EXERCISES 3

GAUSSIAN VECTORS

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space.

1. Linearity, Characteristic function

Exercise 1. Show that the moments of a random variable X of Gaussian law $\mathcal{N}(0,1)$ are given by

$$\forall n \geq 0,\ \mathbb{E}(X^{2n}) = \frac{(2n)!}{2^n n!},\ \mathbb{E}(X^{2n+1}) = 0.$$

Hint: Use the characteristic function of X.

Exercise 2. Let $m = (m_i)_{1\leq i\leq n} \in \mathbb{R}^n$ and $K = (K_{i,j})_{1\leq i,j\leq n}$ be a non-negative symmetric matix. What is the law of $m + K^{1/2}(X_1, \dots, X_n)^t$, where $X_1, \dots, X_n$ are n I.I.D. random variables of $\mathcal{N}(0,1)$ law?

Exercise 3. Let $(X_1, \dots, X_n)$ be a Gaussian vector and $(i_1, \dots, i_m) \in \{1, \dots, n\}^m$. What we can say about the law of $(X_{i_1}, \dots, X_{i_m})$?

Exercise 4. Let X be an $\mathcal{N}(0,1)$ r.v. and Z be a uniformly distributed r.v. on $\{-1,1\}$, independent of X.

(1) Show that ZX is Gaussian.
(2) Considering $X + ZX$, show that the pair (X, ZX) isn't Gaussian.
(3) Prove that X and ZX aren't independent, but that their covariance is zero.

Exercise 5. Let $X_1, \dots, X_n$ be n Gaussian independant r.v. Check that the sum $\sum_{i=1}^n X_i$ is a Gaussian r.v., whose mean and variance are respectively given by the sum of the means and the sum of the variances of the $(X_i)_{1\leq i\leq n}$.

Exercise 6. Let $(X_1, \dots, X_n)$ be a Gaussian random vector with mean $m = (m_j)_{1\leq j\leq n}$ and covariance matrix $K = (K_{j,k})_{1\leq j,k\leq n}$.

(1) For some $(t_j)_{1\leq j\leq n} \in \mathbb{R}^n$, what is the law of $\sum_{j=1}^n t_j X_j$?
(2) Deduce that

$$\mathbb{E}\big[\exp\big(i\sum_{j=1}^n t_j X_j\big)\big] = \exp\big(i\sum_{j=1}^n t_j m_j - \frac{1}{2}\sum_{j,k=1}^n t_j K_{j,k} t_k\big)\big].$$

(3) What can we say about two Gaussian vectors with the same mean and the same covariance?

2. Independence

Exercise 7. Let $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_n)$ be two Gaussian vectors such that **the vector** $(X_1, \dots, X_m, Y_1, \dots, Y_n)$ **is Gaussian**. Show that $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_n)$ are independent if and only if the covariance matrix of $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ is diagonal by block, i.e. has the form

$$\begin{pmatrix} \times & \ldots & \ldots & \times & 0 & \ldots & 0 \\ \times & \ldots & \ldots & \times & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \times & \ldots & \ldots & \times & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 & \times & \ldots & \times \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & \times & \ldots & \times \end{pmatrix}.$$

Exercise 8. Let $(X_i)_{1 \le i \le n}$, $n \ge 2$, be n independent and identically distributed r.v. of Gaussian law $\mathcal{N}(0,1)$. Prove that the r.v. $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $\max_{1 \le i \le n} X_i - \min_{1 \le i \le n} X_i$ are independent.

Hint: Consider the vector $(\bar{X}_n, X_1 - \bar{X}_n, \ldots, X_n - \bar{X}_n)^t$.

Exercise 9. Let $(X_n)_{n \ge 1}$ be a sequence of I.I.D. r.v. of Gaussian law $\mathcal{N}(0,1)$. We set:

$$B_0 = 0,\ \forall n \ge 1,\ B_n = \sum_{k=1}^n X_k.$$

1. Give the covariance matrix of $(B_1, \ldots, B_n)$ as well as its probability density (if exists).
2. For $1 \le m \le n$, set $Z_m = B_m - (m/n)B_n$. Prove that Z_m and B_n are independent.

(Above, the first diagonal block is of size $m \times m$ and the second one of size $n \times n$.

3. Conditional Expectation

Exercise 10. *(Independence case.)*
Let $\mathcal{B}$ a σ-field of $\mathcal{A}$, X and Y be two r.v., and $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel mapping. We assume that X is $\mathcal{B}$-measurable and that Y is independent of $\mathcal{B}$.

Prove that:

$$\mathbb{P}\text{−a.s.},\ \mathbb{E}\big[f(X,Y)|\mathcal{B}\big] = \phi(X),$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is given by:

$$\forall x \in \mathbb{R}^n,\ \phi(x) = \mathbb{E}[f(x,Y)].$$

Exercise 11. Let $\mathcal{B}$ a σ-field of $\mathcal{A}$ and X be an independent r.v. of $\mathcal{B}$ of law $\mathcal{N}(0,\sigma^2)$.

1. What is $\mathbb{E}(Z|\mathcal{B})$?
2. Show that for every $\mathcal{B}$-measurable r.v. Y, the r.v.

   $$Z = \exp(-\frac{\sigma^2}{2}Y^2 + XY),$$

   has 1 as expectation.

Exercise 12. Let X and Y be two independent r.v. of uniform law on $[0,1]$. We set $U = \inf(X,Y)$ and $V = \sup(X,Y)$. What is $\mathbb{E}(U|V)$?