RECENT ADVANCES IN OPTIMAL TRANSPORTATION AND APPLICATIONS

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Luigi Ambrosio
Scuola Normale Superiore di Pisa, Italy

Flows in the space of probability measures and convergence of Wigner transforms

Motivated by the classical problem of convergence of Wigner transforms, in the derivation of classical dynamics from quantum dynamics, we introduce the concept of flow in the space of probability measures and we derive existence, uniqueness and stability results for this notion. We apply these concepts to the Liouville equation with a non-smooth vectorfield, deriving uniqueness of solution for "almost every" measure initial condition. Joint work with A. Figalli, G. Friesecke, J. Giannoulis.

Guy Bouchitté
Université du Sud Toulon-Var, France

Optimization in the class of 1-rectifiable transports

It is a work in collaboration with P. Seppecher (University of Toulon). Given two mass distributions μ_+, μ_- in $\mathbb{R}^3$, we consider optimization problems of the kind

$$\inf \{\mathcal{F}(\lambda) \ : \ \mathrm{d}iv\, \lambda = \mu_+ - \mu_-\} \ ,$$

where the unknown λ is a vector measure (*transport* measure) and $\mathcal{F}$ is a cost functional.

In this talk we will focus on models where the admissible transports measures λ (those for which $\mathcal{F}(\lambda) < +\infty$) have to be concentrated on one-dimensional rectifiable subsets of $\mathbb{R}^3$, that is of the form

$$\lambda \ = \ \theta(s)\, \tau_S\, \mathcal{H}^1 \llcorner S \ ,$$

being $\theta(x) > 0$ a local flux intensity, τ_S an tangent vector and $\mathcal{H}^1 \llcorner S$ the one -dimensional Hausdorff measure on S.

We present a method of existence which allows to extend the case

$$\mathcal{F}(\lambda) = \int_S \theta^\alpha \, d\mathcal{H}^1 \quad \text{with} \quad 0 < \alpha < 1 \ ,$$

considered by many authors for modelling irrigation problems.

As a by-product we are able to consider well posed dynamic formulation of transport problems for countably many travellers taking into account the speed (with possibly positive parking cost) and the economy due to sharing the same convoy.

Lorenzo Brasco
Università di Pisa, Italy

Regularity issues in a continuous model of transportation

I will consider the problem of minimizing a convex energy under a divergence constraint: in the context of mass transportation problems, this has been first proposed by M. Beckmann in the 50's. Some regularity results are presented for the optimizers, strongly relying on the dual problem: due to some precise assumptions on the convex energy, the dual problem turns out to be a very degenerate one, which makes the regularity issue a non-trivial one and interesting in itself. Then, thanks to this results and to the Ambrosio-DiPerna-Lions theory of flows, one can show the original problem to be equivalent to a more mass transportation-oriented one, recently addressed by G. Carlier, C. Jimenez and F. Santambrogio.

Giuseppe Buttazzo
Università di Pisa, Italy

On some problems in spectral optimization

We present some problems where a function of the eigenvalues of the Dirichlet Laplacian has to be minimized over a class of admissible domains. The case of constraints of geometrical type, of volume, of perimeter, and some new ones that we call "of torsion", are presented and discussed, together with the available results and open problems.

Guillaume Carlier
Université Paris-Dauphine, France

From Knothe's transport to Brenier's map and a continuation method for optimal transport

A simple procedure to map two probability measures in $\mathbb{R}^d$ is the so-called Knothe-Rosenblatt rearrangement, which consists in rearranging monotonically the marginal distributions of the last coordinate, and then the conditional distributions, iteratively. We show that this mapping is the limit of solutions to a class of Monge-Kantorovich mass transportation problems with quadratic costs, with the weights of the coordinates asymptotically dominating one another. This enables us to design a continuation method for numerically solving the optimal transport problem (joint with Alfred Galichon and Filippo Santambrogio).

Thierry Champion
Université du Sud Toulon-Var, France

The Monge problem in $\mathbb{R}^{\mathbf{d}}$

We consider the Monge problem in a convex bounded subset of $\mathbb{R}^d$. The cost is given by a general norm, and we prove the existence of an optimal transport map under the classical assumption that the first marginal is absolutely continuous with respect to the Lebesgue measure. The approach we propose to solve this problem does not use the disintegration of measures. (This is a joint work with Luigi De Pascale)

Antoine Lemenant
Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy

Regularity for the average distance problem : main results and open questions

The problem under consideration, also called "irrigation problem", is the following : Let Ω be a bounded open subset of $\mathbb{R}^N$, μ a positive measure on Ω, and $a > 0$ a constant. We minimize the quantity $\int_\Omega dist(x, S)d\mu(x)$ over all the closed and connected one-dimensional sets S of Ω, of length less than a. This problem, introduced by G. Buttazzo, E. Oudet and E. Stepanov (2002), can describe for instance an optimal urban transporation network in a city with a given density of population. Despite of the quite elementary formulation of the problem, the question of topological and analytical regularity for the minimizers is still not completely solved, even under very simple asumptions (dimension 2, μ= Lebesgue measure, Ω convex and smooth). The aim of this talk is to present an overview of the principal regularity results including the more recent ones. We will try as far as possible to explain the key ideas of the proofs without insisting much on technicalities.

Francesco Maggi
Università di Firenze, Italy

Stability problems for anisotropic surface tensions

The equilibrium shape of a crystal is determined by the minimization under a volume constraint of its free energy, consisting of an anisotropic interfacial surface energy plus a bulk potential energy. In the absence of the potential term, the equilibrium shape can be directly characterized in terms of the surface tension and turns out to be a convex set, the Wulff shape of the crystal.

Our first result is a sharp quantitative inequality implying that any shape with almost-optimal surface energy is close in the proper sense to the Wulff shape. This is a joint work with Aldo Pratelli (Pavia) and Alessio Figalli (Paris).

Under the action of a weak potential or, equivalently, if the total mass of the crystal is small enough, the surface energy of the equilibrium shape is actually close to that of the corresponding Wulff shape, and the previous result applies. However, stronger geometric properties are now expected, due to the fact that the considered shapes are minimizers. Indeed we can prove their convexity, as well as their proximity to the Wulff shape with respect to a stronger notion of distance. This is a joint work with Alessio Figalli (Paris).

Edoardo Mainini

Scuola Normale Superiore di Pisa, Italy

A class of nonlinear diffusion equations in Hilbert spaces

We study a class of non linear diffusion equations in a Hilbert space X, with respect to a log-concave reference probability measure γ. We obtain existence, uniqueness and stability properties, in the framework of gradient flows in the space of probability measures with respect to the quadratic optimal transportation distance. Joint work with Luigi Ambrosio.

Aldo Pratelli

Università di Pavia, Italy

Minimizing the area or maximizing the perimeter of convex sets with given minimal bisecting length

Let us consider a convex planar set E: among all the chords bi- secting the area, there is a minimal one. An old problem (presented also in the book Unsolved problems in Geometry, 1991, and known as Santaló problem) is the following: if we fix this minimal length, which set has the smallest possible area? It is reasonable to guess that this set must be the ball, but thanks to the work of Zindler (1921) it is known that it is not so. In particular, Zindler shows that there are sets which have all the bisecting chords of the same length, and which are smaller than the ball: these sets are often referred to as Zindler sets. A big work has been done in last decades to study the properties of the convex sets with respect to the minimal bisecting length, in particular in the case of Zindler sets, but the problem of Santaló is still open. We present a solution of this problem in the class of Zindler sets. Joint work with Nicola Fusco.

Ludovic Rifford

Université de Nice Sophia-Antipolis, France

On Riemannian manifolds satisfying the Transport Continuity Property

We present some results concerning the regularity of optimal transport maps associated with quadratic costs on Riemannian manifolds. We will give necessary and sufficient conditions related to the so-called Ma-Trudinger-Wang and extended Ma-Trudinger-Wang conditions. We will illustrate the results with examples and counterexamples. This is a joint work with Alessio Figalli and Cédric Villani.

Filippo Santambrogio
Université Paris-Dauphine, France

Optimal maps for non-strictly convex transport costs: a general strategy and a case where it succeeds (the power of a norm in dimension two)

We discuss the existence of optimal transport maps for some optimal transport problems with a convex but non strictly convex cost. We present a decomposition strategy to address this issue. As part of our strategy, we have to treat some transport problems, of independent interest, with a convex constraint on the displacement. As an illustration of our strategy, we prove existence of optimal transport maps in the case where the source measure is absolutely continuous with respect to the Lebesgue measure and the transportation cost is of the form $h(||x - y||)$ with h strictly convex increasing and $|| \cdot ||$ an arbitrary norm in $\mathbb{R}^2$. Joint work with Guillaume Carlier and Luigi De Pascale.

Giuseppe Savaré
Università di Pavia, Italy

"Displacement extrapolation" and the one-dimensional sticky particle system

We present a simple approach to study the one-dimensional pressureless Euler system via adhesion dynamics in the Wasserstein spaces of probability measures. Starting from a discrete system of a nite number of sticky particles, we obtain explicit estimates of the solution in terms of the initial mass and momentum, which are useful to construct an evolution semigroup in a measure-theoretic phase space. We investigate various properties of this semigroup, in particular its link with the gradient flow of the (opposite) squared Wasserstein distance and with the "displacement extrapolation", i.e. the problem to extend geodesics in Wasserstein space beyond the formation of singularities. (Joint result in collaboration with Luca Natile, Pavia)