_i a_e^i = 1. \quad (2.6d)$$

We introduce the following notation:  $U_e := (\rho_e, \rho_e w_e, \rho_e \vec{a}_e)^T$  and  $F_e(U_e) = (\rho_e v_e, \rho_e v_e w_e, \rho_e v_e \vec{a}_e)^T$ .

The multiclass version of the original ‘‘Aw-Rascle’’ model [2] was introduced in [3] for a single commodity, i.e.,  $m = 1$ . We extended this to the multicommodity case by introducing the vector  $\vec{a}_e$ . However, the properties of the now  $m + 2 \times m + 2$ -model are clearly similar to the properties of the commodity model introduced in [3]. In particular, for this model, the first characteristic field is genuinely nonlinear whereas the second and the  $m$  fields corresponding to  $a_e^i$  are linearly degenerate. The solution to the Riemann problem associated to the system (2.6) consists of two waves: rarefaction or shock wave associated with the first characteristic field, followed by a contact discontinuities associated with the other characteristic fields. Furthermore, shock curves and rarefaction curves coincide for (2.6). Therefore the system (2.6) belongs to the class of Temple systems [28] and its Riemann invariants are  $v_e, w_e$  and  $\vec{a}_e$ .

### 3. Solution to the Riemann Problem

For later analysis we recall the notion of Riemann problems for (2.6): A Riemann problem of (2.6) is a Cauchy problem with piecewise constant initial data  $U_e(x, 0) = U_l$  for  $x < 0$  and  $U_e(x, 0) = U_r$  for  $x \geq 0$ . We refer the reader to [3] for a general discussion of solutions to Riemann problems associated to (2.6). A derivation of the necessary conditions at the junction, i.e., the coupling conditions, can be found in [10, 16, 17]. We define weak solutions of the network problem in the following sense: A set of functions  $\{U_e = (\rho_e, \rho_e v_e)\}_{e \in \mathcal{E}}$  is called a weak solution of (2.6) if and only if

$$\begin{aligned} & \sum_{e=1}^{\mathbf{E}} \int_0^\infty \int_{y_e^l}^{y_e^r} \begin{pmatrix} \rho_e \\ \rho_e w_e \\ \rho_e \vec{a}_e \end{pmatrix} \cdot \partial_t \phi_e + \begin{pmatrix} \rho_e v_e \\ \rho_e v_e w_e \\ \rho_e v_e \vec{a}_e \end{pmatrix} \cdot \partial_x \phi_e dx dt \\ & + \int_{y_e^l}^{y_e^r} \begin{pmatrix} \rho_{e,0} \\ \rho_{e,0} w_{e,0} \\ \rho_{e,0} \vec{a}_{e,0} \end{pmatrix} \cdot \phi_e(x, 0) dx = 0 \end{aligned} \quad (3.1)$$

holds for any set of  $C^1$ -functions  $\{\phi_e\}_{e \in \mathcal{E}} : I_e \times [0, +\infty[ \rightarrow \mathbb{R}^3$  having compact support. Additionally, the family  $\{\phi_e\}_{e \in \mathcal{E}}$  needs to be continuous across a junction  $n$ , i.e.,

$$\phi_k(y_k^r) = \phi_j(y_j^l) \forall k \in \delta_n^- \text{ and } \forall j \in \delta_n^+. \quad (3.2)$$

Herein,  $U_{e,0}(x) = (\rho_{e,0}(x), (\rho_{e,0} v_{e,0})(x), (\rho_{e,0} v_{e,0} \vec{a}_{e,0})(x))$  is the initial data. Furthermore, the set of functions  $U_e$  satisfies for all  $e$  the relation

$$w_e(x, t) = v_e(x, t) + p_e^\dagger(\rho_e(x, t)), \quad (3.3)$$

where the function  $p_e^\dagger(\cdot)$  is initially unknown.

On an outgoing road its explicitly depends on the mixture of the cars. On any incoming road  $k \in \delta_n^-$  the relation  $p_k^\dagger \equiv p_k$  is valid. The precise definition can be found e.g. in [16] and an explicit construction can be found in [15].

Next, we extend the previous discussion for networks to the system including the property  $\vec{a}_e, \forall e$ . With the previous remarks in mind, we consider a single intersection  $n$  with incoming and outgoing arcs  $\delta_n^-$  and  $\delta_n^+$ , respectively. We assume the initial

data  $U_{e,0}$  for  $e \in \delta_n^- \cup \delta_n^+$  to be constant. At the vertex  $n$  located at  $x = x_0$  we consider the following (half-)Riemann problems

$$\partial_t U_e + \partial_x F_e(U_e) = 0, U_e(x, 0) = \begin{pmatrix} U_e^- & x < x_0 \\ U_e^+ & x > x_0 \end{pmatrix}, \quad (3.4)$$

for each arc  $e \in \delta_n^- \cup \delta_n^+$ .

Depending on the road, only one of the Riemann data in (3.4) is given, namely, if  $e \in \delta_n^- : U_e^- = U_{e,0}, x_0 = y_e^r$  and if  $e \in \delta_n^+ : U_e^+ = U_{e,0}, x_0 = y_e^l$ .

We construct a weak solution to (2.6) such that the generated waves have non-positive speed (resp. non-negative speed) if  $e \in \delta_n^-$  (resp. if  $e \in \delta_n^+$ ). For each road the unknown state  $U_e^+$  (resp.  $U_e^-$ ) for  $e \in \delta_n^-$  (resp.  $e \in \delta_n^+$ ) will be determined such a way that the coupling conditions

$$\sum_{e \in \delta_n^-} (\rho_e v_e)(y_e^r-, t) = \sum_{e \in \delta_n^+} (\rho_e v_e)(y_e^l+, t), \quad (3.5a)$$

$$\sum_{e \in \delta_n^-} (\rho_e v_e w_e)(y_e^r-, t) = \sum_{e \in \delta_n^+} (\rho_e v_e w_e)(y_e^l+, t), \quad (3.5b)$$

$$\sum_{e \in \delta_n^-} (\rho_e v_e \vec{a}_e)(y_e^r-, t) = \sum_{e \in \delta_n^+} (\rho_e v_e \vec{a}_e)(y_e^l+, t), \quad (3.5c)$$

are satisfied. Those coupling conditions are the conservation of the momentum of the conservation laws, i.e., conservation of mass and the ‘‘pseudo’’-momentum  $\rho v w$  as well as the property  $\rho v \vec{a}$ . Even in the case  $\vec{a} \equiv 1, \forall e$ , the conditions (3.5) are not sufficient to obtain a unique family of solutions  $\{U_e\}_{e \in \mathcal{E}}$  at the intersection, see e.g. for the simpler case of a single commodity flow and the Lighthill–Whitham equation [17] and [16] for single commodity Aw–Rascle flow. Therefore, we impose additional constraints which imply (3.5), see the definition of solutions at an intersection below. Of course, there is freedom in the modelling of additional conditions.

For notational convenience we denote by  $q_{jk}$  the initially unknown mass flux of cars going from road  $k$  to road  $j$ . We introduce the quantities  $q_k := \sum_{j \in \delta_n^+} q_{jk}$  and  $q_j := \sum_{k \in \delta_n^-} q_{jk}$  describing the total mass-flux on incoming and outgoing roads  $k$  and  $j$ , respectively. Due to the construction of solutions to the (half-)Riemann problems we have  $\rho_k v_k(y_k^r-, t) = q_k$  for  $k \in \delta_n^-$  and  $\rho_j v_j(y_j^l+, t) = q_j$  for  $j \in \delta_n^+$  and  $t > 0$ .

Now, we present a straightforward approach to obtain suitable coupling conditions first, before turning to more sophisticated examples.

First, we denote by  $\alpha_{jk}$  the percentage of cars on road  $k$  going to road  $j$  at an intersection  $n$ . The corresponding matrix  $A := (\alpha_{jk})_{j \in \delta_n^+, k \in \delta_n^-, \forall n}$  is assumed to be **known**, see [4, 10, 16]. By definition we have

$$\sum_{j \in \delta_n^+} \alpha_{jk} = 1 \forall k \in \delta_n^-. \quad (3.6)$$

Then, we obtain  $\alpha_{jk} = q_{jk}/q_k$  and clearly, any solution  $\{\rho_k v_k\}_{k \in \delta_n^\pm}$  which satisfies

$$q_j = \sum_{k \in \delta_n^-} \alpha_{jk} q_k \equiv \sum_{k \in \delta_n^-} q_{jk} \quad (3.7)$$

also complies with (3.5a).

Next, we introduce the quantity  $\beta_{jk} := q_{jk}/q_j$ . This quantity describes the mixture of cars on the outgoing road. Several approaches exist for determining  $\beta_{jk}$ , but in any case we have  $\sum_{k \in \delta_n^-} \beta_{jk} = 1$ . In [16] the proportions  $\beta_{jk}$  are assumed to be known and an existence result is valid for an arbitrary number of connected roads. On the contrary, in [15] the proportions  $\beta_{jk}$  are obtained from a maximization problem but the result is limited to vertices of degree three. To keep the discussion as simple as possible, we postulate the quantities  $\beta_{jk}$  to be known a priori. More precisely, as in [9], we assume that at any junction  $n$ , all the cars on the incoming roads with destination  $i$  will take the same path and therefore the same outgoing road.

We introduce a set of parameters  $\gamma_j^i, i = 1, \dots, m$ , such that  $\forall j \in \delta_n^+$  we have

$$\gamma_j^i = \begin{cases} 1 & \text{if } j \text{ belongs to a unique "shortest path" to reach the destination } i, \\ 0 & \text{otherwise.} \end{cases} \quad (3.8)$$

The notion of "shortest path" depends on the context under consideration. Therefore, the path can be the shortest according to the distance, the travel cost, the travel time etc... The coefficients  $\gamma_j^i$  can be determined a priori according to the topology of the network, the prescribed behavior of the drivers, etc... In particular, we assume that they are **known** at any junction. A mathematical description is to say that  $j$  belongs to a path  $\mathbb{P}$  where  $\mathbb{P}$  belongs to a predefined set of possible paths in the network.

A path  $P$  in the network  $G$  is a sequence of vertices  $n_l \in \mathcal{N}$ ,  $l = 1, \dots, N-1$  such that  $n_l$  and  $n_{l+1}$  are connected by the directed arc  $e_{l,l+1} \in \mathcal{E}$ . The set of all paths in  $G$  is denoted by  $\mathcal{P}$  and we note  $P^i$  the path that should follow cars of commodity  $i$ .

**REMARK 3.1.** *We emphasize that the restriction to considering the shortest path as condition for defining  $\gamma_j^i$  is for ease of presentation. Other choices are possible, for instance the path with the actual shortest travel time to the destination  $i$ . Then the definition (3.8) becomes more complicated and can vary as time goes.*

*From a computational point of view defining appropriate values  $\gamma_j^i$  at the intersection for a given network is an NP-hard problem.*

Next, we assume that the particular choice of  $\gamma_j^i$  and the proportions  $a_k^i$  define the coefficients  $\alpha_{jk} \beta_{jk}$  at an intersection and the proportions  $a_k^i$  on each outgoing road. We summarize all assumptions on the distribution at an intersection as follows:

**A1** The behavioral rule (3.8) implies a distribution of the flux.

$$\alpha_{jk} = \frac{\sum_{i=1}^m \gamma_j^i a_k^i(y_k^r-, t)}{\sum_{i=1}^m a_k^i} = \sum_{i=1}^m \gamma_j^i a_k^i(y_k^r-, t), \quad \forall k \in \delta_n^-, \quad \forall j \in \delta_n^+. \quad (3.9)$$

Hence, we require that

$$(\rho_j v_j)(y_j^l+, t) = \sum_{k \in \delta_n^-} \alpha_{jk} (\rho_k v_k)(y_k^r+, t) \quad \forall j \in \delta_n^+.$$

**A2** The behavioral rule (3.8) implies a mixture on the outgoing roads.

$$\beta_{jk} = \frac{\sum_{i=1}^m \gamma_j^i a_k^i(y_k^r-, t)}{\sum_{i=1}^m \sum_{e \in \delta_n^-} \gamma_j^i a_e^i(y_e^r-, t)}, \quad \forall k \in \delta_n^-, \quad \forall j \in \delta_n^+. \quad (3.10)$$

Then, we require that (3.13) holds:

$$(w_j)(y_j^l+, t) = \sum_{k \in \delta_v^-} \beta_{jk} w_k(y_k^r-, t).$$

**A3** The behavioral rule (3.8) implies a distribution of the quantities  $a_j^i$ .

$$a_j^i(y_j^l+, t) = \frac{\sum_{k \in \delta_n^-} \gamma_j^i a_k^i(y_k^r-, t)}{\sum_{\sigma=1}^m \sum_{k \in \delta_n^-} \gamma_j^\sigma a_k^\sigma(y_k^r-, t)} \quad \forall j \in \delta_n^+, \quad \forall i \in \{1, \dots, m\}. \quad (3.11)$$

The coupling conditions (3.9) and (3.11) are similar to the ones proposed in [9] where the dynamics on the network are governed by the LWR-equations. Moreover, it is easy to check that

**PROPOSITION 3.1.** *Under the assumptions **A1** – **A3** the coupling conditions (3.5) are satisfied since we have*

$$\sum_{j \in \delta_n^+} \alpha_{jk} = 1; \quad \sum_{k \in \delta_n^-} \beta_{jk} = 1; \quad \sum_{i=1}^m a_j^i = 1. \quad (3.12)$$

From the Lagrangian point of view it is intuitive to prescribe inflow conditions on the outgoing arcs. In fact,  $w_k(y_k^r-, t) = v_{k,0} + p_k(\rho_{k,0}) =: w_k(U_{k,0})$  for all  $t > 0$  and  $k \in \delta_n^-$ . Assuming that (2.6) and (3.10) are satisfied, the quantity

$$w_j(y_j^l+, t) = \bar{w}_j = \sum_{k \in \delta_v^-} \beta_{jk} w_k(U_{k,0}) \quad (3.13)$$

is constant for all  $t > 0$ . However, since the variable  $w$  is coupled by  $p(\rho)$  to the dynamics of the system, one has to be careful on the precise relation of  $w_j$  and the function  $p_j$  for  $j \in \delta_n^+$ . The correct formulation includes the homogenization process. This can be understood as follows: Consider constant initial data  $U_{j,0}$ . Since  $w_l$  is Lagrangian quantity and transported with the velocity  $v_l$ . At an intersection we therefore obtain that  $w_k(t, 0+) = w_{k,0}$  for  $k \in \delta_v^-$ . On the outgoing roads  $j \in \delta_v^+$  we can have mixture of quantities  $w_{k,0}$  with cars from road  $k$  enter road  $j$ . This mixture is described by  $\beta_{jk}$ . This is described in condition (3.13). However, the  $w_j$  is not just a quantity which is transported but has itself an influence on the dynamics since it has to satisfy  $w = v + p(\rho)$ . Hence, the mixture  $\beta_{jk}$  therefore implies new flow conditions  $p_j^\dagger$ . We introduce  $P_j(\xi) = p_j(1/\xi)$ . Then, the new flow conditions in Lagrangian coordinates are  $1/\rho^*(X, t) = \sum_{k \in \delta_v^-} \beta_{jk} P_j^{-1}(w_{k,0} - v^*(X, t))$  which we

choose to rewrite as  $w = v + p_j^\dagger(1/\rho)$  for  $w = \sum_{k \in \delta_v^-} \beta_{jk} w_{k,0} \equiv \bar{w}_j$ . This reformulation holds true for  $w \equiv \bar{w}_j$  only and defines the homogenized  $p_j^\dagger$  for a given mixture rules and initial data  $w_{k,0}$ . The solution on the outgoing road  $j \in \delta_v^+$  finally satisfies  $w_j(t, 0+) = v(t, 0+) + p_j^\dagger(t, 0+) \equiv \bar{w}_j$ . We refer to section 5 and section 6 of [16] for a more precise mathematical derivation of the previous ideas.

We conclude this section with the definition of a solution for the multicommodity network model.

**DEFINITION 3.2.** *Consider a junction  $n$  with an arbitrary number of incoming and outgoing roads and denote by  $\{U_{e,0}\}_{e \in \delta_n^\pm}$  given constant initial data. We are given the coefficients  $\gamma_j^i$  are given (for example determined according to the drivers traveling strategies).*

*We call a family  $\{U_e\}_{e \in \delta_n^\pm}$  an admissible solution at the junction, if and only if it satisfies*

(C1)  $\forall e \in \delta^- \cup \delta^+$ ,  $U_e(x, t)$  is a weak entropy solution in the sense of (3.1) of the network problem (3.3), where  $p_e^\dagger \equiv p_e, \forall e \in \delta_n^-$ .

*On an outgoing road  $j \in \delta^+$ , the solution  $U_j(x, t)$  is constructed as in [15][page 12]: in the triangle  $\{(x, t); y_j^l < x < y_j^l + tv_{j,0}\}$ ,  $U_j$  is the homogenized solution defined below with  $p_j^\dagger \equiv p_j^*$ , whereas for  $x > y_j^l + tv_{j,0}$ ,  $p_j^\dagger \equiv p_j$ .*

(C2) *The flux distribution satisfies*

$$(\rho_j v_j)(y_j^l +, t) = \sum_{k \in \delta_n^-} \alpha_{jk}(t) (\rho_k v_k)(y_k^r +, t) \quad \forall j \in \delta_n^+$$

wherein

$$\alpha_{jk}(t) := \sum_{i=1}^m \gamma_j^i a_k^i(y_k^r -, t), \quad \forall k \in \delta_n^-, \quad \forall j \in \delta_n^+.$$

(C3) *The car properties are mixed according to*

$$(w_j)(y_j^l +, t) = \sum_{k \in \delta_v^-} \beta_{jk}(t) w_k(y_k^r -, t).$$

wherein

$$\beta_{jk}(t) = \frac{\sum_{i=1}^m \gamma_j^i a_k^i(y_k^r -, t)}{\sum_{i=1}^m \sum_{e \in \delta_n^-} \gamma_j^i a_e^i(y_e^r -, t)}, \quad \forall k \in \delta_n^-, \quad \forall j \in \delta_n^+.$$

(C4) *The sum of the incoming fluxes  $\sum_{k \in \delta^-} \rho_k v_k(y_k^r -, t) = \sum_{k \in \delta^-} q_k$  is maximal subject to (C2) and (C3).*

Some remarks are in order. The conditions (C2) and (C3) inspired by the behavioral rule (3.8). In (C1) a weak entropy solution on each arc is a weak entropic solution in the sense of Lax. A family of functions  $\{U_e\}$  satisfies (3.5) a.e. in  $t$  and furthermore  $\sum_{i=1}^m a_e^i = 1$  for all roads  $e$ . Using [16][Theorem 7.1] the existence of solutions to constant initial data  $U_{e,0}$  is straightforward.

#### 4. The coupling conditions: a general framework

In order to treat the case of multicommodity flows at a general junction, we present two distribution schemes of traffic flow based on the approximation of the homogenization coefficients  $\beta_{jk}$  on the outgoing roads (see equation (3.10) for instance).

Furthermore, we compare the numerical results obtained with both schemes in Section 5.

To distribute the vehicle flux at the intersection we propose two approaches: The first one consists of first distributing the incoming flux on the outgoing roads and then merging or homogenizing the flux on the respective outgoing roads, whereas the second approach amounts to do the converse, i.e., we first homogenize all the incoming fluxes and then we distribute the homogenized flux on the outgoing roads.

For both schemes we assume the following rule: at a given junction  $n$  the distribution of the flux on the outgoing roads is induced by the path with the actual shortest traveling time from  $n$  to the different destinations  $i$ . Before we describe the two distribution schemes, let us recall the notion of supply and demand functions.

The demand function  $d_C(\rho)$  and the supply function  $s_C(\rho)$  as introduced in [23] and [6] are the non-decreasing (see Figure 4.1) and the non-increasing (Figure 4.1) parts of the curve  $w = \text{constant}$ , respectively. We remark that for any constant  $cste \geq 0$  there is a level curve  $w = cste$  and the demand  $d = d_C$  and supply  $s = s_C$  function depend on the chosen constant. However, the shape of the demand and supply function are similar to the ones shown in the figure below.

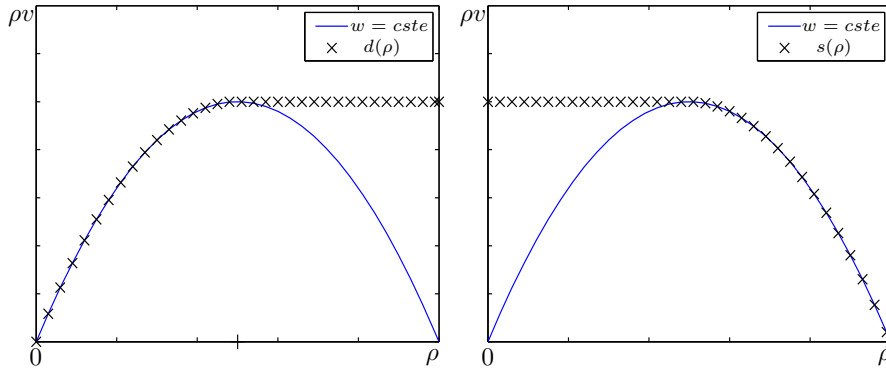


FIG. 4.1. The demand (left) and supply (right) function for a given constant  $cste$

#### 4.1. Scheme 1: Distribution before homogenization

We first distribute the incoming flux on the outgoing roads and then we homogenize the total flux on each outgoing road. We assume that the coefficients  $\gamma_j^i$  are known.

Hence, on each outgoing road  $j \in \delta_n^+$ , near the junction, the property  $w_j$  belongs to the curve  $w_j = \text{const}$  where the constant  $w_j^*$  is given by

$$w_j^*(y_j^l+, t) := \sum_{k \in \delta_n^-} \beta_{jk} w_k(y_k^r-, t), \quad (4.1)$$

where  $\beta_{jk}$  are given by (3.10).

Then the quantities  $a_j^i$  on the outgoing roads are given by (3.11).

For the outflow and the inflow at a given junction, we have the following constraints in terms of the supply and the demand:

$$q_j = \sum_{i=1}^m q_j a_j^i \leq s_j(y_j^l+, t) = s_j \quad (4.2)$$

$$q_k = \sum_{i=1}^m q_k a_k^i \leq d_k(y_k^r-, t) = d_k \quad (4.3)$$

The supply on each outgoing road  $j$  is  $s_j = s_j(y_j^l+, t) = \mathbf{s}(U_j^*)$  where  $U_j^* = (\rho_j^*, v_j^*)$  is the state on the road  $j$  such that:

$$\rho_j^* = p_j^{-1}(w_j^* - v_j^*) \quad \text{and} \quad v_j^* = v_j^+. \quad (4.4)$$

The mentioned approximation consists precisely in replacing the exact equation  $\rho_j^* = (p_j^\dagger)^{-1}(w_j^* - v_j^+)$  where  $p_j^\dagger$  is the homogenized “pressure” see [16, 15], with (4.4).

In order to compute the optimal fluxes that satisfy the Rankine–Hugoniot jump conditions (3.5) on each road at a given junction, we solve the following problem

$$\max \sum_{j \in \delta^-} q_j, \quad (4.5a)$$

$$\text{subject to:} \quad (4.5b)$$

$$0 \leq q_k a_k^i \leq d_k a_k^i, \quad \forall k \in \delta^-, \forall i, \quad (4.5c)$$

$$0 \leq q_j \leq s_j \quad \forall j \in \delta^+, \quad (4.5d)$$

$$\sum_{j \in \delta^+} q_j a_j^i = \sum_{k \in \delta^-} q_k a_k^i, \quad \forall i. \quad (4.5e)$$

Note that here, the coefficient  $\beta_{jk}$  only plays a role through the supply  $s_j$ . The problem is a convex optimization problem on a bounded domain and has therefore at least one solution. Since the mixture rates  $\beta_{jk}$  are fixed by the choice of  $\gamma_{jk}^i$  we conclude by applying [16][Theorem 7.1] that there exists a unique solution. By construction of  $\beta_{jk}$  and  $\alpha_{jk}$ , this solution is also a solution in the sense of definition 3.2.

#### 4.2. Scheme 2: Homogenization before distribution

Contrarily to what we did above, in this approach we first homogenize all the incoming fluxes and then distribute the homogenized flux on the outgoing roads. In other words, we decompose a junction into a juxtaposition of a “merging junction” and a “diverging junction”. Let us label  $z$  the virtual intermediate road which is both the outgoing road for the “merging junction” and the incoming road for the “diverging junction” as depicted in Figure 4.2, see also [24].

Therefore, we have

$$a_z^i = \frac{\sum_{k \in \delta_n^-} a_k^i}{\sum_{o=1}^m \sum_{k \in \delta_n^-} a_k^o}, \quad \forall i = 1, \dots, m, \quad (4.6)$$

$$\beta_{zk} = \frac{\sum_{i=1}^m a_k^i}{\sum_{o=1}^m \sum_{k \in \delta_n^-} a_k^o}, \quad \forall k \in \delta_n^- \quad (4.7)$$

and the quantities  $a_j^i, \forall i=1 \dots m, \forall j \in \delta_n^+$  are determined by (3.11).

The traffic condition on the virtual intermediate road  $z$  is the same on all the outgoing roads nearby the junction and is given by the curve

$$\{w_j^*(y_j^l+, t) = w_z = \sum_{k \in \delta_n^-} \beta_{zk} w_k(y_k^r-, t)\}. \quad (4.8)$$

Therefore, on each outgoing road  $j \in \delta^+$ , the supply  $s_j$  nearby the junction is given by the supply  $\mathbf{s}(U_j^*)$  at the intermediate state  $U_j^* = (\rho_j^*, v_j^*)$  such that:

$$\rho_j^* = p_j^{-1}(w_j^* - v_j^*) \quad \text{and} \quad v_j^* = v_j^+. \quad (4.9)$$

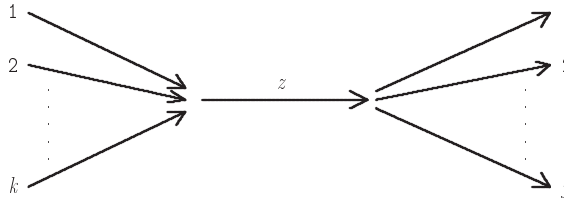


FIG. 4.2. Decomposition of a junction

Instead of solving (4.5) we now consider the following maximization problem:

$$\max \sum_{j \in \delta_n^-} q_j, \quad (4.10a)$$

$$\text{subject to:} \quad (4.10b)$$

$$0 \leq q_k a_k^i \leq d_k a_k^i, \quad \forall k \in \delta^-, \forall i, \quad (4.10c)$$

$$0 \leq q_j \leq s_j \quad \forall j \in \delta^+, \quad (4.10d)$$

$$\sum_{j \in \delta_n^+} q_j a_j^i = \sum_{k \in \delta_n^-} q_k a_k^i, \quad \forall i. \quad (4.10e)$$

As in Subsection 4.1 the solution to (4.10) enables us to define a solution at an intersection in the sense of Definition 3.2.

REMARK 4.1. Note that although the optimization problems (4.5) and (4.10) formally look alike, they yield in general different optimal solutions. The supply  $s_j$  differs in the formulations due to the changes in the definitions of  $w_j^*$  in (4.1) and in (4.8). In (4.1) the quantities  $\beta_{jk}$ , cf. (3.10), are used, whereas in (4.8) we use  $\beta_{zk}$  which are given in (4.7). This leads to different values  $\rho_j^*$  in (4.4) and (4.9), respectively.

### 4.3. Coupling conditions for the LWR multicommodity model

In Section 5, we include a brief description of the coupling conditions we use for the LWR multiclass model [9] given by

$$\partial_t \rho_e + \partial_x v_e(\rho_e) \rho_e = 0, \quad (4.11a)$$

$$\partial \vec{a}_e + v_e \partial_x \vec{a}_e = 0, \quad (4.11b)$$

where the flux function  $f(\rho) := v_e(\rho_e) \rho_e$  is a strictly concave function with a single maximum. The demand function is defined as the non decreasing part of the flux-function  $f$ . The supply function is the non increasing part of the flux function  $f$ . We consider here the following setup of a diverging junction with two commodities. Then as in [4, 9, 17] we obtain a solution satisfying the coupling conditions (3.5a) and (3.5c) by considering the following optimization problem

$$\begin{aligned} & \max f(\rho_1^r), \\ & \text{subject to:} \\ & 0 \leq f(\rho_1^r) \leq d_1, \\ & 0 \leq \alpha_{21} f(\rho_1^r) \leq s_2, \\ & 0 \leq (1 - \alpha_{21}) f(\rho_1^r) \leq s_3. \end{aligned}$$

Here  $\rho_1^r$  denotes the right state in the half-Riemann problem on the incoming road,  $d_1 = d_1(\rho_l)$  denotes the demand on the incoming road 1 at the given state  $\rho_l$ ,  $s_2 = s_2(\rho_2^r)$  and  $s_3 = s_3(\rho_3^r)$  denote the supplies for the given right states  $\rho_2^r$  and  $\rho_3^r$  on the outgoing roads 2 and 3 respectively. On the one hand, we can determine the value  $\alpha_{21}$  using the coupling condition (3.9) as for the AR model, which enables us to compare the two different models.

Next, we slightly adjust our notation considering now distribution rates  $\gamma_{jk}^i$  for distributing the  $i$ th commodity coming from road  $k$  and moving to road  $j$ . Clearly, this is just another convenient way to rewrite the path condition (3.8).

$$\alpha_{jk} = \frac{\sum_{i=1}^m \gamma_{jk}^i a_k^i(y_k^r, t)}{\sum_{i=1}^m a_k^i} = \sum_{i=1}^m \gamma_{jk}^i a_k^i(y_k^r, t).$$

For the computations in Section 5 we choose a value for  $\gamma_{21}^1$  and  $a_1^1$ . Then  $\gamma_{31}^1$  and  $a_1^2$  are determined. It was shown in [4] that the optimization problem under consideration is well posed for all  $\alpha_{21} \in [0, 1]$  and constant left and right states  $\rho_1^l, \rho_2^r, \rho_3^r$ . The main difference between our approach and the one in [9] is that here the path from the source to the destination is not fixed in advance because of the variation of the control parameters  $\gamma_{jk}^i$ . Therefore drivers are allowed to change their itinerary according to the traffic conditions.

## 5. Numerical results

In this section we present numerical simulations for the different schemes constructed above for the AR model and the LWR model. First we solve the Riemann problem for the multicommodity AR model with some given initial data using the distribution

schemes proposed in subsections 4.1 and 4.2 respectively and then we compare the obtained results. In the second part of this section, we simulate the distribution scheme for the multicommodity LWR model presented in subsection 4.3. Moreover, we compare the results of the AR multicommodity model with those obtained by the multicommodity LWR model for the same initial and distribution settings.

### 5.1. Investigation of the different distribution schemes

In this subsection we compare the results obtained by the distribution schemes presented in subsections 4.1 and 4.2 respectively. To this end we consider the junction depicted in Figure 5.1. We assume only two commodities  $i = 1$  and  $i = 2$  on the incoming roads 1 and 2. To simplify the presentation, we assume that the outgoing road 3 belongs to the “shortest” path for commodity 1 and that drivers with commodity 2 can reach their destination faster via the outgoing road 4. Consequently, we require  $a_3^1 = 1, a_3^2 = 0, a_4^1 = 0$  and  $a_4^2 = 1$ . In all our simulations we obtain this result for the values of quantities  $a_e^i$  independently of the used distribution scheme and the used model. The controls corresponding to the defined “shortest” paths read

$$\gamma_{13}^1 = 1, \quad \gamma_{13}^2 = 0, \quad \gamma_{23}^1 = 1, \quad \gamma_{23}^2 = 0.$$

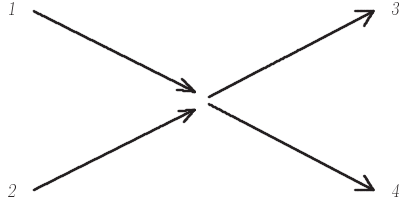


FIG. 5.1. A junction with two incoming and two outgoing roads.

For some given initial values of  $a_k^i$ ,  $k, i = 1, 2$ , we define controls on the outgoing roads

$$\alpha_{3k} := \gamma_{k3}^1 a_k^1 + \gamma_{k3}^2 a_k^2.$$

The LWR coupling conditions are then

$$\begin{aligned} f_3(\rho_3^-) &= \alpha_{31} f_1(\rho_1^+) + \alpha_{32} f_2(\rho_2^+), \\ f_4(\rho_4^-) &= (1 - \alpha_{31}) f_1(\rho_1^+) + (1 - \alpha_{32}) f_2(\rho_2^+). \end{aligned}$$

For the LWR model we solve for some given states  $\rho_1^-, \rho_2^-, \rho_3^+$  and  $\rho_4^+$  the following optimization problem

$$\max q_1 + q_2, \tag{5.1a}$$

subject to:

$$\alpha_{31} q_1 + \alpha_{32} q_2 \leq s_3(\rho_3^+), \tag{5.1b}$$

$$(1 - \alpha_{31}) q_1 + (1 - \alpha_{32}) q_2 \leq s_4(\rho_4^+), \tag{5.1c}$$

$$q_1 \leq d_1(\rho_1^-), \tag{5.1d}$$

$$q_2 \leq d_2(\rho_2^-), \tag{5.1e}$$

where the  $s_3$  and  $s_4$  are the supplies on the outgoing roads and  $d_1$  and  $d_2$  are the demands on the incoming roads. In the sequel, we compare the results of solutions at a junction for the LWR optimization problem with those obtained by the distribution schemes for the AR model presented in subsections 4.1 and 4.2. We pose for the LWR model  $v(\rho) = 2 \cdot (1 - \rho)$  and for the AR model we set  $p_e(\rho) = \rho$ ,  $\forall e$ . The initial densities are the following

$$\rho_1^- = 0.6, \quad \rho_2^- = 0.7, \quad \rho_3^+ = 0.5, \quad \rho_4^+ = 0.4,$$

therefore

$$v_1^- = 0.8, \quad v_2^- = 0.6, \quad v_3^+ = 1, \quad v_4^+ = 1.2.$$

For the AR model, on the incoming roads 1 and 2 we have

$$w_1 = w_1^- = v_1^- + p_1(\rho_1^-), \quad w_2 = w_2^- = v_2^- + p_2(\rho_2^-).$$

The the LWR model the fundamental diagram is identical on each road and is given by the function

$$\rho \mapsto \rho v(\rho) = 2(\rho - \rho^2). \quad (5.2)$$

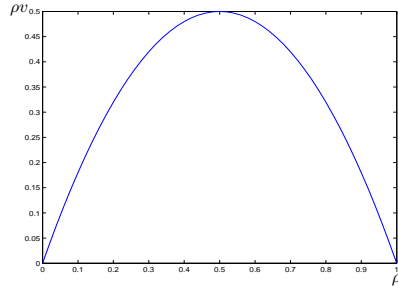


FIG. 5.2. The fundamental diagram for the LWR model on each road anywhere.

Whereas for the AR model it depends on the actual value of  $w$  on the considered road. Therefore, on an incoming road  $k$ ,  $k=1,2$  we have

$$\rho \mapsto \rho w_k - \rho p_k(\rho) = \rho w_k - \rho^2, \quad (5.3)$$

and on an outgoing road  $j$ ,  $j=3,4$  **far away from the junction** we have

$$\rho \mapsto \rho w_j - \rho p_j(\rho) = \rho w_j - \rho^2. \quad (5.4)$$

However on an outgoing road  $j$ ,  $j=3,4$  **nearby the junction**, due to the traffic homogenization, the fundamental diagram is given by

$$\rho \mapsto \rho w_j^* - \rho p_j(\rho) = \rho w_j^* - \rho^2, \quad (5.5)$$

where  $w_j^* = \sum_{k=1}^2 \beta_{jk} w_k$  with  $\beta_{jk}$  obtained by (3.10).

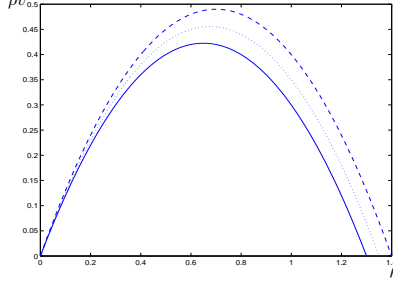


FIG. 5.3. Fundamental diagrams for the AR model: on the incoming road 1, the dashed line; on the incoming road 2, the solid line; on an outgoing road nearby the junction, the fundamental diagram follows from the homogenization of traffics on incoming roads and therefore lies between the fundamental diagrams on incoming roads, the dotted line.

The values of the quantities  $a_k^i$  on the incoming roads 1 and 2 are for the first and second test, respectively,

$$\text{Test 1. } a_1^1 = 0.6, \quad a_1^2 = 0.4, \quad a_2^1 = 0.3, \quad a_2^2 = 0.7 \quad (5.6)$$

$$\text{Test 2. } a_1^1 = 0.2, \quad a_1^2 = 0.8, \quad a_2^1 = 0.9, \quad a_2^2 = 0.1 \quad (5.7)$$

We summarize the results for the LWR model and the AR model (the schemes *homogenize then distribute* (HD) and *distribute then homogenize* (DH)) in Table 5.1.

	LWR	HD	DH		LWR	HD	DH
$q_1$	0.5	0.36	0.36	$q_1$	0.5	0.48	0.48
$q_2$	0.42857	0.4225	0.4225	$q_2$	0.4	0.376	0.4
$q_3$	0.42857	0.34275	0.34275	$q_3$	0.5	0.4344	0.455625
$q_4$	0.5	0.43975	0.43975	$q_4$	0.4	0.4216	0.424

TABLE 5.1. Solution of the optimization problems for the different models and schemes. The left table corresponds to the initial data (5.6) and the right one to (5.7)

We observe that in the three cases we have flow conservation through the junction:  $q_1 + q_2 = q_3 + q_4$ . For the LWR model  $q_1 = f(\rho_1^+)$  and  $\rho_1^+$  is obtained in a way that the wave connecting  $\rho_1^-$  and  $\rho_1^+$  has a non positive speed, cf. [4]. In Table 5.2 we report the density values corresponding to the fluxes of Table 5.1. Although the flux for the distribution schemes HD and DH are the same in both test cases whereas the corresponding densities are not always the same.

	LWR	HD	DH		LWR	HD	DH
$\rho_1^+$	0.5	0.34	0.34	$\rho_1^+$	0.5	0.6	0.6
$\rho_2^+$	0.69	0.7	0.7	$\rho_2^+$	0.6	0.8656	0.8
$\rho_3^-$	0.311	0.33	0.34	$\rho_3^-$	0.5	0.318	0.35
$\rho_3^*$	–	0.36	0.35	$\rho_3^*$	–	0.318	0.35
$\rho_4^-$	0.5	0.586	0.549	$\rho_4^-$	0.3	0.448	0.497
$\rho_4^*$	–	0.136	0.15	$\rho_4^*$	–	0.18	0.15

TABLE 5.2. Complete solution to the Riemann problem at the junction depicted Figure 5.1. The left table captures the results for the initial data (5.6), the right one for (5.7).

In the first test setup the results from the LWR and the AR model are rather similar. We have the same traffic behavior. However in the second example we find a considerable disagreement. In the AR model we have a jam on road 2 indicated by a density  $\rho_2^+ \geq 0.8 > 0.7 = \rho_2^-$  whereas in the LWR model the traffic still flows without backing up. That means the dynamics of multicommodity flows depends strongly on the used model. Additionally, in Table 5.2 we show that the mixing strategies at an intersection with more than three roads influences the solution to the Riemann problem.

## 5.2. Simulations on a test network

We now consider the network depicted in Figure 5.4. We simulate for this network, the dynamics for a constant inflow using the LWR and the AR multicommodity, respectively. The two distribution schemes for the AR model do not produce different results since the network under consideration has only nodes of at most degree three. For the LWR model we assume that  $v(\rho) = 4 \cdot (1 - \rho)$ .

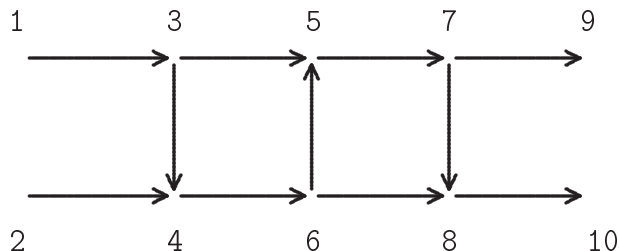


FIG. 5.4. Example of road network with 4 commodities

We have only 2 commodities for the network considered in Figure 5.4 and they are summarized in Table 5.3. We also assume that the “shortest” paths are always the same and given by the node sequence stated in the third column of Table 5.3:

Commodities	“Shortest” Paths
Commodity 1: Vehicles with destination 9	1-3-5-7-9, 2-4-6-5-7-9
Commodity 2: Vehicles with destination 10	1-3-4-6-8-10, 2-4-6-8-10

TABLE 5.3. Commodities for the network given in Figure 5.4.

Due to the topology of the network of Figure 5.4 and the assumption in Table 5.3

among the control parameters  $\gamma$  only four can be chosen freely for instance see the Table 5.4 below.

$$\begin{aligned}\gamma_{(1,3)(3,4)}^1 &\in [0, 1] \\ \gamma_{(1,3)(3,4)}^2 &\in [0, 1] \\ \gamma_{(4,6)(6,5)}^1 &\in [0, 1] \\ \gamma_{(4,6)(6,5)}^2 &\in [0, 1]\end{aligned}$$

TABLE 5.4. Controls for the network in Figure 5.4.

In particular, we can choose the value 0 or 1 for these controls. To model the shortest paths indicated in Table 5.3 we set  $\gamma_{(1,3)(3,4)}^1 = \gamma_{(4,6)(6,5)}^2 = 0$  and  $\gamma_{(1,3)(3,4)}^2 = \gamma_{(4,6)(6,5)}^1 = 1$ .

For the simulation, we assume that the network is initially empty and the inflows on the incoming roads (1,3) and (2,4) are constant in time and given by  $\rho_{(1,3)}(y_{(1,3)}^l, t) = 0.2$  and  $\rho_{(2,4)}(y_{(2,4)}^l, t) = 0.25$ , respectively. Furthermore we assume that the supply in infinity on the outgoing roads (7,9) and (8,10), respectively. For the quantities  $a_e^i$ , we assume (initially and over time) the parameters given in Table 5.5

Commodity 1	Commodity 2
$a_{(1,3)}^1 = 0.7$	$a_{(1,3)}^2 = 0.3$
$a_{(2,4)}^1 = 0.4$	$a_{(2,4)}^2 = 0.6$

TABLE 5.5. Initial values of the quantities  $a_e^i$  for the network given in Figure 5.4.

In Table 5.6 we compare the density and the quantities  $a_e^i$  distributions in the steady state at  $T = 4$  for selected roads.

Arc ( $e$ )	$\rho_e$		$a_e^1$		$a_e^2$	
	LWR	AR	LWR	AR	LWR	AR
(1,3)	0.2	0.2	0.7	0.7	0.3	0.3
(2,4)	0.25	0.25	0.4	0.4	0.6	0.6
(4,6)	0.38	0.1198	0.32	0.2	0.68	0.8
(3,5)	0.1285	0.1373	1	1	0	0
(6,5)	0.08167	0.0233	1	1	0	0

TABLE 5.6. Comparison at  $T = 4$  for the simulation with the LWR model and AR model for the network in Figure 5.4.

As in the previous section the models produce different results in particular for roads (4,6), (3,5) and (6,5).

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