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Fluctuation relations for semiclassical single-mode laser

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Abstract. Over the last few decades, the study of laser fluctuations has shown that laser theory may be regarded as a prototypical example of a nonlinear nonequilibrium problem. The present paper discusses the fluctuation relations, recently derived in nonequilibrium statistical mechanics, in the context of the semiclassical laser theory.

Keywords: stationary states, diffusion

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Contents

1. Introduction

Nonequilibrium statistical mechanics aims at a statistical description of closed and open systems evolving under the action of time-dependent conservative forces or under time-independent or time-dependent non-conservative ones. *Fluctuation relations* are robust identities involving the statistics of entropy production or performed work in such systems. They hold arbitrarily far from thermal equilibrium, reducing close to equilibrium to Green–Kubo or fluctuation–dissipation relations usually obtained in the scope of linear response theory [12, 20, 25, 13, 6, 3]. In a previous paper [2], we presented a unified approach to fluctuation relations in classical nonequilibrium systems described by diffusion processes. We traced the origin of different fluctuation relations to the freedom of choice of the time inversion. The purpose of this paper is to illustrate the results of [2] for the example of a phenomenological model of a laser described by a stochastic differential equation. The semiclassical theory of lasers describes the regime where, due to a large number of photons in the laser cavity, one may treat the electrical field classically, but the two-level atoms are treated quantum mechanically [21, 11]. The dynamical behavior of a single-mode laser is then described by the equation of motion for the complex amplitude of the electric field E_t:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (a_t - bE\bar{E})E, \tag{1}$$

where $\bar{E}_t$ is the complex conjugate of E_t. The function a_t is called the net gain coefficient and it takes into account the coherent emission and absorption of atoms and the losses. In the general case, a_t may have an explicit dependence on time. b is called the self-saturation coefficient. In most instances, it has a positive real part. There exist cases (with an absorber) [22] where b has a negative real part, but we shall not consider them below. If the resonance frequency $\omega_{\rm c}$ of the laser cavity and the atomic frequency $\omega_{\rm a}$ are exactly tuned then both a_t and b are real. In the case of detuning [23], a_t and b are both complex. The equation of motion (1) describes the dynamical behavior of the laser field in a completely deterministic manner with the properties like coherence or spectral width lying outside the domain of the theory. The key to an understanding of such questions resides in the fluctuations of the electric field which are caused by random spontaneous atomic emissions. Such fluctuations may be accounted for by replacing equation (1) by the stochastic differential equation

$$\frac{{\rm d}E}{{\rm d}t} = (a_t - bE\bar{E})E + \eta(t, E, \bar{E}), \tag{2}$$

with the noise $\eta(t, E, \bar{E})$ mimicking the effect of the random spontaneous emission of atoms in other modes, a purely quantum effect neglected in semiclassical theory, but also the effect of vibrations of the cavity [21, 11]. We shall take $\eta(t, E, \bar{E})$ as a random Gaussian field with zero mean and delta-correlated in time. In the following, we shall look at two possible forms for η, one additive and the other one multiplicative. The present paper is organized as follows. In section 2, we recall the main results of [2]. In section 3.1, we study the most elementary model of lasers—the stationary tuned laser with an additive noise—and show that its dynamics satisfies the detailed balance. Section 3.2 is devoted to the fluctuation relations for a non-stationary tuned laser. In section 4.1, we examine the case of a stationary laser with detuning. The detailed balance is broken here, but we show that its slight generalization, the modified detailed balance, still holds. In section 4.2, we study the non-stationary detuned case. In section 5, we look at a slightly different case with multiplicative noise.

2. Fluctuation relation in diffusive systems

In [2], we dealt with arbitrary diffusion processes in $\boldsymbol{R}^{\boldsymbol{d}}$ defined by a stochastic differential equation (SDE):

$$\dot{x} = u_t(x) + v_t(x), \tag{3}$$

where $\dot{x} \equiv {\rm d}x/{\rm d}t$ and, on the right-hand side, $u_t(x)$ is a time-dependent deterministic vector field (a drift), and $v_t(x)$ is a Gaussian random vector field with mean zero and covariance:

$$\left\langle v_t^i(x) v_s^j(y) \right\rangle = \delta(t-s) D_t^{ij}(x, y). \tag{4}$$

For the process solving the SDE (3) defined using the Stratonovich convention, we showed a detailed fluctuation relation (DFR):

$$\mu_0({\rm d}x)\, P_{0,T}(x; {\rm d}y, {\rm d}W)\, \exp(-W) = \mu_0'({\rm d}y^*)\, P_{0,T}'(y^*; {\rm d}x^*, {\rm d}(-W)), \tag{5}$$

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where:

- $\mu_0(\mathrm{d}x) = \exp(-\varphi_0(x))\,\mathrm{d}x$ is the initial distribution of the original (forward) process,
- $\mu'_0(\mathrm{d}x) = \exp(-\varphi'_0(x))\,\mathrm{d}x$ is the initial distribution of the backward process obtained from the forward process by applying a time inversion (see below),
- $P_{0,T}(x;\mathrm{d}y,\mathrm{d}W)$ is the joint probability distribution of the time T and position x_T of the forward process starting at time zero at x and of the functional $W_T[x]$ of the process (to be given later) that has the interpretation of the entropy production.
- $P'_{0,T}(x;\mathrm{d}y,\mathrm{d}W)$ is the similar joint probability distribution for the backward process.

The key behind the DFR (5) is the action of the time inversion on the forward system. First, the time inversion acts on time and space by an involutive transformation $(t, x) \to (t^* = T - t, x^*)$. Second, to recover a variety of fluctuation relations discussed in the literature [16, 17, 4, 5, 15, 24, 1, 8, 9], we allow for a non-trivial behavior of the drift u_t under the time inversion, dividing it into two parts:

$$u_t = u_{t,+} + u_{t,-} \tag{6}$$

with $u_{t,+}$ transforming as a vector field under time inversion, i.e. $u'^{i}_{t^*,+}(x^*) = +(\partial_k x^{*,i})(x)\,u^k_{t,+}(x)$, and $u_{t,-}$ transforming as a pseudo-vector field, i.e. $u'^{i}_{t^*,-}(x^*) = -(\partial_k x^{*,i})(x)\,u^k_{t,-}(x)$. The random field v_t may be transformed with either rule: $v'^{i}_{t^*}(x^*) = \pm(\partial_k x^{*,i})(x)v^k_t(x)$. By definition, the backward process then satisfies the SDE:

$$\dot{x} = u'_t(x) + v'_t(x) \tag{7}$$

taken again with the Stratonovich convention. The functional W_T which appears in the DFR depends explicitly on the functions φ_0, φ'_0 and on the time inversion and has the explicit form

$$W_T = \Delta_T\varphi + \int_0^T J_t\,\mathrm{d}t, \tag{8}$$

where $\Delta_T\varphi = \varphi_T(x_T) - \varphi_0(x_0)$ with

$$\mu_T(\mathrm{d}x) \equiv \exp(-\varphi_T(x))\,\mathrm{d}x \equiv \exp(-\varphi'_0(x^*))\,\mathrm{d}x^* = \mu'_0(\mathrm{d}x^*), \tag{9}$$

and where

$$J_t = 2\widehat{u}_{t,+}\cdot d_t^{-1}(x_t)(\dot{x}_t - u_{t,-}(x_t)) - \nabla\cdot u_{t,-}(x_t) \tag{10}$$

with $d_t(x) = D_t(x,x)$ and $\widehat{u}^i_{t,+} = u^i_{t,+} - \frac{1}{2}\partial_{y^i}D^{ij}_t(x,y)|_{y=x}$. The time integral in equation (8) should be taken in the Stratonovich sense.

The measures μ_0 and μ'_0 in the DFR (5) do not have to be normalized or even normalizable. If they are, then distributing the initial points of the forward and the backward processes with probabilities $\mu_0(\mathrm{d}x)$ and $\mu'_0(x)$, respectively, we may define the averages

$$\langle F\rangle = \int \mu_0(\mathrm{d}x)\,\boldsymbol{E}_x\,F[x], \qquad \langle F\rangle' \equiv \int \mu'_0(\mathrm{d}x)\,\boldsymbol{E}'_x\,F[x], \tag{11}$$

where $\boldsymbol{E}_x$ ($\boldsymbol{E}'_x$) stands for the expectation value for the forward (backward) process starting at x. From the DFR one may derive a generalization of the celebrated Jarzynski

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equality [14, 15]:

$$\langle \exp(-W_T) \rangle = 1, \tag{12}$$

which may be viewed as an extension of the fluctuation–dissipation theorem to the situations arbitrarily far from equilibrium. Note that the relation (12) implies the inequality $\langle W_T \rangle \geq 0$.

To reformulate the DFR in a form where the entropic interpretation of W_T is clearer, consider the probability measures $M[\mathrm{d}x]$ and $M'[\mathrm{d}x]$ on the spaces of trajectories of the forward and the backward process, respectively, such that

$$\langle F \rangle = \int F[x]\, M[\mathrm{d}x], \qquad \langle F \rangle' = \int F[x]\, M'[\mathrm{d}x]. \tag{13}$$

The DFR may be reformulated in the Crooks form [5] as the identity

$$\langle F \exp(-W_T) \rangle = \langle \widetilde{F} \rangle', \tag{14}$$

where $\widetilde{F}[x] = F[\widetilde{x}]$ with $\widetilde{x}_t = x^*_{T-t}$, and the relation (14) implies the equality

$$\widetilde{M'}[\mathrm{d}x] = \exp(-W_T[x])\, M[\mathrm{d}x], \tag{15}$$

for the trajectory measures with $\widetilde{M'}[\mathrm{d}x] = M'[\mathrm{d}\widetilde{x}]$. By introducing the relative entropy $S(M|\widetilde{M'}) = \int \ln(M[\mathrm{d}x]/(\widetilde{M'}[\mathrm{d}x]))\, M[\mathrm{d}x]$ of the measure $\widetilde{M'}$ with respect to M, we infer that

$$\langle W_T \rangle = S(M|\widetilde{M'}). \tag{16}$$

Thus the inequality $\langle W_T \rangle \geq 0$ follows also from the positivity of relative entropy. One may postulate that $\int_0^T \langle J_t \rangle\, \mathrm{d}t$ describes the mean entropy production in the environment modeled by the stochastic noise:

$$\int_0^T \langle J_t \rangle \ \mathrm{d}t = \Delta_T S_{\mathrm{env}}. \tag{17}$$

This is coherent with the previous result and particular cases, see [7, 10, 18]. We may then interpret $\int_0^T \ J_t\, \mathrm{d}t$ as the fluctuating entropy production in the environment. An easy calculation leads to the relation

$$\langle W_T \rangle = S(\hat{\mu}_T) - S(\mu_0) + \Delta_T S_{\mathrm{env}} + S(\hat{\mu}_T|\mu_T), \tag{18}$$

where $\hat{\mu}_t(\mathrm{d}x) = \exp(-\hat{\varphi}_t(x))\, \mathrm{d}x$ is the measure describing the time t distribution of the forward process if its initial distribution were $\mu_0(\mathrm{d}x)$. $S(\hat{\mu}_t) = \int \hat{\varphi}_t(x)\hat{\mu}_t(\mathrm{d}x)$ is the mean instantaneous entropy of the forward process x_t and $S(\hat{\mu}_T) - S(\mu_0)$ is its change over time T. We could interpret $\hat{\varphi}_t(x_t)$ as the fluctuating instantaneous entropy. In general, $\hat{\mu}_T$ is not linked to μ_T of formula (9). The relative entropy $S(\hat{\mu}_T|\mu_T)$ is a penalty due to the use at time T of a measure different than $\hat{\mu}_T$. In the case where $\hat{\mu}_T = \mu_T$, $\langle W_T \rangle$ is the mean entropy production in the system and environment during time T and we could interpret

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W_T as the corresponding fluctuating quantity. After a simple calculation [19], one gets

$$\Delta_T S_{\rm env} = \int_0^T \langle J_t\rangle\,{\rm d}t = \int_0^T {\rm d}t \int [2\widehat{u}_{t,+}(x)\cdot d_t^{-1}(x)(\hat{\jmath}_t(x){\rm d}x - u_{t,-}(x)\hat{\mu}_t({\rm d}x)) \\ - (\nabla\cdot u_{t,-})(x)\hat{\mu}_t({\rm d}x)], \tag{19}$$

$$S(\hat{\mu}_T) - S(\mu_0) = \int_0^T {\rm d}t \int \hat{\jmath}_t(x)\cdot\nabla\hat{\varphi}_t(x)\,{\rm d}x, \tag{20}$$

where $\hat{\jmath}_t$ is the probability current at time t with the components

$$\hat{\jmath}_t^i = (\widehat{u}_t^i - \tfrac{1}{2}d_t^{ij}\partial_j)\exp(-\hat{\varphi}_t) \tag{21}$$

that satisfies the continuity equation

$$\partial_t\exp(-\hat{\varphi}_t) + \partial_i\hat{\jmath}_t^i = 0.$$

We shall now apply these results to three type of semiclassical single-mode laser.

3. Tuned laser with additive noise

3.1. Stationary case

Let us consider the most common model of a stationary laser with no detuning and with an additive form of the noise [21, 11]. Its dynamics is described by the SDE

$$\frac{{\rm d}E}{{\rm d}t} = (a - bE\bar{E})E + \eta, \tag{22}$$

with a and b real, $b > 0$, and with white noise η with mean zero and covariance

$$\begin{aligned} \langle\eta_t\bar{\eta}_{t'}\rangle &= D\,\delta(t-t'),\\ \langle\eta_t\eta_{t'}\rangle &= \langle\bar{\eta}_t\bar{\eta}_{t'}\rangle = 0. \end{aligned} \tag{23}$$

We can write the covariance matrix in the $(E, \bar{E})$ space as

$$d = D\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}. \tag{24}$$

Equation (22) then has the form of the Langevin equation describing equilibrium dynamics of the process $\boldsymbol{E}_t = (E_t, \bar{E}_t)$:

$$\frac{{\rm d}\boldsymbol{E}}{{\rm d}t} = -\frac{1}{2}d\,\boldsymbol{\nabla} H_{ab} + \boldsymbol{\eta} \tag{25}$$

for $H_{ab}(\boldsymbol{E}) = (1/D)[b(E\bar{E})^2 - 2aE\bar{E}]$. The Einstein relation is satisfied for the inverse temperature equal to 1, implying that the Gibbs measure

$$\mu_{ab}({\rm d}\boldsymbol{E}) = Z_{ab}^{-1}\exp(-H_{ab}(\boldsymbol{E})\,{\rm d}\boldsymbol{E} \tag{26}$$

is invariant, has a vanishing probability current $\boldsymbol{j}$ and satisfies the detailed balance:

$$\mu_{ab}({\rm d}\boldsymbol{E}_0)\;P_{0,T}(\boldsymbol{E}_0;{\rm d}\boldsymbol{E}) = \mu_{ab}({\rm d}\boldsymbol{E})\;P_{0,T}(\boldsymbol{E};{\rm d}\boldsymbol{E}_0). \tag{27}$$

This relation is a particular case of the detailed fluctuation relation (5) where the time inversion acts trivially in the spatial sector, i.e. $\boldsymbol{E}^* = \boldsymbol{E}$, the pseudo-vector part of the drift is taken as zero, and we start with the Gibbs measure μ_{ab} for the forward and backward processes. In this case both processes have the same distribution and $W_T \equiv 0$. The relation (27) may be projected to the one for the process $I_t = E_t\bar{E}_t$ describing the intensity of the laser:

$$\mu_{ab}(\mathrm{d}I_0)\; P_{0,T}(I_0;\mathrm{d}I) = \mu_{ab}(\mathrm{d}I)\; P_{0,T}(I;\mathrm{d}I_0). \tag{28}$$

The fluctuating entropy production in the environment may be identified with the heat production $\Delta_T Q$ which is a state function here:

$$\Delta_T Q = \int_0^T J_t\,\mathrm{d}t = -H(\boldsymbol{E}_T) + H(\boldsymbol{E}_0). \tag{29}$$

This relation is the first law of thermodynamics in the case with no work applied to the system. If we start with the Gibbs density then the mean entropy production in the environment $\Delta_T S_{\mathrm{env}} = \langle \Delta_T Q\rangle$ vanishes (19) as well as the instantaneous entropy production and W_T. If the process starts with an arbitrary measure $\mu_0(\mathrm{d}\boldsymbol{E})$ then at subsequent times the measure is

$$\hat{\mu}_t(\mathrm{d}\boldsymbol{E}) = \int \mu_0(\mathrm{d}\boldsymbol{E}_0)\; P_{0,t}(\boldsymbol{E}_0;\mathrm{d}\boldsymbol{E}) \tag{30}$$

converging at long times to the invariant measure $\mu_{ab}(\mathrm{d}\boldsymbol{E})$. During this process the mean rate of heat production $\langle q_t\rangle$ in the environment is (19)

$$\langle q_t\rangle = \langle J_t\rangle = -\int (\boldsymbol{\nabla} H_{ab}\cdot \hat{\boldsymbol{j}}_t)(\boldsymbol{E})\,\mathrm{d}\boldsymbol{E}. \tag{31}$$

After an integration by parts, this may be written as

$$\langle q_t\rangle = -\int H_{ab}(\boldsymbol{E})\partial_t\hat{\mu}_t(\mathrm{d}\boldsymbol{E}). \tag{32}$$

3.2. Non-stationary case

3.2.1. Non-stationary net gain coefficient. Let us consider now the SDE

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (a_t - bE\bar{E})E + \eta \tag{33}$$

with an explicit time dependence for the (real) net gain coefficient a_t, with $b > 0$, and with the white noise η as before. The explicit time dependence a_t may result from an external manipulation. In the matrix notation, the last equation takes the form

$$\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} = -\frac{1}{2}d\boldsymbol{\nabla} H_t + \boldsymbol{\eta} \tag{34}$$

with $H_t \equiv H_{a_t b}$. Here, we are outside the scope of the detailed balance and we enter into the world of transient fluctuation relations. To find an interesting DFR in this case, let us search for an appropriate time inversion. For example, we may impose that the backward process is still described by a Langevin equation but with the Hamiltonian $H'_t(\boldsymbol{E}) = H_{t^*}(\boldsymbol{E}^*)$. By assuming a linear relation $\boldsymbol{E}^* = M\boldsymbol{E}$ and by transforming the drift with the vector rule, we obtain for the drift of the backward process the relation

$$u'_t(x) = -\tfrac{1}{2}M\,\mathrm{d}M^T\,(\boldsymbol{\nabla} H'_t)(\boldsymbol{E}). \tag{35}$$

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To ensure that $M\,\mathrm{d}M^T = d$, we shall take $M = 1$ or $M = D^{-1}d$, i.e. $\boldsymbol{E}^* = \boldsymbol{E}$ or $\boldsymbol{E}^* = \bar{\boldsymbol{E}} = (\bar{E}, E)$. In these two cases, $H_t(\boldsymbol{E}^*) = H_t(\boldsymbol{E})$ so that $H_t'(\boldsymbol{E}) = H_{t^*}(\boldsymbol{E})$ and the backward process satisfies the same SDE as the forward process but with the time dependence of the Hamiltonian reparameterized. With this choice, a small calculation gives

$$\int_0^T J_t\,\mathrm{d}t = -\int_0^T \boldsymbol{\nabla} H_t(\boldsymbol{E}_t)\cdot \mathrm{d}\boldsymbol{E}_t = -H_T(\boldsymbol{E}_T) + H_0(\boldsymbol{E}_0) + \int_0^T (\partial_t H_t)(\boldsymbol{E}_t)\,\mathrm{d}t. \tag{36}$$

The first law of thermodynamics implies then that $\int_0^T (\partial_t H_t)(\boldsymbol{E}_t)\,\mathrm{d}t$, is the work performed on the laser during a time T. But the relation between this work and the thermodynamic work is not clear currently. Starting from the Gibbs measure for the forward and backward processes, we obtain the relation

$$W_T = -\Delta_T F + \int_0^T (\partial_t H_t)(\boldsymbol{E}_t)\,\mathrm{d}t = -\Delta_T F - \frac{2}{D}\int_0^T (\partial_t a_t) I_t\,\mathrm{d}t, \tag{37}$$

where $\Delta_T F = F_T - F_0$ is the change of the Helmholtz free energy $F_t = -\ln\int \exp(-H_t(\boldsymbol{E}))\,\mathrm{d}\boldsymbol{E}$. The DFR (5) takes here the form

$$\mu_0(\mathrm{d}\boldsymbol{E}_0)\ P_{0,T}(\boldsymbol{E}_0; \mathrm{d}\boldsymbol{E}, \mathrm{d}W)\ \exp(-W) = \mu_T(\mathrm{d}\boldsymbol{E})\ P'_{0,T}(\boldsymbol{E}^*; \mathrm{d}\boldsymbol{E}_0^*, \mathrm{d}(-W)), \tag{38}$$

where μ_t denotes the Gibbs measure corresponding to H_t. In this case, there is a non-vanishing entropy production in the environment given by

$$\Delta_T S_{\mathrm{env}} = \langle \Delta_T Q\rangle = \int_0^T \langle J_t\rangle\,\mathrm{d}t = \int_0^T \mathrm{d}t \int H_t(\boldsymbol{E})\,(\partial_t\hat{\varphi}_t)(\boldsymbol{E})\ \exp(-\hat{\varphi}_t(\boldsymbol{E}))\,\mathrm{d}\boldsymbol{E}, \tag{39}$$

where $\hat{\mu}_t(\mathrm{d}\boldsymbol{E}) = \exp(-\hat{\varphi}(\boldsymbol{E}))\,\mathrm{d}\boldsymbol{E}$ is the distribution of $\boldsymbol{E}_t$ if $\boldsymbol{E}_0$ is distributed with the Gibbs measure $\mu_0(\mathrm{d}\boldsymbol{E})$. Note that, in general, $\hat{\mu}_t \neq \mu_t$. The associated Jarzynski equality (12) takes the form

$$\left\langle \exp\left[-\int_0^T (\partial_t H_t)(\boldsymbol{E}_t)\,\mathrm{d}t\right]\right\rangle = \exp(-\Delta F), \tag{40}$$

that is, explicitly,

$$\left\langle \exp\left[\frac{2}{D}\int_0^T (\partial_t a_t)\, I_t\,\mathrm{d}t\right]\right\rangle = \exp\left(\frac{a_T^2 - a_0^2}{bD}\right)\frac{1+\mathrm{erfc}(a_T/\sqrt{bD})}{1+\mathrm{erfc}(a_0/\sqrt{bD})}. \tag{41}$$

In fact, there is an infinity of Jarzynski equalities that correspond to different splittings of the drift $\boldsymbol{u}_t = -\frac{1}{2}d\boldsymbol{\nabla} H_t$ into $\boldsymbol{u}_{t,\pm}$ parts. The peculiarity of the Jarzynski equality with the functional W_T of (37) is that, upon its expansion to second order in the small time variation $a_t = a + h_t$ with $h_t \ll a$, one obtains the standard fluctuation–dissipation theorem [12, 20, 25, 13, 6, 3]

$$\frac{\delta\langle I_t\rangle}{\delta h_s}|_{h\equiv 0} = \frac{2}{D}\,\partial_s\langle I_s I_t\rangle_0 \tag{42}$$

for $s \leq t$, where $\langle\cdots\rangle_0$ is the equilibrium average in the stationary state with $h \equiv 0$.

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3.2.2. External coherent field. Another frequent way to induce a non-stationary behavior of the laser is to add an external coherent field at the laser frequency, modulated with a time-dependent amplitude $E_t^{\rm ext}$, which is injected into the cavity [12]. The gain and the self-saturation of the laser now depends on the total field $E_t + E_t^{\rm ext}$, but the losses depend just on E_t, so equation (22) becomes

$$\frac{{\rm d}E}{{\rm d}t} = \left(a - b|E + E_t^{\rm ext}|^2\right)(E + E^{\rm ext}) - \alpha E_t^{\rm ext} + \eta, \tag{43}$$

where α is the part of the dissipation in the net gain coefficient a. This equation takes for $E_t^{\rm tot} = E_t + E_t^{\rm ext}$ the form

$$\frac{{\rm d}E^{\rm tot}}{{\rm d}t} = \left(a - b\left|E^{\rm tot}\right|^2\right)E^{\rm tot} - \alpha E^{\rm ext} + \frac{{\rm d}E^{\rm ext}}{{\rm d}t} + \eta. \tag{44}$$

Upon denoting $-\alpha E_t^{\rm ext} + \frac{{\rm d}E_t^{\rm ext}}{{\rm d}t} = f_t$, this may be rewritten as

$$\frac{{\rm d}\boldsymbol{E}^{\rm tot}}{{\rm d}t} = -\frac{1}{2}d\,\boldsymbol{\nabla} H_t + \boldsymbol{\eta}. \tag{45}$$

with

$$H_t(\boldsymbol{E}^{\rm tot}) \equiv H_{ab}(\boldsymbol{E}^{\rm tot}) - \frac{2}{D}\left(\bar{f}_t E^{\rm tot} + f_t \bar{E}^{\rm tot}\right).$$

In the case where $E_t^{\rm ext}$ is not infinitesimal, we are outside the linear response regime, but the Jarzynski relation (40) is always true with

$$\partial_t H_t(\boldsymbol{E}^{\rm tot}) = -\frac{2}{D}\left((\partial_t \bar{f}_t)E^{\rm tot} + (\partial_t f_t)\bar{E}^{\rm tot}\right).$$

In the limit of infinitesimal f_t, this Jarzynski relation gives once again the fluctuation–dissipation theorem [12]:

$$\left.\frac{\delta\langle A_t\rangle}{\delta f_s}\right|_{h\equiv 0} = \frac{2}{D}\,\partial_s\langle \bar{E}_s^{\rm tot} A_t\rangle_0, \tag{46}$$

$$\left.\frac{\delta\langle A_t\rangle}{\delta \bar{f}_s}\right|_{h\equiv 0} = \frac{2}{D}\,\partial_s\langle E_s^{\rm tot} A_t\rangle_0. \tag{47}$$

4. Detuned laser with additive noise

4.1. Stationary case

For the stationary case with no tuning [23]

$$\frac{{\rm d}E}{{\rm d}t} = (a - bE\bar{E})E + \eta, \tag{48}$$

with $a = a_1 + {\rm i}a_2$ and $b = b_1 + {\rm i}b_2$ complex, $b_2 > 0$, and with covariance of the noise η given by equation (23). The detuning destroys the Langevin form of the equation because the drift cannot be put any more in the form $\boldsymbol{u} = -(d/2)\boldsymbol{\nabla} H$ but, instead,

$$\boldsymbol{u} = -\frac{d}{2}\boldsymbol{\nabla} H_{a_1b_1} + {\rm i}D\,\Pi\boldsymbol{\nabla} H_{a_2b_2}, \tag{49}$$

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with $\Pi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It is easy to see that the probability current of the Gibbs measure $\mu_{a_1 b_1}(\mathrm{d}\boldsymbol{E})$ is

$$\boldsymbol{j}(\boldsymbol{E}) = \mathrm{i}\, Z^{-1}_{a_1 b_1} \Pi \boldsymbol{\nabla} H_{a_2 b_2} \exp(-H_{a_1 b_1}) = Z^{-1}_{a_1 b_1}(-\mathrm{i} b_2 E^2 \bar{E} + \mathrm{i} a_2 E, \\ - \mathrm{i} b_2 E \bar{E}^2 - \mathrm{i} a_2 \bar{E}) \exp(-H_{a_1 b_1}) \tag{50}$$

and that it is conserved: $\boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$ because H depends only on the intensity I. It follows that the measure $\mu_{a_1 b_1}(\mathrm{d}\boldsymbol{E})$ is preserved by the dynamics. We are in a steady state [6]. The detailed balance breaks down due to the non-vanishing of current $\boldsymbol{j}$. It is replaced by the modified detailed balance:

$$\mu_{a_1 b_1}(\mathrm{d}\boldsymbol{E}_0)\; P_{0,T}(\boldsymbol{E}_0; \mathrm{d}\boldsymbol{E}) = \mu_{a_1 b_1}(\mathrm{d}\boldsymbol{E})\; P_{0,T}(\bar{\boldsymbol{E}}, \mathrm{d}\bar{\boldsymbol{E}}_0). \tag{51}$$

This relation, once again, implies a detailed balance for the process for intensity:

$$\mu_{a_1 b_1}(\mathrm{d}I_0)\; P_{0,T}(I_0; \mathrm{d}I) = \mu_{a_1 b_1}(\mathrm{d}I)\; P_{0,T}(I; \mathrm{d}I_0). \tag{52}$$

The relation (51) is a particular case of the DFR (5) where the time inversion acts in the spatial sector as the complex conjugation $\boldsymbol{E}^* = \bar{\boldsymbol{E}}$, with the vector and pseudo-vector parts of the drift equal to

$$\boldsymbol{u}_+ = ((a_1 - b_1 E\bar{E})E,\, (a_1 - b_1 E\bar{E})\bar{E}), \qquad \boldsymbol{u}_- = (\mathrm{i}(a_2 - b_2 E\bar{E})E,\, -\mathrm{i}(a_2 - b_2 E\bar{E})\bar{E}). \tag{53}$$

Here again the backward process that we obtain with this choice of time inversion has the same distribution as the forward one and the heat production

$$\Delta_T Q = \int_0^T J_t \,\mathrm{d}t = -H_{a_1 b_1}(\boldsymbol{E}_T) + H_{a_1 b_1}(\boldsymbol{E}_0) \tag{54}$$

is a state function. If the forward and backward processes are distributed initially with the Gibbs density $\exp(-H_{a_1 b_1})$ then, on average, there is no entropy production in the environment:

$$\Delta_T S_{\mathrm{env}} = \langle \Delta_T Q \rangle = \int_0^T \langle J_t \rangle \,\mathrm{d}t = -\int_0^T \mathrm{d}t \int \boldsymbol{\nabla} H_{a_1 b_1} \cdot \boldsymbol{j}(\boldsymbol{E})\,\mathrm{d}\boldsymbol{E} \\ = -\int_0^T \mathrm{d}t \int H_{a_1 b_1} \cdot \boldsymbol{\nabla} \boldsymbol{j}(\boldsymbol{E})\,\mathrm{d}\boldsymbol{E} = 0 \tag{55}$$

and $W_T = 0$. We have the usual features of equilibrium.

4.2. Non-stationary case

Introduction of a time dependence of the net gain coefficient to the previous model leads to the SDE

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (a_t - bE\bar{E})E + \eta \tag{56}$$

with an explicit time dependence for the net gain coefficient $a_t = a_{1,t} + \mathrm{i} a_{2,t}$ and $b = b_1 + \mathrm{i} b_2$ with $b_2 > 0$. Here, the fluctuation relation can be developed exactly as in section 3.2 but now (38) becomes for $\boldsymbol{E}^* = \bar{\boldsymbol{E}}$

$$\mu_{a_{1,0}b_1}(\mathrm{d}\boldsymbol{E}_0)\ P_{0,T}(\boldsymbol{E}_0; \mathrm{d}\boldsymbol{E}, \mathrm{d}W) = \mu_{a_{1,T}b_1}(\mathrm{d}\boldsymbol{E}) P_{0,T}(\bar{\boldsymbol{E}}, \mathrm{d}\bar{\boldsymbol{E}}_0, \mathrm{d}(-W)), \tag{57}$$

where $\mu_{a_{1,t}b_1}$ denotes the Gibbs measure corresponding to $H_{a_{1,t}b_1}$ and

$$W_T = -\Delta_T F_{a_1 b_1} + \int_0^T (\partial_t H_{a_{1,t}b_1})(\boldsymbol{E}_t)\,\mathrm{d}t = -\Delta_T F_{a_1 b_1} - \frac{2}{D}\int_0^T (\partial_t a_{1,t})\, I_t\,\mathrm{d}t. \tag{58}$$

The corresponding Jarzynski relation takes the form

$$\left\langle \exp\left[\frac{2}{D}\int_0^T (\partial_t a_{1,t})\, I_t\,\mathrm{d}t\right]\right\rangle = \exp\left(\frac{a_{1,T}^2 - a_{1,0}^2}{b_1 D}\right)\frac{1+\mathrm{erfc}(a_{1,T}/\sqrt{b_1 D})}{1+\mathrm{erfc}(a_{1,0}/\sqrt{b_1 D})}, \tag{59}$$

compared to (41). The second-order expansion in the small-time variation $a_t = a + h_t$ with $h_t = h_{1,t} + \mathrm{i}h_{2,t}$ now gives the fluctuation–dissipation relations

$$\frac{\delta\langle I_t\rangle}{\delta h_{1,s}}\Big|_{h\equiv 0} = \frac{2}{D}\,\partial_s\langle I_s I_t\rangle_0, \qquad \frac{\delta\,\langle I_t\rangle}{\delta h_{2,s}}\Big|_{h\equiv 0} = 0, \tag{60}$$

see [3] for the details.

5. Tuned laser with multiplicative noise

5.1. Stationary case

It is not always clear *a priori* whether the noise is better represented by a multiplicative or an additive model. In laser theory, when the randomness is due to pumping, it is more reasonable to use a multiplicative model of noise [21]. The stationary laser dynamics is then described by the non-Langevin SDE for the complex amplitude E_t:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (a - bE\bar{E})E + \eta_t E, \tag{61}$$

with a real, b positive and the white noise η_t as before. In complex coordinates, the covariance matrix (4) now takes the form

$$D(\boldsymbol{E}, \boldsymbol{E}') = \begin{pmatrix} 0 & DE\bar{E}' \\ DE'\bar{E} & 0 \end{pmatrix} \tag{62}$$

and, on the diagonal,

$$\mathrm{d}(\boldsymbol{E}) = D(\boldsymbol{E}, \boldsymbol{E}) = DE\bar{E}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{63}$$

One can show directly that the density $\exp(-\varphi(I))$, where

$$\varphi(I) = \frac{2b}{D}I + \left(1 - \frac{2a}{D}\right)\ln I \tag{64}$$

and $I = E\bar{E}$, is preserved by the dynamics and corresponds to the vanishing current, leading to the detailed balance

$$\exp(-\varphi(\boldsymbol{E}_0))\,\mathrm{d}\boldsymbol{E}_0\ P_{0,T}(\boldsymbol{E}_0; \mathrm{d}\boldsymbol{E}) = \exp(-\varphi(\boldsymbol{E}))\,\mathrm{d}\boldsymbol{E} P_{0,T}(\boldsymbol{E}; \mathrm{d}\boldsymbol{E}_0). \tag{65}$$

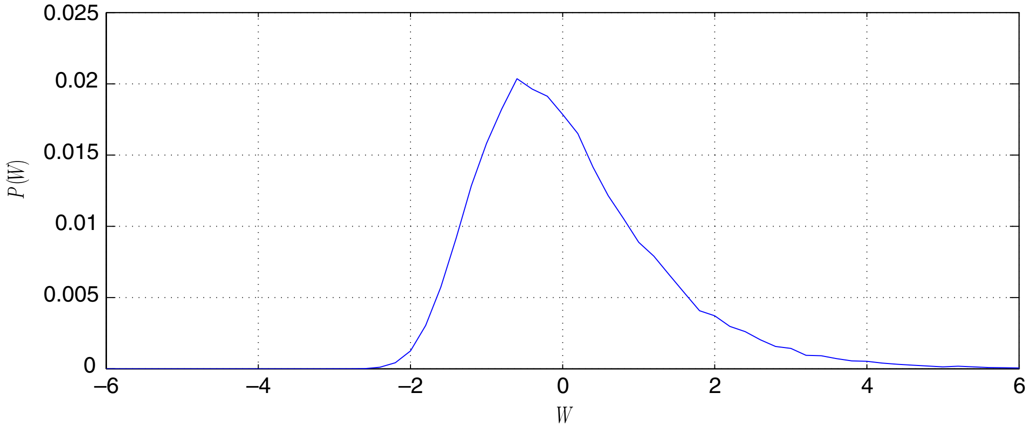

Figure 1. $P_{0,1}(W)$ as a function of W. Here $\langle W_1\rangle = 0.1023$.

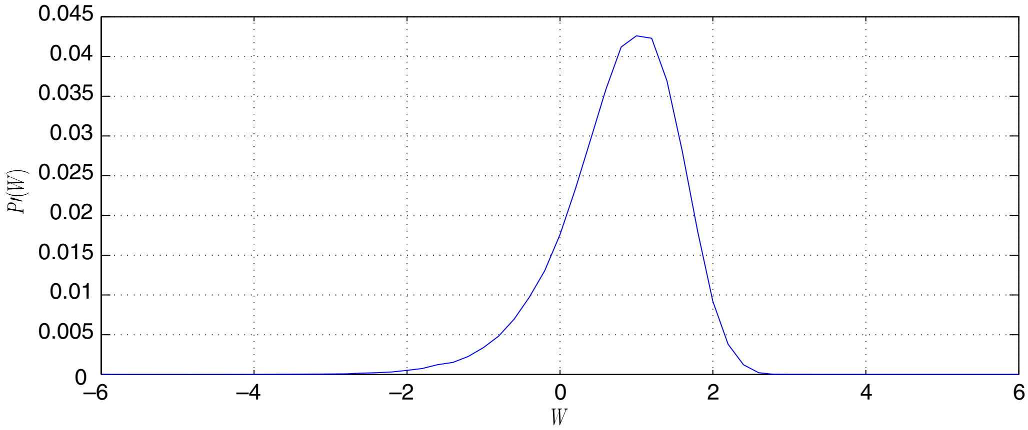

Figure 2. $P'_{0,1}(W)$ as a function of W. Here $\langle W'_1\rangle = 0.784$.

It is normalizable if $a > 0$. In this case, the normalized measure $\mu(\mathrm{d}\boldsymbol{E}) = Z^{-1}\exp(-\varphi(I))\,\mathrm{d}\boldsymbol{E}$, is invariant and we are once again in an equilibrium case. There is no invariant probability measure when $a \le 0$. Note the intensity I satisfies here a closed SDE

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 2(a - bI)I + (\eta_t + \bar{\eta}_t)I \tag{66}$$

that should be taken with the Stratonovich convention.

5.2. Non-stationary case

Introduction of a time dependence of the net gain coefficient to the previous model results in the SDE

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (a_t - bE\bar{E})E + \eta_t E. \tag{67}$$

With $\boldsymbol{E}^* = \boldsymbol{E}$ or $\boldsymbol{E}^* = \bar{\boldsymbol{E}}$ and the vector rule for the time inversion of the drift, the backward process solves the same SDE with a_t and η_t replaced by a_{t^*} and η_{t^*}. This time reversal corresponds both to the so-called reversed protocol and to the current reversal of the articles [1, 2]. The DFR (5) now takes the form

$$\exp(-\varphi_0(I_0))\,\mathrm{d}\boldsymbol{E}_0\ P_{0,T}(\boldsymbol{E}_0;\mathrm{d}\boldsymbol{E},\mathrm{d}W)\ \exp(-W) = \exp(-\varphi_T(I))\,\mathrm{d}\boldsymbol{E}\ P'_{0,T}(\boldsymbol{E}^*;\mathrm{d}\boldsymbol{E}_0^*,\mathrm{d}(-W)) \tag{68}$$

with

$$\varphi_t(I) = \frac{2b}{D}I + \left(1-\frac{2a_t}{D}\right)\ln I \tag{69}$$

and

$$W_T = \int_0^T (\partial_t\varphi_t)(\boldsymbol{E}_t)\,\mathrm{d}t = -\frac{2}{D}\int_0^T (\partial_t a_t)\ln I_t\,\mathrm{d}t. \tag{70}$$

The intensity process I_t satisfies the SDE (66) with the net gain coefficient a replaced by a_t. The backward intensity process is given by the same SDE with a_t and η_t replaced by a_{t^*} and η_{t^*}, leading to the DFR (5):

$$\exp(-\varphi_0(I_0)))\,\mathrm{d}I_0\ P_{0,T}(I_0;\mathrm{d}I,\mathrm{d}W)\ \exp(-W) = \exp(-\varphi_T(I))\,\mathrm{d}I\ P'_{0,T}(I;\mathrm{d}I_0,\mathrm{d}(-W)). \tag{71}$$

Introducing the distribution of W_T in the forward and backward processes by the relations

$$P_{0,T}(W)\,\mathrm{d}W = \frac{\int \exp(-\varphi_0(I_0))\,\mathrm{d}I_0\,P_{0,T}(I_0;\mathrm{d}I,\mathrm{d}W)\mathrm{d}I}{\int \exp(-\varphi_0(I_0))\,\mathrm{d}I_0}$$

and

$$P'_{0,T}(W)\,\mathrm{d}W = \frac{\int \exp(-\varphi_T(I_0))\,\mathrm{d}I_0\,P'_{0,T}(I_0;\mathrm{d}I,\mathrm{d}W)\,\mathrm{d}I}{\int \exp(-\varphi_T(I_0))\,\mathrm{d}I_0}$$

we obtain by integration (68) the Crooks relation [4]

$$P_{0,T}(W) = P'_{0,T}(-W)\exp(W-\Delta_T F) \qquad \text{with } \Delta_T F = F_T - F_0, \tag{72}$$

where $F_t = -\ln\int\exp(-\varphi_t(I))\,\mathrm{d}I$. In the case with positive a_0 and a_T, we may derive the associated Jarzynski equality:

$$\langle\exp(-W_T)\rangle = \exp(-\Delta_T F), \tag{73}$$

where

$$\exp(-\Delta_T F) = \frac{\int\exp(-\varphi_T(\boldsymbol{E}))\,\mathrm{d}\boldsymbol{E}}{\int\exp(-\varphi_0(\boldsymbol{E}))\,\mathrm{d}\boldsymbol{E}} \tag{74}$$

or, explicitly,

$$\left\langle \exp\left[\frac{2}{D}\int_0^T(\partial_t a_t)\ln I_t\,\mathrm{d}t\right]\right\rangle = \left(\frac{2b}{D}\right)^{-2(a_T-a_0)/D}\frac{\Gamma(2a_T/D)}{\Gamma(2a_0/D)}. \tag{75}$$

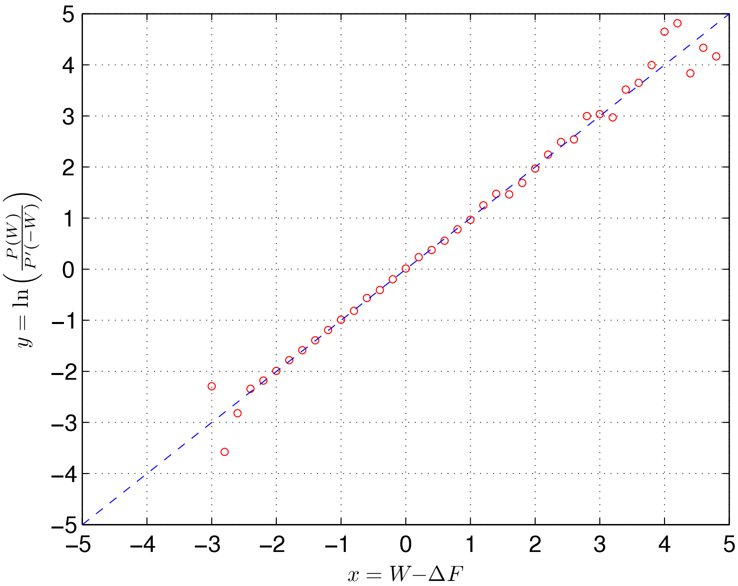

Figure 3. $\ln(P_{0,1}(W)/(P'_{0,1}(-W)))$ as a function of $W - \Delta_1 F$. The continuum line is the identity function.

Expanded to the second order in $h_t = a_t - a$, the identity (73) induces the generalized fluctuation–dissipation theorem (for a non-Langevin case):

$$\frac{\delta \langle \ln I_t \rangle}{\delta h_s}\bigg|_{h\equiv 0} = \frac{2}{D}\, \partial_s \langle \ln I_t \, \ln I_s \rangle_0 \tag{76}$$

for $s < t$. Once again, it is the fluctuation–dissipation theorem associated with the stochastic equation (67), as was demonstrated in [3].

We did a numerical verification of the Crooks relation (72) for the case $T = 1$, $a_t = 1 + t$, $b = 1$ and $D = 1$. We realized with Patrick Loiseau[1] a Matlab computation. We draw $P_{0,1}(W)$ (see figure 1), $P'_{0,1}(W)$ (see figure 2) as a function of W and $\ln(P_{0,1}(W)/(P'_{0,1}(-W)))$ as a function of $W - \Delta_1 F$ (see figure 3). The simulation was done on 5000 initial conditions between 0 and 10. For each initial condition, we considered 50 realizations of the noise. The interval of discretization in time was 2^{-15}.

6. Conclusion

We have discussed different fluctuation relations for a stochastic model of the semiclassical regime in a single-mode laser. In particular, we showed that the stationary tuned laser with additive noise has an equilibrium state with detailed balance (27) and that the detuning preserves the features (51) and (55) of equilibrium. We also studied the non-stationary case, showing for the tuned and the detuned laser close to equilibrium the standard fluctuation–dissipation theorems (42) and (60) that extend to the appropriate Jarzynski equality (59) far from equilibrium. Finally we studied a laser with multiplicative noise. We specified in this case the detailed balance relation (65) satisfied in the stationary case

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and the fluctuation–dissipation theorem (76). We also verified numerically the Crooks relation (72) in the non-stationary case.

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