

ISODIAMETRIC SETS IN THE HEISENBERG GROUP

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ABSTRACT. In the sub-Riemannian Heisenberg group equipped with its Carnot-Carathéodory metric and with a Haar measure, we consider *isodiametric sets*, i.e. sets maximizing the measure among all sets with a given diameter. In particular, given an isodiametric set, and up to negligible sets, we prove that its boundary is given by the graphs of two locally Lipschitz functions. Moreover, in the restricted class of *rotationally invariant sets*, we give a quite complete characterization of any compact (rotationally invariant) isodiametric set. More specifically, its Steiner symmetrization with respect to the \mathbb{C}^n -plane is shown to coincide with the Euclidean convex hull of a CC-ball. At the same time, we also prove quite unexpected non-uniqueness results.

1. INTRODUCTION

The classical isodiametric inequality in the Euclidean space says that balls maximize the volume among all sets with a given diameter. This was originally proved by Bieberbach [5] in \mathbb{R}^2 and by Urysohn [14] in \mathbb{R}^n , see also [6]. In this paper we are interested in the case of the Heisenberg group \mathbb{H}^n equipped with its Carnot-Carathéodory distance d and with the Haar measure \mathcal{L}^{2n+1} (see Section 2 for the definitions). Our aim is to study *isodiametric sets*, i.e. sets maximizing the measure among sets with a given diameter.

Recalling that the homogeneous dimension of \mathbb{H}^n is $2n + 2$, we define the isodiametric constant C_I by

$$C_I = \sup \mathcal{L}^{2n+1}(F) / (\text{diam } F)^{2n+2}$$

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where the supremum is taken among all sets $F \subset \mathbb{H}^n$ with positive and finite diameter. Sets realizing the supremum do exist, see [12] or Theorem 3.1 below. Since the closure of any such set is a compact set that still realizes the supremum, we consider the class \mathcal{I} of compact isodiametric sets,

$$\mathcal{I} = \{E \subset \mathbb{H}^n ; E \text{ compact, } \text{diam } E > 0, \mathcal{L}^{2n+1}(E) = C_I (\text{diam } E)^{2n+2}\} .$$

In other words, due to the presence of dilations in \mathbb{H}^n , \mathcal{I} denotes the class of compact sets that maximize the \mathcal{L}^{2n+1} -measure among all sets with the same diameter.

In contrast to the Euclidean case, balls in (\mathbb{H}^n, d) are not isodiametric (see [12]) and we shall give in this paper some further and refined evidence that the situation is indeed quite different from the Euclidean one.

Before describing our main results let us recall some classical motivations and consequences coming from the study of isodiametric type problems. First the isodiametric constant C_I coincides with the ratio between the measure \mathcal{L}^{2n+1} and the $(2n+2)$ -dimensional Hausdorff measure \mathcal{H}^{2n+2} in (\mathbb{H}^n, d) , namely,

$$\mathcal{L}^{2n+1} = C_I \mathcal{H}^{2n+2} ,$$

where $\mathcal{H}^{2n+2}(A) = \lim_{\delta \downarrow 0} \inf \{ \sum_i (\text{diam } A_i)^{2n+2} ; A \subset \cup_i A_i , \text{diam } A_i \leq \delta \}$. This can actually be generalized to any Carnot group equipped with a homogeneous distance (see [12]), and for abelian Carnot groups one recovers the well-known Euclidean situation. We also refer the interested reader to [1] where some relationships between different intrinsic volumes that can be defined in sub-Riemannian geometry are studied.

As a consequence, the knowledge of the numerical value of the isodiametric constant C_I , or equivalently the explicit description of isodiametric sets, gives non trivial information about the geometry of the metric space (\mathbb{H}^n, d) and about the measure \mathcal{H}^{2n+2} which may be considered as a natural measure from the metric point of view.

There are also some links with the Besicovitch 1/2-problem which is in turn related to the study of the connections between densities and rectifiability. Let us only briefly sketch here this connection. We refer to [11] for a more detailed introduction and known results about the Besicovitch 1/2-problem and [12] for more details about the connection between the isodiametric problem in Carnot groups and the Besicovitch 1/2-problem. Let $\sigma_n(M, d)$ denote the density constant of the metric space (M, d) . It is the smallest number such that every subset with finite \mathcal{H}^n -measure having n -dimensional lower density strictly greater than $\sigma_n(M, d)$ at \mathcal{H}^n -almost all of its points is n -rectifiable (see [11] for the precise definition). The validity of the bound $\sigma_n(M, d) \leq 1/2$ for any separable metric space (M, d) , which was conjectured long ago by A.S. Besicovitch for the one-dimensional density constant in \mathbb{R}^2 (see [4]), is known as the generalized Besicovitch 1/2-problem.

It turns out that for a Carnot group equipped with a homogeneous distance (G, d) and with homogeneous dimension Q , the density constant $\sigma_Q(G, d)$ can be easily related to the inverse of the isodiametric constant. Then upper bounds on the isodiametric constant give lower bounds on $\sigma_Q(G, d)$. Following these ideas it is in particular proved in [12] that $\sigma_{2n+2}(\mathbb{H}^n, d) > 1/2$ for $n \in \{1, \dots, 8\}$ thus giving non trivial counter examples to the Besicovitch 1/2-conjecture. It is clear that the precise knowledge of the numerical value of the isodiametric constant would allow to get further and more complete results about the Besicovitch 1/2-problem.

Our main results in the present paper are a regularity property for sets in \mathcal{I} and a rather complete solution to a restricted isodiametric problem within the class of so-called rotationally invariant sets.

Let us first describe our regularity result. We shall prove that given $E \in \mathcal{I}$ then $\text{int } \overline{E}$ is still a compact isodiametric set with the same diameter as E and with locally Lipschitz boundary. More precisely, identifying \mathbb{H}^n with $\mathbb{C}^n \times \mathbb{R}$ (see Section 2), we prove that

$$\overline{\text{int } E} = \{[z, t] \in \mathbb{H}^n ; z \in \overline{U}, f^-(z) \leq t \leq f^+(z)\}$$

for some open set U in \mathbb{C}^n and some continuous maps $f^-, f^+ : \overline{U} \rightarrow \mathbb{R}$ that are locally Lipschitz continuous on U . See Theorem 3.12 for a complete statement.

This regularity property will actually follow from a slightly more general result. We will prove that a set $E \in \mathcal{I}$ must satisfy the following necessary condition,

(NC) for all $p \in \partial E$, there exists $q \in \partial E$ such that $d(p, q) = \text{diam } E$,

see Proposition 3.2, and is t -convex, see Subsection 2.4 for the definition of t -convexity and Proposition 3.11. Independently from the isodiametric problem, the property (NC) together with t -convexity turn out to imply the regularity properties sketched above. See Theorem 3.3.

As already mentioned, one knows that balls in (\mathbb{H}^n, d) are not isodiametric and isodiametric sets in (\mathbb{H}^n, d) are actually not explicitly known so far. This question turns out to be a challenging and rather delicate one. However, restricting ourselves to the family \mathcal{R} of so-called rotationally invariant sets, we are able to give a rather complete picture of the situation for compact isodiametric sets within this class. As we shall explain below, this picture will give some further information about the class \mathcal{I} . This may also hopefully give some insight towards a complete solution of the general isodiametric problem in (\mathbb{H}^n, d) .

We shall denote by $\mathcal{I}_{\mathcal{R}}$ the class of compact sets in \mathcal{R} that are isodiametric within the class \mathcal{R} . See Section 3 for the definition of the class \mathcal{R} and of this restricted isodiametric problem. First it is not hard to check that sets in

$\mathcal{I}_{\mathcal{R}}$ satisfy (NC) and are t -convex, see Proposition 3.2 and Proposition 3.11, and hence satisfy the regularity properties of Theorem 3.3. Next our main specific result concerning sets $E \in \mathcal{I}_{\mathcal{R}}$ is the characterization of their Steiner symmetrization $\text{St } E$ with respect to the \mathbb{C}^n -plane (see Subsection 2.4 for the definition of $\text{St } E$). We prove that if $E \in \mathcal{I}_{\mathcal{R}}$ then $\text{St } E$ belongs to $\mathcal{I}_{\mathcal{R}}$ and has the same diameter as E . Moreover we prove that $\text{St } E$ is actually uniquely determined once the diameter of E is fixed, i.e., $\text{St } E = A_{\text{diam } E}$ for some peculiar set $A_{\text{diam } E}$, see Theorem 4.4.

Given $\lambda > 0$, the set A_λ can be guessed via the following argument. One starts with the ball in (\mathbb{H}^n, d) centered at the origin and with diameter λ . As already said it does not satisfy (NC). Thus one can enlarge it around points where (NC) fails and get a set with still the same diameter but with greater measure. One can actually try to enlarge it as much as possible without increasing the diameter, remaining in the class \mathcal{R} , and preserving the property that it coincides with its Steiner symmetrization with respect to the \mathbb{C}^n -plane. In such a way, one ends up with a maximal set A_λ that satisfies (NC). It turns out that this set is the closed convex hull (in the Euclidean sense when identifying \mathbb{H}^n with \mathbb{R}^{2n+1}) of the ball in (\mathbb{H}^n, d) centered at the origin and with diameter λ . See Figure 1.

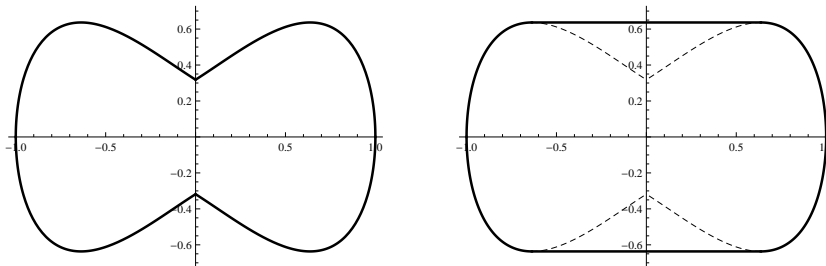


FIGURE 1. The intersection of the boundary $\partial B(0, 1)$ of the unit ball (left) and of the boundary ∂A_2 of its Euclidean convex hull (right) with a plane containing the vertical t -axis.

We also construct small suitable perturbations of the set A_λ that preserve its Lebesgue measure and its diameter, see Proposition 4.5. Considering rotationally invariant perturbations, this gives the non uniqueness of sets in $\mathcal{I}_{\mathcal{R}}$. This non uniqueness has to be understood in an “essential” sense, i.e., also up to left translations and dilations. See Corollary 4.6.

Finally, considering non rotationally invariant perturbations, one gets the existence of sets in \mathcal{I} that are not rotationally invariant, even modulo left translations, see Corollary 4.7. This gives one more significant difference with the Euclidean situation. See Figure 2.

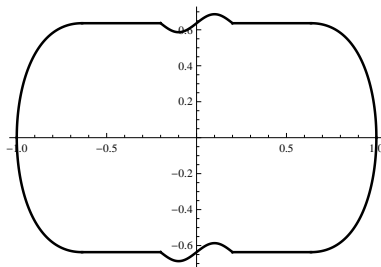


FIGURE 2. The intersection of the boundary of a perturbation of A_2 with a plane containing the vertical t -axis.

The paper is organized as follows. In Section 2 we introduce the Heisenberg group \mathbb{H}^n and recall basic facts about the Carnot-Carathéodory distance d and balls in (\mathbb{H}^n, d) . We also introduce t -convexification and the Steiner symmetrization with respect to the \mathbb{C}^n -plane. Section 3 is devoted to existence and regularity results, while in Section 4 we prove our more specific results about the isodiametric problem restricted to the class \mathcal{R} .

2. NOTATIONS AND PRELIMINARY RESULTS

2.1. The Heisenberg group. The Heisenberg group \mathbb{H}^n is a connected and simply connected Lie group with stratified Lie algebra (see e.g. [13, 7]). We identify it with $\mathbb{C}^n \times \mathbb{R}$ and denote points in \mathbb{H}^n by $[z, t]$, where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $t \in \mathbb{R}$. The group law is

$$[z, t] \cdot [z', t'] = [z + z', t + t' + 2 \operatorname{Im} z \bar{z}']$$

where $\operatorname{Im} z \bar{z}' = \sum_{j=1}^n \operatorname{Im} z_j \bar{z}'_j$. The unit element is $0 = [0, 0]$.

There is a natural family of dilations δ_λ on \mathbb{H}^n defined by $\delta_\lambda([z, t]) = [\lambda z, \lambda^2 t]$ for $\lambda \geq 0$.

We define the canonical projection $\pi : \mathbb{H}^n \rightarrow \mathbb{C}^n$ as

$$(2.1) \quad \pi([z, t]) = z$$

for any $[z, t] \in \mathbb{H}^n$. Given $p \in \mathbb{H}^n$, we shall frequently denote by $[z_p, t_p]$ its coordinates in $\mathbb{C}^n \times \mathbb{R}$.

We also represent \mathbb{H}^n as $\mathbb{R}^{2n+1} \approx \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ through the identification $[z, t] \approx (x, y, t)$, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ with $z_j = x_j + iy_j$.

The $(2n + 1)$ -dimensional Lebesgue measure \mathcal{L}^{2n+1} on $\mathbb{H}^n \approx \mathbb{R}^{2n+1}$ is a Haar measure of the group. It is $(2n + 2)$ -homogeneous with respect to dilations,

$$\mathcal{L}^{2n+1}(\delta_\lambda(F)) = \lambda^{2n+2} \mathcal{L}^{2n+1}(F)$$

for all measurable $F \subset \mathbb{H}^n$ and $\lambda \geq 0$.

The horizontal subbundle of the tangent bundle is defined by

$$\mathcal{H}^n = \operatorname{span} \{X_j, Y_j ; j = 1, \dots, n\}$$

where the left invariant vectors fields X_j and Y_j are given by

$$X_j = \partial_{x_j} + 2y_j \partial_t, \quad Y_j = \partial_{y_j} - 2x_j \partial_t.$$

Setting $T = \partial_t$ the only non trivial bracket relations are $[X_j, Y_j] = -4T$, hence the Lie algebra of \mathbb{H}^n admits the stratification $\mathcal{H}^n \oplus \text{span}\{T\}$.

2.2. Carnot-Carathéodory distance. We fix a left invariant Riemannian metric g on \mathbb{H}^n that makes $(X_1, \dots, X_n, Y_1, \dots, Y_n, T)$ an orthonormal basis. The Carnot-Carathéodory distance between any two points p and $q \in \mathbb{H}^n$ is then defined by

$$d(p, q) = \inf\{\text{length}_g(\gamma); \gamma \text{ horizontal curve joining } p \text{ to } q\}$$

where a curve is said to be horizontal if it is absolutely continuous and such that at a.e. every point its tangent vector belongs to the horizontal subbundle \mathcal{H}^n of the tangent bundle. Recall that by Chow-Rashevsky theorem any two points can be joined by a horizontal curve of finite length. Therefore the function d turns out to be a distance. It induces the original topology of the group, it is left-invariant, i.e.,

$$d(p \cdot q, p \cdot q') = d(q, q')$$

for all $p, q, q' \in \mathbb{H}^n$, and one-homogeneous with respect to dilations, i.e.,

$$d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q)$$

for all $p, q \in \mathbb{H}^n$ and $\lambda \geq 0$.

Equipped with this distance, \mathbb{H}^n is a separable and complete metric space in which closed bounded sets are compact. We will denote by $B(p, r)$, respectively $\overline{B}(p, r)$, the open, respectively closed, ball with center $p \in \mathbb{H}^n$ and radius $r > 0$. Note that the diameter of any ball in (\mathbb{H}^n, d) is given by twice its radius.

Lemma 2.1. *Let $p \in \mathbb{H}^n$. The distance function $d_p : \mathbb{H}^n \setminus \{p\} \rightarrow \mathbb{R}$ from p defined by $d_p(q) = d(p, q)$ is an open map.*

Proof. We prove that $d_p(B)$ is open for any open ball $B \subset \mathbb{H}^n \setminus \{p\}$. Since balls in (\mathbb{H}^n, d) are connected (this is more generally true in any length space) it follows that $d_p(B)$ is a bounded interval. Setting $m = \inf(d_p(q) ; q \in B)$ and $M = \sup(d_p(q) ; q \in B)$ it is thus enough to prove that $m \notin d_p(B)$ and $M \notin d_p(B)$.

If $m = 0$ then $m \notin d_p(B)$ because $p \notin B$. If $m > 0$ we assume by contradiction that $m \in d_p(B)$. Then we can find $q \in B$ such that $d_p(q) = m$. The map $\lambda \in [0, +\infty) \mapsto p \cdot \delta_\lambda(p^{-1} \cdot q)$ being continuous, one has $p \cdot \delta_\lambda(p^{-1} \cdot q) \in B$ for all λ close enough to 1. It follows that

$$d_p(q) \leq d_p(p \cdot \delta_\lambda(p^{-1} \cdot q)) = \lambda d_p(q) < d_p(q)$$

provided $\lambda < 1$ is close enough to 1, which gives a contradiction. The fact that $M \notin d_p(B)$ can be proved in a similar way and this concludes the proof. \square

Remark 2.2. As an immediate consequence of Lemma 2.1 we get that for any set $F \subset \mathbb{H}^n$ and any $p \in F$,

$$d(p, q) < \text{diam } F \text{ for all } q \in \text{int } F.$$

Moreover, if F is bounded then $\text{diam } F = \text{diam } \partial F$.

Although the Carnot-Carathéodory distance between any two points is in general hardly explicitly computable, we recall for further reference the following well-known special cases. One has

$$(2.2) \quad d([z, t], [z', t]) = \|z' - z\|$$

for all $z, z' \in \mathbb{C}^n$ such that $\text{Im } z\bar{z}' = 0$ and all $t \in \mathbb{R}$. Here $\|z\| = (\sum_{j=1}^n |z_j|^2)^{1/2}$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. We have also

$$(2.3) \quad d([z, t], [z, t']) = (\pi|t' - t|)^{1/2}$$

for all $z \in \mathbb{C}^n$ and $t, t' \in \mathbb{R}$.

2.3. Description of balls and consequences. Explicit descriptions of balls in (\mathbb{H}^n, d) are well-known, see [2], [9], [10]. One has

$$\begin{aligned} \overline{B}(0, 1) = \\ \left\{ \left[\frac{\sin \varphi}{\varphi} \chi, \frac{2\varphi - \sin(2\varphi)}{2\varphi^2} \|\chi\|^2 \right] \in \mathbb{H}^n ; \chi \in \mathbb{C}^n, \|\chi\| \leq 1, \varphi \in [-\pi, \pi] \right\}. \end{aligned}$$

We set

$$g(\varphi) = \begin{cases} \frac{2\varphi - \sin(2\varphi)}{2\varphi^2} & \text{for } 0 < \varphi \leq \pi \\ 0 & \text{if } \varphi = 0. \end{cases}$$

The function g admits a maximum at $\varphi = \pi/2$ with $g(\pi/2) = 2/\pi$. It is increasing from $[0, \pi/2]$ onto $[0, 2/\pi]$ and decreasing from $[\pi/2, \pi]$ onto $[1/\pi, 2/\pi]$.

We set

$$\rho(\varphi) = \begin{cases} \frac{\sin \varphi}{\varphi} & \text{for } 0 < \varphi \leq \pi \\ 1 & \text{if } \varphi = 0. \end{cases}$$

The function ρ is decreasing from $[0, \pi]$ onto $[0, 1]$. Let $\rho^{-1} : [0, 1] \rightarrow [0, \pi]$ denote its inverse and set

$$(2.4) \quad h = g \circ \rho^{-1}.$$

We have the following other description of the closed unit ball

$$(2.5) \quad \overline{B}(0, 1) = \{[z, t] \in \mathbb{H}^n ; \|z\| \leq 1, |t| \leq h(\|z\|)\} .$$

Using dilations we have

$$\overline{B}(0, \lambda) = \{[z, t] \in \mathbb{H}^n ; \|z\| \leq \lambda, |t| \leq h_\lambda(\|z\|)\}$$

for all $\lambda > 0$, where

$$(2.6) \quad h_\lambda(\|z\|) = \lambda^2 h(\|z\|/\lambda).$$

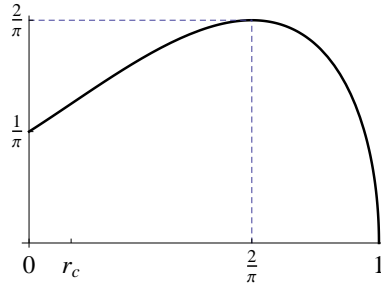


FIGURE 3. The profile function h .

We list some properties of the function h that will be needed in the sequel. See Figure 3 for a picture.

- (i) The map h is increasing from $[0, 2/\pi]$ onto $[1/\pi, 2/\pi]$ and decreasing from $[2/\pi, 1]$ onto $[0, 2/\pi]$.
- (ii) There exists $r_c \in (0, 2/\pi)$ such that $h'' > 0$ on $(0, r_c)$, $h''(r_c) = 0$, and $h'' < 0$ on $(r_c, 1)$.

Indeed we have

$$h'(r) = -2 \frac{\cos \varphi}{\varphi}$$

and

$$h''(r) = 2 \frac{\varphi \sin \varphi + \cos \varphi}{\varphi^2 \rho'(\varphi)}$$

for all $r \in [0, 1)$ and where $\varphi = \rho^{-1}(r) \in (0, \pi]$. Then statement (i) follows from the expression of h' together with the properties of ρ^{-1} . Statement (ii) follows noting that there exists a unique $\varphi_c \in (\pi/2, \pi)$ such that $\varphi_c \sin \varphi_c + \cos \varphi_c = 0$ and that one has $\varphi \sin \varphi + \cos \varphi > 0$ if and only if $0 < \varphi < \varphi_c$.

We call vertical segment any set of the form $\{[z, t] \in \mathbb{H}^n ; t^- \leq t \leq t^+\}$ for some $t^- \leq t^+$. Given $p, q \in \mathbb{H}^n$ with $z_p = z_q$, we denote by $L_{p,q}$ the vertical segment joining p to q ,

$$L_{p,q} = \{[z_p, t] \in \mathbb{H}^n ; \min(t_p, t_q) \leq t \leq \max(t_p, t_q)\} .$$

In the next proposition we state an elementary geometric property of balls in \mathbb{H}^n for further reference. When not specified, by ball we mean a ball that can be indifferently taken as open or closed.

Proposition 2.3. *The following statements hold.*

- (i) *Let B denote a ball in \mathbb{H}^n . For any $p, q \in B$ such that $z_p = z_q$, we have $L_{p,q} \subset B$.*
- (ii) *For any $p \in \mathbb{H}^n$ and any $p_1, p_2 \in \mathbb{H}^n$ such that $z_{p_1} = z_{p_2}$, we have*

$$d(p, q) \leq \max(d(p, p_1), d(p, p_2))$$

for all $q \in L_{p_1, p_2}$.

Proof. Property (i) holds for the (closed or open) unit ball by (2.5). Then this property follows for any ball using dilations and translations and noting that these maps are bijective maps that send vertical segments onto vertical segments.

To prove property (ii) set $r = \max(d(p, p_1), d(p, p_2))$. We have $p_1, p_2 \in \overline{B}(p, r)$. Hence $L_{p_1, p_2} \subset \overline{B}(p, r)$ by (i) and thus (ii) follows. \square

In the next lemma we deal with an outer vertical cone property for balls centered at the origin. Its proof (that we provide for the reader's convenience) follows from the local Lipschitz continuity of the profile function h on $[0, 1)$.

Lemma 2.4. *Let $d > 0$ and $\delta > 0$ be fixed. There exists $\alpha(d, \delta) > 0$ such that the following holds. If $p \in \partial B(0, d)$ is such that $t_p \geq \delta$, respectively $t_p \leq -\delta$, and $[w, s] \in \mathbb{H}^n$ is such that*

$$s > t_p + \alpha(d, \delta) \|w - z_p\|, \text{ respectively } s < t_p - \alpha(d, \delta) \|w - z_p\|,$$

then $[w, s] \notin \overline{B}(0, d)$.

Proof. Let $\alpha = \alpha(d, \delta) > 0$ to be chosen later. Let us consider the case where $p \in \partial B(0, d)$ and $[w, s] \in \mathbb{H}^n$ are such that $t_p \geq \delta$ and $s > t_p + \alpha \|w - z_p\|$, the other case being analogous. If $\|w\| > d$ then we obviously have $[w, s] \notin \overline{B}(0, d)$. If $\|w\| \leq d$ we want to prove that $s > h_d(\|w\|)$. We have

$$s > t_p + \alpha \|w - z_p\| = h_d(\|z_p\|) + \alpha \|w - z_p\|$$

and thus it will be sufficient to show that

$$h_d(\|z_p\|) + \alpha \|w - z_p\| \geq h_d(\|w\|).$$

Since $\lim_{r \rightarrow d^-} h_d(r) = 0$, one can find $\bar{r} = \bar{r}(d, \delta) \in (0, d)$ such that $h_d(r) < \delta$ for all $r \in (\bar{r}, d]$. In particular $\|z_p\| \leq \bar{r}$ because $h_d(\|z_p\|) = t_p \geq \delta$. We choose $\alpha > 0$ to be the Lipschitz constant of h_d on $[0, \bar{r}]$. If $\|w\| \leq \bar{r}$ it follows

$$h_d(\|w\|) \leq h_d(\|z_p\|) + \alpha \left| \|w\| - \|z_p\| \right| \leq h_d(\|z_p\|) + \alpha \|w - z_p\|$$

as wanted. Whereas

$$h_d(\|w\|) < \delta \leq h_d(\|z_p\|) \leq h_d(\|z_p\|) + \alpha \|w - z_p\|$$

if $\|w\| \in (\bar{r}, d]$ which concludes the proof. \square

Remark 2.5. We note that, for any fixed $d > 0$, the function $\alpha(d, \delta)$ can be taken continuous w.r.t. the variable δ . This follows from the definition of $\alpha(d, \delta) = \|h'_d\|_{L^\infty([0, \bar{r}(d, \delta)])}$ together with the fact that $\delta \mapsto \bar{r}(d, \delta)$ can be chosen to be continuous and the fact that the map $r \in (0, d) \mapsto \|h'_d\|_{L^\infty([0, r])}$ is continuous.

2.4. Two geometric transformations in \mathbb{H}^n . In this subsection we introduce two geometric transformations, namely the convexification along the vertical t -axis and the Steiner symmetrization with respect to the \mathbb{C}^n -plane. They will play a crucial role in the sequel.

Given $F \subset \mathbb{H}^n$ we define its t -convex hull $t\text{-co } F$ by

$$t\text{-co } F = \{p \in \mathbb{H}^n ; p \in L_{p_1, p_2} \text{ for some } p_1, p_2 \in F \text{ with } z_{p_1} = z_{p_2}\}.$$

We say that F is t -convex if $F = t\text{-co } F$.

Lemma 2.6. *Let $F \subset \mathbb{H}^n$. We have $F \subset t\text{-co } F$ and $\text{diam}(t\text{-co } F) = \text{diam } F$.*

Proof. We obviously have $F \subset t\text{-co } F$ and hence $\text{diam } F \leq \text{diam}(t\text{-co } F)$. Conversely let $p, p' \in t\text{-co } F$. One can find $p_1, p_2 \in F$ with $z_{p_1} = z_{p_2}$ such that $p \in L_{p_1, p_2}$. Then it follows from Proposition 2.3(ii) that

$$d(p', p) \leq \max(d(p', p_1), d(p', p_2)).$$

Similarly one can find $p'_1, p'_2 \in F$ with $z_{p'_1} = z_{p'_2}$ such that $p' \in L_{p'_1, p'_2}$. Then it follows once again from Proposition 2.3(ii) that

$$\max(d(p', p_1), d(p', p_2)) \leq \max_{i,j=1,2} d(p_i, p'_j) \leq \text{diam } F$$

which concludes the proof. \square

Given $F \subset \mathbb{H}^n$ measurable, its Steiner symmetrization $\text{St } F$ with respect to the \mathbb{C}^n -plane is defined by

$$\text{St } F = \{[z, t] \in \mathbb{H}^n ; z \in \pi(F), 2|t| \leq \mathcal{L}^1(\{s \in \mathbb{R} ; [z, s] \in F\})\}$$

where \mathcal{L}^1 denotes the one-dimensional Lebesgue measure and $\pi : \mathbb{H}^n \rightarrow \mathbb{C}^n$ is the canonical projection defined in (2.1). We define the reflection map $\sigma : \mathbb{H}^n \rightarrow \mathbb{H}^n$ as

$$(2.7) \quad \sigma([z, t]) = [\bar{z}, t].$$

For the sake of simplicity, the following lemma is stated for compact sets. This will be the only case needed in this paper. It can however be easily generalized to non compact sets.

Lemma 2.7. *Let $F \subset \mathbb{H}^n$ be compact and such that $\sigma(F) = F$. Then $\text{diam}(\text{St } F) \leq \text{diam } F$.*

Proof. Since F is a compact subset of \mathbb{H}^n , then $t\text{-co } F$ is compact and is obviously t -convex. We have $\sigma(t\text{-co } F) = t\text{-co } F$ as soon as $\sigma(F) = F$. Since $\text{St } F \subset \text{St}(t\text{-co } F)$ and $\text{diam}(t\text{-co } F) = \text{diam } F$ by Lemma 2.6, it is thus sufficient to consider t -convex compact sets F such that $\sigma(F) = F$. Then we can describe F as

$$F = \{[z, t] \in \mathbb{H}^n ; z \in \pi(F), a(z) - \mathcal{L}(z) \leq t \leq a(z) + \mathcal{L}(z)\}$$

for some map $a : \pi(F) \rightarrow \mathbb{R}$ which satisfies $a(\bar{z}) = a(z)$ for all $z \in \pi(F)$ and where $\mathcal{L}(z) = \mathcal{L}^1(\{s \in \mathbb{R} ; [z, s] \in F\})/2$. Note that we also have $\mathcal{L}(\bar{z}) = \mathcal{L}(z)$ for any $z \in \pi(F)$.

Let $p_1 = [z_1, t_1] \in \text{St } F$ and $p_2 = [z_2, t_2] \in \text{St } F$. Set

$$\begin{aligned} F_1 &= \{[z, t] \in F ; z = z_1 \text{ or } z = \bar{z}_1\} , \\ F_2 &= \{[z, t] \in F ; z = z_2 \text{ or } z = \bar{z}_2\} . \end{aligned}$$

We will prove that

$$(2.8) \quad d(p_1, p_2) \leq \max(d(q_1, q_2) ; q_1 \in F_1, q_2 \in F_2) .$$

Since $F_1 \cup F_2 \subset F$ this will imply $d(p_1, p_2) \leq \text{diam } F$ as wanted.

Set $\hat{F}_i = [0, -a(z_1)] \cdot F_i$, $i = 1, 2$, i.e.,

$$\begin{aligned} \hat{F}_1 &= \{z_1, \bar{z}_1\} \times [-\mathcal{L}(z_1), \mathcal{L}(z_1)] \\ \hat{F}_2 &= \{z_2, \bar{z}_2\} \times [b - \mathcal{L}(z_2), b + \mathcal{L}(z_2)] \end{aligned}$$

where $b = a(z_2) - a(z_1)$. The distance being left invariant, we have

$$(2.9) \quad \max(d(q_1, q_2) ; q_1 \in F_1, q_2 \in F_2) = \max(d(q_1, q_2) ; q_1 \in \hat{F}_1, q_2 \in \hat{F}_2) .$$

Next set $T_2 = |b| + \mathcal{L}(z_2)$. We have $p_2 \in \{z_2\} \times [-T_2, T_2]$ hence it follows from Proposition 2.3(ii) that

$$(2.10) \quad d(p_1, p_2) \leq \max(d(p_1, [z_2, T_2]), d(p_1, [z_2, -T_2])) .$$

Assume that $b \geq 0$. Then $[z_2, T_2] \in \hat{F}_2$ and since $p_1 \in \hat{F}_1$ we get

$$(2.11) \quad d(p_1, [z_2, T_2]) \leq \max(d(q_1, q_2) ; q_1 \in \hat{F}_1, q_2 \in \hat{F}_2) .$$

Let ι denote the isometry in (\mathbb{H}^n, d) defined by

$$(2.12) \quad \iota([z, t]) = [\bar{z}, -t] .$$

We have $\iota(\hat{F}_1) = \hat{F}_1$ and $\iota([z_2, -T_2]) = [\bar{z}_2, T_2] \in \hat{F}_2$, hence

$$(2.13) \quad d(p_1, [z_2, -T_2]) = d(\iota(p_1), \iota([z_2, -T_2])) \leq \max(d(q_1, q_2) ; q_1 \in \hat{F}_1, q_2 \in \hat{F}_2) .$$

Inequalities (2.10), (2.11) and (2.13) together with (2.9) give (2.8). The case where $b \leq 0$ can be treated in a similar way and this concludes the proof. \square

3. ISODIAMETRIC PROBLEM

We recall the definitions of the isodiametric constant

$$C_I = \sup\{\mathcal{L}^{2n+1}(F)/(\text{diam } F)^{2n+2} ; 0 < \text{diam } F < +\infty \}$$

and of the class of compact isodiametric sets

$$\mathcal{I} = \{E \subset \mathbb{H}^n ; E \text{ compact, } \text{diam } E > 0, \mathcal{L}^{2n+1}(E) = C_I (\text{diam } E)^{2n+2}\} .$$

Recall that it is not restrictive to ask isodiametric sets to be compact as the closure of any set which realizes the supremum in the right-hand side of the definition of C_I is a compact set that still realizes the supremum.

We also introduce the class of so-called rotationally invariant sets. Given $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$, we define the rotation $r_\theta : \mathbb{H}^n \rightarrow \mathbb{H}^n$ around the \mathbb{R} -axis by

$$r_\theta([z, t]) = [(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n), t] .$$

Any such r_θ is an isometry in (\mathbb{H}^n, d) . We denote by \mathcal{R} the class of rotationally invariant sets,

$$\mathcal{R} = \{F \subset \mathbb{H}^n ; r_\theta(F) \subset F \text{ for all } \theta \in \mathbb{R}^n\} .$$

We set

$$C_{I,\mathcal{R}} = \sup\{\mathcal{L}^{2n+1}(F)/(\text{diam } F)^{2n+2} ; F \in \mathcal{R}, 0 < \text{diam } F < +\infty \}$$

and denote by $\mathcal{I}_{\mathcal{R}}$ the family of compact rotationally invariant sets that are isodiametric within the class of rotationally invariant sets,

$$\mathcal{I}_{\mathcal{R}} = \{E \in \mathcal{R} ; E \text{ compact, } \text{diam } E > 0, \mathcal{L}^{2n+1}(E) = C_{I,\mathcal{R}} (\text{diam } E)^{2n+2}\} .$$

In other words, \mathcal{I} , resp. $\mathcal{I}_{\mathcal{R}}$, denotes the class of compact sets, resp. compact sets in \mathcal{R} , that maximize the \mathcal{L}^{2n+1} -measure among all subsets of \mathbb{H}^n , resp. among all sets in \mathcal{R} , with the same diameter.

We first prove the existence of sets in \mathcal{I} and $\mathcal{I}_{\mathcal{R}}$.

Theorem 3.1. *Both families \mathcal{I} and $\mathcal{I}_{\mathcal{R}}$ are nonempty.*

Proof. The proof that \mathcal{I} is non empty relies on the compactness of equibounded sequences of non empty compact sets with respect to the Hausdorff metric (see 2.10.21 in [8]) together with the upper-semicontinuity of the Lebesgue measure (see Theorem 3.2 in [3]). The fact that $\mathcal{I}_{\mathcal{R}} \neq \emptyset$ as well can be proved in a similar way noting that \mathcal{R} is closed with respect to the convergence of sets in the Hausdorff metric. \square

The rest of this section is devoted to the study of the regularity of sets in \mathcal{I} and $\mathcal{I}_{\mathcal{R}}$. The necessary condition (NC) introduced in Section 1 will be one of the key ingredients in this study and we prove in the next proposition that sets in \mathcal{I} and $\mathcal{I}_{\mathcal{R}}$ do satisfy this condition.

Proposition 3.2. *Let $E \in \mathcal{I} \cup \mathcal{I}_{\mathcal{R}}$. Then E satisfies the necessary condition (NC).*

Proof. We argue by contradiction. Assume that $E \in \mathcal{I}$ and $p \in \partial E$ is such that $d(p, q) < \text{diam } E$ for all $q \in \partial E$. The distance from p being a continuous and open map and E being compact, we have $\max_{q \in E} d(p, q) = \max_{q \in \partial E} d(p, q)$ (see Lemma 2.1 and Remark 2.2) and hence

$$\max_{q \in E} d(p, q) < \text{diam } E .$$

Choosing $r = (\text{diam } E - \max_{q \in E} d(p, q))/2 > 0$, it follows that

$$\text{diam}(E \cup \overline{B}(p, r)) = \text{diam } E .$$

On the other hand, since E is closed and $p \in \partial E$, we have $\text{int}(B(p, r) \setminus E) \neq \emptyset$. This implies in particular that

$$\mathcal{L}^{2n+1}(E \cup \overline{B}(p, r)) > \mathcal{L}^{2n+1}(E)$$

which contradicts the fact that $E \in \mathcal{I}$.

When $E \in \mathcal{I}_{\mathcal{R}}$ we modify the argument as follows. We set

$$F = E \cup \bigcup_{\theta \in \mathbb{R}^n} \overline{B}(r_{\theta}(p), r)$$

where $r = (\text{diam } E - \max_{q \in E} d(p, q))/2 > 0$ as before. We have $F \in \mathcal{R}$ and

$$(3.1) \quad \text{diam } F = \text{diam } E .$$

Indeed, to prove (3.1) we fix $q_1, q_2 \in F$. If $q_1, q_2 \in E$ then $d(q_1, q_2) \leq \text{diam } E$. If $q_1, q_2 \in F \setminus E$ then there exist $\theta_1, \theta_2 \in \mathbb{R}^n$ such that $d(q_i, r_{\theta_i}(p)) \leq r$, $i = 1, 2$. Recalling that any rotation r_{θ} is an isometry in (\mathbb{H}^n, d) and that $E \in \mathcal{R}$, it follows that

$$\begin{aligned} d(q_1, q_2) &\leq d(r_{\theta_1}(p), r_{\theta_2}(p)) + 2r \\ &= d(p, r_{\theta_1 - \theta_2}(p)) + 2r \\ &\leq \max_{q \in E} d(p, q) + 2r = \text{diam } E . \end{aligned}$$

If $q_1 \in E$ and $q_2 \in F \setminus E$ with $d(q_2, r_{\theta_2}(p)) \leq r$ for some $\theta_2 \in \mathbb{R}^n$, we have $d(q_1, q_2) \leq d(q_1, r_{\theta_2}(p)) + r \leq d(p, r_{-\theta_2}(q_1)) + r \leq \max_{q \in E} d(p, q) + r \leq \text{diam } E$.

On the other hand, similarly as before, we have

$$\mathcal{L}^{2n+1}(F) > \mathcal{L}^{2n+1}(E)$$

and this contradicts the fact that $E \in \mathcal{I}_{\mathcal{R}}$. □

Let us introduce some notations. Given a compact set E , we define f^+, f^-, U and \hat{E} as follows. We set

$$\begin{aligned} f^+(z) &= \max(t \in \mathbb{R} ; [z, t] \in E) , \\ f^-(z) &= \min(t \in \mathbb{R} ; [z, t] \in E) \end{aligned}$$

for all $z \in \pi(E)$. Clearly, $[z, f^\pm(z)] \in \partial E$ for all $z \in \pi(E)$. Recalling the definition of t -co E given in Subsection 2.4, one has

$$t\text{-co } E = \{[z, t] \in \mathbb{H}^n ; z \in \pi(E), f^-(z) \leq t \leq f^+(z)\},$$

and t -co E is itself compact.

We set

$$U = \{z \in \pi(E) ; f^-(z) < f^+(z)\}$$

and

$$\hat{E} = \{[z, t] \in \mathbb{H}^n ; z \in \bar{U}, f^-(z) \leq t \leq f^+(z)\} .$$

Since $\pi(E)$ is closed, we have $\bar{U} \subset \pi(E)$. In particular f^+ and f^- are well defined on \bar{U} . Moreover $\hat{E} = t\text{-co } E \cap \pi^{-1}(\bar{U})$ is compact and contained in $t\text{-co } E$.

We are now ready to state our key regularity result. It concerns t -convex and compact sets satisfying (NC).

Theorem 3.3. *For any t -convex and compact set E satisfying (NC) the following properties hold.*

- (i) *The set U is open in \mathbb{C}^n and the maps f^- and f^+ are locally Lipschitz on U and continuous on $\pi(E)$.*
- (ii) $\mathcal{L}^{2n+1}(E) = \mathcal{L}^{2n+1}(\hat{E})$.
- (iii) *We have*

$$\begin{aligned} \text{int } E &= \text{int } \hat{E} = \{[z, t] \in \mathbb{H}^n ; z \in U, f^-(z) < t < f^+(z)\} , \\ \partial \hat{E} &= \{[z, f^\pm(z)] \in \mathbb{H}^n ; z \in \bar{U}\} , \\ \hat{E} &= \overline{\text{int } E} . \end{aligned}$$

Before starting with the proof of Theorem 3.3, we introduce some notations and give a technical lemma.

Given $r > 0$ and $p_1, p_2 \in \mathbb{H}^n$ with $\pi(p_1) = \pi(p_2)$, $p_1 = [z_{12}, t_1]$ and $p_2 = [z_{12}, t_2]$ for some $z_{12} \in \mathbb{C}^n$ and $t_1, t_2 \in \mathbb{R}$, we set $\delta_{12} = (t_2 - t_1)/2$ and

$$F_{p_1, p_2}(r) = \left\{ [w, s] \in \mathbb{H}^n ; \|w - z_{12}\| \leq r, \left| s - \frac{t_1 + t_2}{2} \right| \leq \delta_{12} \left(1 - \frac{\|w - z_{12}\|}{r} \right) \right\}$$

whenever $\delta_{12} > 0$. Clearly, $F_{p_1, p_2}(r_1) \subset F_{p_1, p_2}(r_2)$ provided $r_1 \leq r_2$.

Lemma 3.4. *Let $C > 0$, $d > 0$ and $\delta > 0$ be fixed. There exists $\gamma(C, d, \delta) > 0$ such that the following holds. For any $r \in (0, \gamma(C, d, \delta)]$, $p_1 = [z_{12}, t_1]$ and $p_2 = [z_{12}, t_2] \in \mathbb{H}^n$ such that $\|z_{12}\| \leq C$ and $\delta_{12} = \delta$, we have*

$$F_{p_1, p_2}(r) \subset \overline{B}(p, d)$$

for all $p \in \mathbb{H}^n$ such that $p_1, p_2 \in \overline{B}(p, d)$.

Proof. Set $p_0 = [-z_{12}, -(t_1 + t_2)/2]$. After a left translation by p_0 we need to prove that for all $r > 0$ small enough we have

$$p_0 \cdot F_{p_1, p_2}(r) \subset \overline{B}(p, d)$$

for all $p \in \mathbb{H}^n$ such that $d(p, p_0 \cdot p_1) \leq d$ and $d(p, p_0 \cdot p_2) \leq d$. For such a $p = [z, t] \in \mathbb{H}^n$ we have $(p_0 \cdot p_i)^{-1} \cdot p = [z, t \pm \delta] \in \overline{B}(0, d)$ for $i = 1, 2$, hence $\|z\| \leq d$ and $|t \pm \delta| \leq h_d(\|z\|)$. It follows that $h_d(\|z\|) \geq |t| + \delta \geq \delta$. Recalling that $\lim_{u \rightarrow d^-} h_d(u) = 0$ and considering $\bar{r} = \bar{r}(d, \delta) \in (d/2, d)$ such that $h_d(u) < \delta$ whenever $u \in (\bar{r}, d]$, we get that $\|z\| \leq \bar{r}$.

We have

$$p_0 \cdot F_{p_1, p_2}(r) = \left\{ [w, s] \in \mathbb{H}^n ; \|w\| \leq r, |s + 2 \operatorname{Im} z_{12} \bar{w}| \leq \delta \left(1 - \frac{\|w\|}{r}\right) \right\}.$$

Let $[w, s] \in p_0 \cdot F_{p_1, p_2}(r)$ and let us show that $p^{-1} \cdot [w, s] \in \overline{B}(0, d)$ or equivalently that $\|w - z\| \leq d$ and

$$|s - t - 2 \operatorname{Im} z \bar{w}| \leq h_d(\|w - z\|)$$

provided $r > 0$ is small enough. First note that $\|w - z\| \leq r + \bar{r}$ is less than d provided $r \leq d - \bar{r}$.

Next we have

$$\begin{aligned} s - t - 2 \operatorname{Im} z \bar{w} &\leq \delta \left(1 - \frac{\|w\|}{r}\right) - 2 \operatorname{Im} z_{12} \bar{w} - t - 2 \operatorname{Im} z \bar{w} \\ &\leq (\delta - t) - \frac{\delta}{r} \|w\| + 2 \|z_{12}\| \|w\| + 2 \|z\| \|w\| \\ &\leq h_d(\|z\|) - \left(\frac{\delta}{r} - (2C + 2d)\right) \|w\| \end{aligned}$$

where the last inequality follows from the fact that $[-z, \delta - t] = [z, t - \delta]^{-1} \in \overline{B}(0, d)$. On the other hand if $r \leq (d - \bar{r})/2$, we have $\max(\|w - z\|, \|z\|) \leq (d + \bar{r})/2 < d$. Let $M = M(d, \delta) > 0$ denote the Lipschitz constant of h_d on $[0, (d + \bar{r})/2]$. Then we have

$$h_d(\|z\|) \leq h_d(\|w - z\|) + M \|w\|.$$

It follows that

$$s - t - 2 \operatorname{Im} z \bar{w} \leq h_d(\|w - z\|) - \left(\frac{\delta}{r} - (2C + 2d + M)\right) \|w\| \leq h_d(\|w - z\|)$$

provided $r \leq \min((d - \bar{r})/2, \delta/(2C + 2d + M))$.

Similarly we have

$$\begin{aligned}
s - t - 2 \operatorname{Im} z \bar{w} &\geq -\delta \left(1 - \frac{\|w\|}{r}\right) - 2 \operatorname{Im} z_{12} \bar{w} - t - 2 \operatorname{Im} z \bar{w} \\
&\geq -h_d(\|z\|) + \left(\frac{\delta}{r} - (2C + 2d)\right) \|w\| \\
&\geq -h_d(\|w - z\|) + \left(\frac{\delta}{r} - (2C + 2d + M)\right) \|w\| \\
&\geq -h_d(\|w - z\|)
\end{aligned}$$

provided $r \leq \min((d - \bar{r})/2, \delta/(2C + 2d + M))$ and where the second inequality follows from the fact that $[-z, -t - \delta] = [z, t + \delta]^{-1} \in \bar{B}(0, d)$. Hence the lemma follows with

$$(3.2) \quad \gamma(C, d, \delta) = \min\left(\frac{d - \bar{r}(d, \delta)}{2}, \frac{\delta}{2C + 2d + M(d, \delta)}\right).$$

□

Remark 3.5. Note that, C and d being fixed, the function $\gamma(C, d, \delta)$ can be taken to be continuous w.r.t. the variable δ . This is a consequence of (3.2) and the fact that $\bar{r}(d, \delta)$ can be chosen continuous w.r.t. δ .

We turn now to the proof of Theorem 3.3.

Proof of Theorem 3.3. We fix a t -convex and compact set E satisfying (NC) and set

$$C = \max_{p \in E} \|\pi(p)\| \quad \text{and} \quad d = \operatorname{diam} E.$$

We begin with two lemmata. The first one is a consequence of Lemma 3.4.

Lemma 3.6. *Let E be as above, $\delta > 0$ be fixed and let $\gamma = \gamma(C, d, \delta) > 0$ be as in Lemma 3.4. Then, for any $r \in (0, \gamma]$, one has*

$$F_{p_1, p_2}(r) \subset E$$

for all $p_1, p_2 \in E$ such that $\pi(p_1) = \pi(p_2)$ and $\delta_{12} = \delta$.

Proof. Since E is closed, it is enough to show that

$$\operatorname{int} F_{p_1, p_2}(r) \subset E.$$

Assume by contradiction that $(\operatorname{int} F_{p_1, p_2}(r)) \setminus E$ is nonempty. By t -convexity of E we know that also $E \cap \operatorname{int} F_{p_1, p_2}(r)$ is nonempty since it contains $L_{p_1, p_2} \setminus \{p_1, p_2\}$. Therefore there exists $p \in \partial E \cap \operatorname{int} F_{p_1, p_2}(r)$. Then by (NC) there exists $q \in \partial E$ such that $d(p, q) = d$. Thanks to Lemma 3.4, we have

$$p \in \operatorname{int} F_{p_1, p_2}(r) \subset B(q, d),$$

thus $d(p, q) < d$ which is a contradiction. □

Lemma 3.7. *Let E be as before and let $\delta > 0$ be fixed. There exists $\beta(C, d, \delta) > 0$ such that the following holds. If $z \in U$ is such that $f^+(z) - f^-(z) = 2\delta$, then*

$$f^+(w) \leq f^+(z) + \beta(C, d, \delta) \|w - z\| \quad \text{and} \quad f^-(w) \geq f^-(z) - \beta(C, d, \delta) \|w - z\|$$

for all $w \in \pi(E)$.

Proof. We claim that the lemma holds for $\beta(C, d, \delta) = \alpha(d, \delta) + 2C$ where $\alpha(d, \delta)$ is given by Lemma 2.4. Assume by contradiction that $w \in \pi(E)$ is such that

$$f^+(w) > f^+(z) + (\alpha(d, \delta) + 2C) \|w - z\| ,$$

the case where $f^-(w) < f^-(z) - (\alpha(d, \delta) + 2C) \|w - z\|$ being analogous. Since $[z, f^+(z)] \in \partial E$, by (NC) there exists $q \in \partial E$ such that $d([z, f^+(z)], q) = d$. Set $p = q^{-1} \cdot [z, f^+(z)]$ and $[w', s'] = q^{-1} \cdot [w, f^+(w)]$. We have

$$p = [z - z_q, f^+(z) - t_q - 2 \operatorname{Im} z_q \bar{z}]$$

and

$$[w', s'] = [w - z_q, f^+(w) - t_q - 2 \operatorname{Im} z_q \bar{w}] .$$

It follows that

$$\begin{aligned} (3.3) \quad s' &= t_p + f^+(w) - f^+(z) - 2 \operatorname{Im} z_q (\overline{w - z}) \\ &> t_p + (\alpha(d, \delta) + 2C) \|w - z\| - 2C \|w - z\| \\ &= t_p + \alpha(d, \delta) \|w' - z_p\| \end{aligned}$$

where the inequality follows by the choice of $w \in \pi(E)$.

On the other hand set $p' = q^{-1} \cdot [z, f^-(z)]$. We have $z_{p'} = z_p$. We also have $[z, f^-(z)] \in E$ and $q \in E$, hence $d(q, [z, f^-(z)]) \leq d$ or equivalently $p' \in \overline{B}(0, d)$. Therefore

$$|t_{p'}| \leq h_d(\|z_p\|) .$$

Recalling that $p \in \partial B(0, d)$ we get

$$(3.4) \quad t_p = h_d(\|z_p\|) \geq \frac{t_p - t_{p'}}{2} = \frac{f^+(z) - f^-(z)}{2} = \delta .$$

Then, taking Lemma 2.4 into account, we infer from (3.3) and (3.4) that $[w', s'] \notin \overline{B}(0, d)$, i.e., $d(q, [w, f^+(w)]) > d$. This implies that $\operatorname{diam} E > d$, a contradiction. \square

Remark 3.8. Taking into account Remark 2.5, one can take the function $\beta(C, d, \delta)$ to be continuous w.r.t. the variable δ .

We prove now the continuity of f^- and f^+ on $\pi(E)$.

Lemma 3.9. *The functions f^- and f^+ are continuous on $\pi(E)$.*

Proof. Let $z \in \pi(E)$ and let us prove that f^+ is continuous at z , the case of the function f^- being similar. Let (z_j) be a sequence of points in $\pi(E)$ such that $z_j \rightarrow z$ as $j \rightarrow \infty$. Since E is compact, f^+ is bounded and to prove the continuity of f^+ at z it is sufficient to prove that any possible limit of the sequence $(f^+(z_j))$ coincides with $f^+(z)$. By contradiction assume that $f^+(z_j) \rightarrow t$ as $j \rightarrow \infty$ for some $t \neq f^+(z)$. Since E is compact, hence closed, and $[z_j, f^+(z_j)] \in E$, we have $[z, t] \in E$. It follows in particular that, by definition of f^+ and f^- , we must have $f^+(z) > t \geq f^-(z)$. Setting $p_1 = [z, f^-(z)]$ and $p_2 = [z, f^+(z)]$, we thus have $\delta_{12} = (f^+(z) - f^-(z))/2 > 0$. Owing to Lemma 3.6, one obtains that $F_{p_1, p_2}(\gamma) \subset E$ where $\gamma = \gamma(C, d, \delta_{12})$ is given by Lemma 3.4. Therefore, recalling once again the definition of f^+ , we get in particular that $\liminf_{j \rightarrow \infty} f^+(z_j) \geq f^+(z)$, a contradiction. \square

We turn now to the proof of the fact that U is open and that f^- , f^+ are locally Lipschitz continuous on U .

Lemma 3.10. *The set $U \subset \mathbb{C}^n$ is open and the maps f^- , f^+ are locally Lipschitz continuous on U .*

Proof. Let us introduce some notations. We will denote by $\mathcal{B}(z, r)$ the open ball in \mathbb{C}^n with center $z \in \mathbb{C}^n$ and radius $r > 0$. Given $z \in U$, we set

$$\delta_z = \frac{f^+(z) - f^-(z)}{2}$$

and

$$\gamma_z = \gamma(C, d, \delta_z), \quad \beta_z = \beta(C, d, \delta_z)$$

where $\gamma(C, d, \delta_z)$ and $\beta(C, d, \delta_z)$ are given, respectively, by Lemma 3.4 and Lemma 3.7.

We first prove that $U \subset \mathbb{C}^n$ is open. Let $z \in U$. Set $t_1 = f^-(z)$, $t_2 = f^+(z)$ and let $p_1 = [z, t_1]$, $p_2 = [z, t_2] \in E$. It follows from Lemma 3.6 that $F_{p_1, p_2}(\gamma_z) \subset E$. In particular for any $w \in \mathbb{C}^n$ with $\|w - z\| < \gamma_z$, we have $[w, s] \in E$ for all

$$\begin{aligned} s &\in \left[\frac{t_1 + t_2}{2} - \delta_z \left(1 - \frac{\|w - z\|}{\gamma_z}\right), \frac{t_1 + t_2}{2} + \delta_z \left(1 - \frac{\|w - z\|}{\gamma_z}\right) \right] \\ &= \left[t_1 + \delta_z \frac{\|w - z\|}{\gamma_z}, t_2 - \delta_z \frac{\|w - z\|}{\gamma_z} \right]. \end{aligned}$$

Since

$$t_1 + \delta_z \frac{\|w - z\|}{\gamma_z} < t_2 - \delta_z \frac{\|w - z\|}{\gamma_z}$$

as soon as $\|w - z\| < \gamma_z$, it follows that $\mathcal{B}(z, \gamma_z) \subset U$. Hence $U \subset \mathbb{C}^n$ is open.

We prove now that f^- and f^+ are locally Lipschitz continuous on U . Let $z \in U$. By the previous argument we already know that

$$(3.5) \quad f^+(w) \geq f^+(z) - \delta_z \frac{\|w - z\|}{\gamma_z} \quad \text{and} \quad f^-(w) \leq f^-(z) + \delta_z \frac{\|w - z\|}{\gamma_z}$$

for all $w \in \mathcal{B}(z, \gamma_z)$. Next, Lemma 3.7 implies that

$$f^+(w) \leq f^+(z) + \beta_z \|w - z\| \quad \text{and} \quad f^-(w) \geq f^-(z) - \beta_z \|w - z\|$$

for all $w \in \mathcal{B}(z, \gamma_z) \subset \pi(E)$.

We fix a compact set $K \subset U$ and define $L = \sup_{z \in K} \max(\beta_z, \delta_z/\gamma_z)$ and $\gamma = \inf_{z \in K} \gamma_z$. Owing to the continuity of δ_z , β_z and γ_z (see Lemma 3.9 and Remarks 3.5 and 3.8), one has $L < +\infty$ and $\gamma > 0$. Therefore, for any $z, w \in K$ satisfying $\|z - w\| < \gamma$ we conclude that

$$\begin{aligned} |f^+(w) - f^+(z)| &\leq L \|w - z\|, \\ |f^-(w) - f^-(z)| &\leq L \|w - z\|. \end{aligned}$$

Hence f^+ and f^- are Lipschitz continuous on K , as desired. \square

We are now ready to conclude the proof of Theorem 3.3. Statement (i) follows from lemmata 3.9 and 3.10. Statement (ii) is a consequence of

$$E \setminus \hat{E} = \{[z, t] \in \mathbb{H}^n ; z \in \pi(E) \setminus \bar{U}, t = f^+(z)\}$$

and of the continuity of f^+ . Finally (iii) is straightforward and follows from (i) by standard arguments. \square

We are going to apply Theorem 3.3 to sets in \mathcal{I} and $\mathcal{I}_{\mathcal{R}}$. In order to do this, we first need to prove that such sets are t -convex.

Proposition 3.11. *Any set $E \in \mathcal{I} \cup \mathcal{I}_{\mathcal{R}}$ is t -convex.*

Proof. Assume by contradiction that one can find $p = [z, t] \in t\text{-co } E \setminus E$. By definition of $t\text{-co } E$, one must have $f^-(z) < t < f^+(z)$ and hence $z \in U$.

We have $E \subset t\text{-co } E$ and $\text{diam } t\text{-co } E = \text{diam } E$ by Lemma 2.6. Since $t\text{-co } E \in \mathcal{R}$ whenever $E \in \mathcal{R}$, this implies that $t\text{-co } E \in \mathcal{I} \cup \mathcal{I}_{\mathcal{R}}$. Hence $t\text{-co } E$ is a compact set that satisfies (NC) (see Proposition 3.2) and is obviously t -convex. Then Theorem 3.3 applies to $t\text{-co } E$. Noting that the maps f^\pm , and consequently the set U , associated to E and $t\text{-co } E$ coincide, we get from Theorem 3.3(iii) that $p \in \text{int}(t\text{-co } E)$.

Since E is closed and $p \notin E$ it follows that $p \in \text{int}(t\text{-co } E \setminus E)$. In particular $\text{int}(t\text{-co } E \setminus E) \neq \emptyset$ and $\mathcal{L}^{2n+1}(t\text{-co } E \setminus E) > 0$. Recalling that $E \subset t\text{-co } E$ and that both E and $t\text{-co } E$ belong to \mathcal{I} , resp. $\mathcal{I}_{\mathcal{R}}$, with $\text{diam}(t\text{-co } E) = \text{diam } E$, this gives a contradiction. \square

Noting that $\hat{E} \subset E$ whenever E is t -convex and that $\hat{E} \in \mathcal{R}$ whenever $E \in \mathcal{R}$, we get from Theorem 3.3(ii) that $\hat{E} \in \mathcal{I}$, resp. $\hat{E} \in \mathcal{I}_{\mathcal{R}}$, whenever $E \in \mathcal{I}$, resp. $E \in \mathcal{I}_{\mathcal{R}}$, with $\text{diam } \hat{E} = \text{diam } E$.

Summing up let us gather in the next theorem properties of sets in $\mathcal{I} \cup \mathcal{I}_{\mathcal{R}}$ proved in this section. Recall that the notations used in the statement to follow are those introduced before Theorem 3.3.

Theorem 3.12. *For any set $E \in \mathcal{I} \cup \mathcal{I}_{\mathcal{R}}$, the following properties hold.*

- (i) E is t -convex.
- (ii) $\mathcal{L}^{2n+1}(\hat{E}) = \mathcal{L}^{2n+1}(E)$ and $\text{diam } \hat{E} = \text{diam } E$.
- (iii) $\hat{E} \in \mathcal{I}$, resp. $\hat{E} \in \mathcal{I}_{\mathcal{R}}$, whenever $E \in \mathcal{I}$, resp. $E \in \mathcal{I}_{\mathcal{R}}$.
- (iv) The set U is open in \mathbb{C}^n and the maps f^- and f^+ are locally Lipschitz on U and continuous on $\pi(E)$.
- (v) We have

$$\begin{aligned} \text{int } E &= \text{int } \hat{E} = \{[z, t] \in \mathbb{H}^n ; z \in U, f^-(z) < t < f^+(z)\} , \\ \partial \hat{E} &= \{[z, f^\pm(z)] \in \mathbb{H}^n ; z \in \overline{U}\} , \\ \hat{E} &= \overline{\text{int } E} . \end{aligned}$$

4. ISODIAMETRIC PROBLEM FOR ROTATIONALLY INVARIANT SETS

In this section we characterize the Steiner symmetrization with respect to the \mathbb{C}^n -plane of sets $E \in \mathcal{I}_{\mathcal{R}}$. Our main result states that the set $\text{St } E$ belongs to $\mathcal{I}_{\mathcal{R}}$ and is uniquely determined once the diameter of E is fixed. It coincides with a peculiar set $A_{\text{diam } E}$, defined below, that consequently also belongs to $\mathcal{I}_{\mathcal{R}}$.

Constructing suitable perturbations of this set that preserve the diameter and the \mathcal{L}^{2n+1} -measure, see Proposition 4.5, we also get two remarkable consequences. First, the essential non uniqueness of sets in $\mathcal{I}_{\mathcal{R}}$, see Corollary 4.6. Second, the existence of sets in \mathcal{I} which are not rotationally invariant even up to isometries, see Corollary 4.7.

Given $\lambda > 0$, we set

$$l_\lambda(r) = \begin{cases} \frac{\lambda^2}{2\pi} & \text{if } r \in [0, \lambda/\pi] \\ h_{\frac{\lambda}{2}}(r) & \text{if } r \in [\lambda/\pi, \lambda/2] \end{cases}$$

(see (2.6) for the definition of $h_{\frac{\lambda}{2}}$) and

$$A_\lambda = \{[z, t] \in \mathbb{H}^n ; 2\|z\| \leq \lambda, |t| \leq l_\lambda(\|z\|)\} .$$

We have

$$A_\lambda = A_\lambda^1 \cup A_\lambda^2$$

where

$$A_\lambda^1 = \{[z, t] \in \mathbb{H}^n ; \|z\| \leq \lambda/\pi, 2\pi|t| \leq \lambda^2\}$$

and

$$\begin{aligned} A_\lambda^2 &= \{[z, t] \in \mathbb{H}^n ; \lambda/\pi \leq \|z\| \leq \lambda/2, |t| \leq h_{\frac{\lambda}{2}}(r)\} \\ &= \overline{B}(0, \lambda/2) \cap \{[z, t] \in \mathbb{H}^n ; \|z\| \geq \lambda/\pi\} . \end{aligned}$$

The set A_λ is the closed convex hull, in the Euclidean sense when identifying \mathbb{H}^n with \mathbb{R}^{2n+1} , of the ball $B(0, \lambda/2)$ in (\mathbb{H}^n, d) centered at the origin and with diameter λ . See Figure 1 in Section 1 for a picture.

We first show that the diameter of A_λ equals λ .

Proposition 4.1. *We have $\text{diam } A_\lambda = \lambda$ for all $\lambda > 0$.*

Proof. Since $A_\lambda = \delta_\lambda(A_1)$ for all $\lambda > 0$, it is enough to prove that $\text{diam } A_1 = 1$. We begin with a technical lemma.

Lemma 4.2. *Let $d > 0$, $r \leq d/\pi$ and $t \in \mathbb{R}$ be fixed. Set*

$$C = \{[z, t] \in \mathbb{H}^n ; \|z\| = r\}$$

and

$$D = \{[z, t] \in \mathbb{H}^n ; \|z\| \leq r\} .$$

Let $p \in \mathbb{H}^n$. Assume that $\text{diam}(C \cup \{p\}) \leq d$ and $\text{diam}(D \cup \{p\}) > d$. Then $\|z_p\| < r$ and there exists $q \in D \setminus C$ such that $z_p = z_q$ and $\text{diam}(D \cup \{p\}) = d(p, q)$.

Proof. We have $d([z, t], [z', t]) \leq d([z, t], [0, t]) + d([0, t], [z', t]) = \|z\| + \|z'\| \leq 2r$ for all $[z, t], [z', t] \in D$ and $d([z, t], [-z, t]) = 2\|z\| = 2r$ for all $[z, t] \in C$ (see (2.2)). It follows that $2r = \text{diam } C = \text{diam } D \leq \text{diam}(C \cup \{p\}) < \text{diam}(D \cup \{p\})$ hence $\text{diam}(D \cup \{p\}) = d(p, q)$ for some $q = [z_q, t] \in D \setminus C$.

Set $\bar{d} = \text{diam}(D \cup \{p\})$ and

$$h_{p, \bar{d}}^\pm(z) = \pm h_{\bar{d}}(\|z - z_p\|) + t_p + 2 \text{Im } z_p \bar{z}$$

so that

$$\overline{B}(p, \bar{d}) = \{[z, s] \in \mathbb{H}^n ; \|z - z_p\| \leq \bar{d}, h_{p, \bar{d}}^-(z) \leq s \leq h_{p, \bar{d}}^+(z)\} .$$

We have $q = [z_q, t] \in \partial B(p, \bar{d})$. Changing if necessary p , C and D into $\iota(p)$, $\iota(C)$ and $\iota(D)$ (see (2.12) for the definition of ι) one can assume with no loss of generality that $t = h_{p, \bar{d}}^+(z_q)$.

We have $D \subset \overline{B}(p, \bar{d})$. First we note that this implies $\|z_q - z_p\| < \bar{d}$. Otherwise, since $\|z_q\| < r$ we could find z close enough to z_q such that $[z, t] \in D$ with $\|z - z_p\| > \bar{d}$. On the other hand we have $\pi(D) \subset \pi(\overline{B}(p, \bar{d})) = \{w \in \mathbb{C}^n ; \|w - z_p\| \leq \bar{d}\}$ which gives a contradiction. It follows that the map $h_{p, \bar{d}}^+$ is well defined on an open neighbourhood of z_q in \mathbb{C}^n . Next since $D \subset \overline{B}(p, \bar{d})$ and q is in the relative interior of D in $\mathbb{C}^n \times \{t\}$ it follows

that $h_{p,\bar{d}}^+$ admits a local minimum at z_q . Assume now by contradiction that $z_q \neq z_p$. Then $h_{p,\bar{d}}^+$ is differentiable at z_q and we get that

$$(4.1) \quad \bar{d} h' \left(\frac{\|z_q - z_p\|}{\bar{d}} \right) \frac{z_q - z_p}{\|z_q - z_p\|} - 2z_p^\perp = 0$$

where $z_p^\perp = iz_p$. This implies that $z_q \neq 0$. Indeed otherwise (4.1) implies that $z_p = 0$ and hence $z_q = z_p$. Next if $z_p \neq 0$ we get from (4.1) that $z_q = z_p + \langle z_q, z_p^\perp \rangle z_p^\perp$ with $\langle z_q, z_p^\perp \rangle \neq 0$ where the scalar product is that of \mathbb{R}^{2n} after identifying points in \mathbb{C}^n with points in \mathbb{R}^{2n} . It follows that $\|z_q\| > \|z_p\|$ which also holds true if $z_p = 0$. On the other hand, restricting to a segment $s \in (-\varepsilon, \varepsilon) \mapsto z_q + s(z_q - z_p)$ for $\varepsilon > 0$ small enough we get that $h''(\|z_q - z_p\|/\bar{d}) \geq 0$ hence $\|z_q - z_p\|/\bar{d} \leq r_c$ where $[0, r_c]$ is the interval where h' is increasing (see Subsection 2.3). Since $\|z_q - z_p\| > 0$ it follows that

$$h' \left(\frac{\|z_q - z_p\|}{\bar{d}} \right) > h'(0) = \frac{2}{\pi}.$$

All together we finally get

$$\frac{2\bar{d}}{\pi} < \bar{d} h' \left(\frac{\|z_q - z_p\|}{\bar{d}} \right) = 2\|z_p^\perp\| = 2\|z_p\| < 2\|z_q\| < 2r \leq \frac{2d}{\pi}.$$

Recalling that $d < \bar{d}$ this gives a contradiction and concludes the proof. \square

We go back now to the proof of Proposition 4.1. We have $\text{diam } A_1 = \text{diam } \partial A_1$ (recall Remark 2.2). If we set

$$C^\pm = \{[z, t] \in \mathbb{H}^n ; \|z\| = 1/\pi, 2\pi t = \pm 1\},$$

$$D^\pm = \{[z, t] \in \mathbb{H}^n ; \|z\| \leq 1/\pi, 2\pi t = \pm 1\},$$

we have $\partial A_1 \subset A_1^2 \cup D^+ \cup D^- \subset A_1$ hence $\text{diam } A_1 = \text{diam}(A_1^2 \cup D^+ \cup D^-)$.

Let $p \in A_1^2$. We have $C^+ \subset A_1^2 \subset \bar{B}(0, 1/2)$ hence $\text{diam}(C^+ \cup \{p\}) \leq 1$. If $\text{diam}(D^+ \cup \{p\}) > 1$ it follows from Lemma 4.2 that $\|z_p\| < 1/\pi$ which contradicts the fact that $p \in A_1^2$. Hence $\text{diam}(D^+ \cup \{p\}) \leq 1$ for all $p \in A_1^2$. Since $\text{diam } A_1^2 = 1$ it follows that $\text{diam}(D^+ \cup A_1^2) = 1$. Similarly we have $\text{diam}(D^- \cup A_1^2) = 1$.

It thus only remains to check that $\text{diam}(D^- \cup D^+) \leq 1$. Let $p \in D^-$. We have $C^+ \cup \{p\} \subset A_1^2 \cup D^-$ hence $\text{diam}(C^+ \cup \{p\}) \leq 1$ by the previous argument. If $\text{diam}(D^+ \cup \{p\}) > 1$ it follows from Lemma 4.2 that one can find $q \in D^+$ such that $z_p = z_q$ and $d(p, q) = \text{diam}(D^+ \cup \{p\}) > 1$. On the other hand we have $-2\pi t_p = 2\pi t_q = 1$ and it follows from (2.3) that $d(p, q) = 1$ which gives a contradiction. Hence $\text{diam}(D^+ \cup \{p\}) \leq 1$ for any $p \in D^-$ and since $\text{diam } D^- \leq 1$ we finally get $\text{diam}(D^- \cup D^+) \leq 1$ as wanted. \square

The next lemma is an elementary remark that will be used later.

Lemma 4.3. *Let $E \in \mathcal{R}$. Assume moreover that E is symmetric with respect to the \mathbb{C}^n -plane, i.e., $[z, -t] \in E$ for all $[z, t] \in E$. Then*

$$E \subset \{[z, t] \in \mathbb{H}^n ; 2\|z\| \leq \text{diam } E, 2\pi|t| \leq (\text{diam } E)^2\} .$$

Proof. Let $[z, t] \in E$. Since $E \in \mathcal{R}$ we have $[-z, t] \in E$ and it follows from (2.2) that $2\|z\| = d([-z, t], [z, t]) \leq \text{diam } E$. We also have $[z, -t] \in E$ by assumption. It follows from (2.3) that $(2\pi|t|)^{1/2} = d([z, t], [z, -t]) \leq \text{diam } E$ and this concludes the proof. \square

We give now the main result of this section.

Theorem 4.4. *Let $E \in \mathcal{I}_{\mathcal{R}}$. Then $\text{St } E \in \mathcal{I}_{\mathcal{R}}$ and $\text{St } E = A_{\text{diam } E}$.*

Proof. First we note that since E is compact $\text{St } E$ is also compact. Indeed, since E is bounded, $\text{St } E$ is also obviously bounded. Next the fact that E is compact implies that the map $z \in \pi(E) \mapsto \mathcal{L}^1(\{s \in \mathbb{R} ; [z, s] \in E\})$ is upper semi-continuous. It follows that $\text{St } E$ is closed and hence compact. Since $E \in \mathcal{R}$, we have $\sigma(E) = E$ and it follows from Lemma 2.7 that $\text{diam}(\text{St } E) \leq \text{diam } E$. On the other hand one has $\mathcal{L}^{2n+1}(\text{St } E) = \mathcal{L}^{2n+1}(E)$. Since $E \in \mathcal{I}_{\mathcal{R}}$ and $\text{St } E \in \mathcal{R}$, one actually has $\text{diam}(\text{St } E) = \text{diam } E$ and $\text{St } E \in \mathcal{I}_{\mathcal{R}}$.

We set $\lambda = \text{diam } E$ and we prove now that $\text{St } E = A_{\lambda}$. Noting that $\pi(\text{St } E) = \pi(E)$ and taking into account the fact that $\text{St } E \in \mathcal{I}_{\mathcal{R}}$ is also symmetric with respect to the \mathbb{C}^n -plane, it follows from Theorem 3.12 that

$$\text{St } E = \{[z, t] \in \mathbb{H}^n ; z \in \pi(E), |t| \leq f(z)\}$$

for some continuous map $f : \pi(E) \rightarrow [0, +\infty)$ and that the set

$$U = \{z \in \pi(E) ; f(z) > 0\}$$

is open in \mathbb{C}^n .

We have $\pi(E) \subset \{z \in \mathbb{C}^n ; 2\|z\| \leq \lambda\}$ by Lemma 4.3 and we prove now that $f(z) \leq l_{\lambda}(\|z\|)$ for all $z \in \pi(E)$. In case $\pi(E) \cap \{z \in \mathbb{C}^n ; 2\|z\| = \lambda\} \neq \emptyset$, we note that since U is open we must have $f(z) = 0 = l_{\lambda}(\|z\|)$ for any $z \in \pi(E)$ such that $2\|z\| = \lambda$. Next we know from Lemma 4.3 that $f(z) \leq \lambda^2/(2\pi) = l_{\lambda}(\|z\|)$ for all $z \in \pi(E)$ such that $\|z\| \leq \lambda/\pi$.

It thus only remains to prove that $f(z) \leq l_{\lambda}(\|z\|)$ for $z \in \pi(E)$ such that $\lambda/\pi < \|z\| < \lambda/2$. Given such a z we assume by contradiction that $f(z) > l_{\lambda}(\|z\|) \geq 0$ and set $p = [z, l_{\lambda}(\|z\|)] \in A_{\lambda} \cap \partial B(0, \lambda/2)$. Since f is continuous and z belongs to the open set U , one can find an open set $V \subset U \subset \pi(E)$ containing z and such that $f(\|w\|) > l_{\lambda}(\|z\|)$ for all $w \in V$. Since $\{[w, s] \in \mathbb{H}^n ; w \in V, |s| \leq f(\|w\|)\} \subset \text{St } E$ it follows that $p \in \text{int}(\text{St } E)$. Next, since $p \in \partial B(0, \lambda/2)$ with $\|z\| \in (\lambda/\pi, \lambda/2)$, we can write p as

$$p = \left[\chi e^{-i\varphi} \frac{\sin \varphi}{\varphi}, \frac{2\varphi - \sin(2\varphi)}{2\varphi^2} \|\chi\|^2 \right]$$

for some $\chi \in \mathbb{C}^n$ such that $\|\chi\| = \lambda/2$ and some $\varphi \in (-\pi/2, \pi/2)$ (see e.g. [10]). It follows from [10, Lemma 1.11] that $d(q, p) = 2d(0, p) = \lambda$ where

$$q = \left[-\chi e^{i\varphi} \frac{\sin \varphi}{\varphi}, -\frac{2\varphi - \sin(2\varphi)}{2\varphi^2} \|\chi\|^2 \right].$$

Since $p \in \text{int}(\text{St } E)$ and since the distance function from q is an open map (see Lemma 2.1) we can find $p' \in \text{St } E$ such that $d(q, p') > \lambda$. On the other hand we have

$$q = \left[e^{i(\pi+2\varphi)} z, -l_\lambda(\|z\|) \right].$$

Since $\text{St } E \in \mathcal{R}$ is symmetric with respect to the \mathbb{C}^n -plane, we get that $q \in \text{St } E$, i.e. a contradiction.

It follows that $\text{St } E \subset A_\lambda$. Since $\text{diam } \text{St } E = \text{diam } E = \lambda = \text{diam } A_\lambda$ by Proposition 4.1, and since $A_\lambda \in \mathcal{R}$ and $\text{St } E \in \mathcal{I}_{\mathcal{R}}$, we get that $\mathcal{L}^{2n+1}(A_\lambda \setminus \text{St } E) = 0$. Being $\text{St } E$ closed, we obtain that $\text{int}(A_\lambda) \setminus \text{St } E = \text{int}(A_\lambda \setminus \text{St } E) = \emptyset$. It then follows that $A_\lambda = \overline{\text{int}(A_\lambda)} \subset \text{St } E$ and finally $\text{St } E = A_\lambda$ as wanted. \square

We now show that A_λ can be perturbed near the t -axis in such a way that the resulting set has the same volume and diameter as A_λ . As we shall see, the class of such perturbations is quite rich. It contains in particular rotationally and non rotationally invariant sets.

Proposition 4.5. *There exists $r \in (0, 1/\pi)$ such that for every $\lambda > 0$ and for any Lipschitz function $f : \mathbb{C}^n \rightarrow \mathbb{R}$ with compact support in $\{z \in \mathbb{C}^n; \|z\| < \lambda r\}$ and with Lipschitz constant $\text{Lip}(f) < \pi \lambda r/4$, the set*

$$A_{\lambda, f} = \{[z, t] \in \mathbb{H}^n; \|z\| \leq \lambda/\pi, 2\pi|t - f(z)| \leq \lambda^2\} \cup A_\lambda^2$$

satisfies $\mathcal{L}^{2n+1}(A_{\lambda, f}) = \mathcal{L}^{2n+1}(A_\lambda)$ and $\text{diam } A_{\lambda, f} = \text{diam } A_\lambda$.

Proof. The fact that $\mathcal{L}^{2n+1}(A_{\lambda, f}) = \mathcal{L}^{2n+1}(A_\lambda)$ is a consequence of the definition of $A_{\lambda, f}$ and of Fubini's theorem. To prove the last part of the lemma, we assume that $\lambda = 1$ without loss of generality. Since $A_1^2 \subset A_{1, f}$ and $\text{diam } A_1^2 = 1$ we have $\text{diam } A_{1, f} \geq 1 = \text{diam } A_1$.

To complete the proof, we will show that the inequality $\text{diam } A_{1, f} \leq 1 = \text{diam } A_1$ holds up to a suitable choice of r . First, for a given $p \in \mathbb{H}^n$, we define

$$h_p^\pm(z) = \pm h(\|z - z_p\|) + t_p + 2 \text{Im } z_p \bar{z},$$

so that $\bar{B}(p, 1) = \{[z, s] \in \mathbb{H}^n; \|z - z_p\| \leq 1, h_p^-(z) \leq s \leq h_p^+(z)\}$. We set $\kappa = h'(0)/2 = 1/\pi$.

Claim 1. There exists $\bar{r} \in (0, 1/2)$ such that, for all $p \in \mathbb{H}^n$ such that $\|z_p\| \leq \kappa/4$, one has

$$(4.2) \quad h_p^+(z) \geq h_p^+(z_p) + \kappa \|z - z_p\| \quad \text{for all } z \in \mathbb{C}^n \text{ such that } \|z - z_p\| \leq 2\bar{r}$$

and, similarly,

$$(4.3) \quad h_p^-(z) \leq h_p^-(z_p) - \kappa \|z - z_p\| \quad \text{for all } z \in \mathbb{C}^n \text{ such that } \|z - z_p\| \leq 2\bar{r}.$$

Indeed one can find $\bar{r} \in (0, 1/2)$ such that

$$|h(r) - h(0) - h'(0)r| \leq \frac{\kappa}{2} r$$

for all $r \leq 2\bar{r}$. Then we get

$$\begin{aligned} h_p^+(z) &= h(\|z - z_p\|) + t_p - 2\langle iz_p, z - z_p \rangle \\ &\geq h_p^+(z_p) + 2\kappa \|z - z_p\| - \frac{\kappa}{2} \|z - z_p\| - 2\|z_p\| \|z - z_p\| \\ &\geq h_p^+(z_p) + \kappa \|z - z_p\| \end{aligned}$$

for all $p \in \mathbb{H}^n$ such that $\|z_p\| \leq \kappa/4$ and all $z \in \mathbb{C}^n$ such that $\|z - z_p\| \leq 2\bar{r}$. Here the scalar product is that of \mathbb{R}^{2n} after identifying points in \mathbb{C}^n with points in \mathbb{R}^{2n} . This gives (4.2). The proof of (4.3) is similar.

We set $p_0 = [0, 1/(2\pi)]$.

Claim 2. For all $\bar{r} > 0$, there exists $\hat{r} > 0$ such that

$$(4.4) \quad \text{if } p \in B(p_0, \hat{r}) \cup B(p_0^{-1}, \hat{r}) \text{ and } q \in \partial A_1 \setminus B(p, 1) \text{ then } \|z_q\| < \bar{r}.$$

To prove this claim, we set

$$K = \partial A_1 \cap \{[z, t] \in \mathbb{H}^n; \|z\| \geq \bar{r}\} \subset \overline{B}(p_0, 1) \cap \overline{B}(p_0^{-1}, 1)$$

(recall that $p_0, p_0^{-1} \in A_1$ and $\text{diam } A_1 = 1$). Since

$$\partial B(p_0, 1) \cap \partial A_1 = \{p_0^{-1}\} \quad \text{and} \quad \partial B(p_0^{-1}, 1) \cap \partial A_1 = \{p_0\},$$

we actually have

$$K \subset B(p_0, 1) \cap B(p_0^{-1}, 1).$$

By compactness of K and continuity of $p \mapsto \max_{q \in K} d(p, q)$, one can then find $\hat{r} > 0$ such that $K \subset B(p, 1) \cap B(p', 1)$ for any $p \in B(p_0, \hat{r})$ and $p' \in B(p_0^{-1}, \hat{r})$. This proves Claim 2.

Fix $\bar{r} \in (0, 1/\pi)$ such that (4.2) and (4.3) hold for all $p \in \mathbb{H}^n$ satisfying $\|z_p\| \leq \kappa/4$. Then choose $\hat{r} > 0$ such that (4.4) holds. Set $r = \min(\bar{r}, 2\hat{r}/\pi, \kappa/4)$. Let $f : \mathbb{C}^n \rightarrow \mathbb{R}$ be a Lipschitz function with compact support in $\{z \in \mathbb{C}^n; \|z\| < r\}$ and with Lipschitz constant $\text{Lip}(f) < \pi r/4$.

We have $\|f\|_\infty \leq \text{Lip}(f)r < \pi r^2/4$ hence

$$\begin{aligned} \partial A_{1,f} \setminus \partial A_1 &\subset \left\{ [z, t] \in \mathbb{H}^n; \|z\| < r, 4 \left| t - \frac{1}{2\pi} \right| < \pi r^2 \right\} \\ &\quad \cup \left\{ [z, t] \in \mathbb{H}^n; \|z\| < r, 4 \left| t + \frac{1}{2\pi} \right| < \pi r^2 \right\} \\ &\subset B(p_0, \pi r/2) \cup B(p_0^{-1}, \pi r/2) \\ &\subset B(p_0, \hat{r}) \cup B(p_0^{-1}, \hat{r}). \end{aligned}$$

Now we take $p \in \partial A_{1,f} \setminus \partial A_1$ and $q \in \partial A_{1,f}$. Without loss of generality, we also assume that $p \in B(p_0, \pi r/2)$, the other case being analogous. Then

$$p = [z_p, \frac{1}{2\pi} + f(z_p)] .$$

If $\|z_q\| \geq \bar{r} \geq r$ then $q \in \partial A_1$ and by (4.4) we get $d(p, q) < 1$. If $\|z_q\| < \bar{r} < 1/\pi$ then

$$t_q = \pm \frac{1}{2\pi} + f(z_q) .$$

If $t_q = \frac{1}{2\pi} + f(z_q)$, since $\|z_q\| < \bar{r}$ and $|f(z_q)| < \pi r^2/4 \leq \pi \bar{r}^2/4$, we have $d(p_0, q) < \pi \bar{r}/2$. Therefore

$$d(p, q) < \frac{\pi}{2}(r + \bar{r}) < 1 .$$

If $t_q = -\frac{1}{2\pi} + f(z_q)$ we note that $\|z_p\| \leq \kappa/4$ and $\|z_p - z_q\| \leq r + \bar{r} \leq 2\bar{r}$. Then by (4.3) we get

$$\begin{aligned} h_p^-(z_q) &\leq h_p^-(z_p) - \kappa \|z_p - z_q\| \\ &= -\frac{1}{\pi} + t_p - \kappa \|z_p - z_q\| \\ &\leq -\frac{1}{2\pi} + f(z_p) - \text{Lip}(f) \|z_p - z_q\| \\ &\leq -\frac{1}{2\pi} + f(z_q) = t_q . \end{aligned}$$

Similarly, using (4.2),

$$h_p^+(z_q) \geq h_p^+(z_p) + \kappa \|z_p - z_q\| \geq \frac{3}{2\pi} + f(z_q) > t_q .$$

Hence we have

$$h_p^-(z_q) \leq t_q \leq h_p^+(z_q) .$$

that is $q \in \bar{B}(p, 1)$.

It follows that $d(p, q) \leq 1$ for all $p \in \partial A_{1,f} \setminus \partial A_1$ and $q \in \partial A_{1,f}$. Recalling that $\text{diam}(\partial A_1) = 1$ and that $\text{diam} A_{1,f} = \text{diam}(\partial A_{1,f})$ this concludes the proof. \square

We get from this proposition the following two consequences.

Corollary 4.6. *There exists $E \in \mathcal{I}_{\mathcal{R}}$ such that $p \cdot E \neq A_{\text{diam } E}$ for all $p \in \mathbb{H}^n$.*

In other words, although there is uniqueness modulo Steiner symmetrization with respect to the \mathbb{C}^n -plane for sets in $\mathcal{I}_{\mathcal{R}}$ with a given diameter, we have essential non uniqueness of sets in $\mathcal{I}_{\mathcal{R}}$.

Proof. Consider a set $A_{\lambda,f}$ given by Proposition 4.5 for some $\lambda > 0$ and where $f \not\equiv 0$ is moreover chosen in such a way that $A_{\lambda,f} \in \mathcal{R}$. By Theorem 4.4, we have $A_\lambda \in \mathcal{I}_{\mathcal{R}}$. On the other hand, by Proposition 4.5, we have $\mathcal{L}^{2n+1}(A_{\lambda,f}) = \mathcal{L}^{2n+1}(A_\lambda)$ and $\text{diam } A_{\lambda,f} = \text{diam } A_\lambda$. Therefore $A_{\lambda,f} \in \mathcal{I}_{\mathcal{R}}$.

Let us prove that $p \cdot A_{\lambda,f} \neq A_\lambda$ for all $p \in \mathbb{H}^n$. First we note that if $F \in \mathcal{R}$ and F is bounded then $p \cdot F \in \mathcal{R}$ if and only if $z_p = 0$. Next it is straightforward from the analytic description of A_λ and $A_{\lambda,f}$ that $p \cdot A_{\lambda,f} \neq A_\lambda$ for any $p \in \mathbb{H}^n$ such that $z_p = 0$. This concludes the proof. \square

Another consequence of Theorem 4.4 and Proposition 4.5 is the existence of non-rotationally invariant isodiametric sets, even modulo left translations.

Corollary 4.7. *There exists $E \in \mathcal{I}$ such that $p \cdot E \notin \mathcal{R}$ for all $p \in \mathbb{H}^n$.*

Proof. If $\mathcal{I} \cap \mathcal{R} = \emptyset$ then there is nothing to prove. If $\mathcal{I} \cap \mathcal{R} \neq \emptyset$ then $\mathcal{I}_{\mathcal{R}} = \mathcal{I} \cap \mathcal{R} \subset \mathcal{I}$. In particular it then follows from Theorem 4.4 that $A_\lambda \in \mathcal{I}$ for any $\lambda > 0$. Let $A_{\lambda,f}$ be given by Proposition 4.5 where f is moreover chosen in such a way that $A_{\lambda,f} \notin \mathcal{R}$. By Proposition 4.5, we have $\mathcal{L}^{2n+1}(A_{\lambda,f}) = \mathcal{L}^{2n+1}(A_\lambda)$ and $\text{diam } A_{\lambda,f} = \text{diam } A_\lambda$, and hence $A_{\lambda,f} \in \mathcal{I}$. Next we check that $p \cdot A_{\lambda,f} \notin \mathcal{R}$ for all $p \in \mathbb{H}^n$. Assume by contradiction that $p \cdot A_{\lambda,f} \in \mathcal{R}$ for some $p \in \mathbb{H}^n$. Then we get in particular that $\rho_\theta(\pi(p \cdot A_{\lambda,f})) = \pi(p \cdot A_{\lambda,f})$ for all $\theta \in \mathbb{R}^n$ where ρ_θ is the rotation in \mathbb{C}^n defined by $\rho_\theta(z) = (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$. On the other hand we have $\pi(p \cdot A_{\lambda,f}) = z_p + \{z \in \mathbb{C}^n ; 2\|z\| \leq \lambda\}$. All together this implies that $z_p = 0$. Then we get that $A_{\lambda,f} = p^{-1} \cdot (p \cdot A_{\lambda,f})$ is the vertical left translation by $p^{-1} = [0, -t_p]$ of $p \cdot A_{\lambda,f} \in \mathcal{R}$ and hence belongs to \mathcal{R} , a contradiction. \square

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