EXERCISES 6

ITO'S FORMULA

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ *denotes the current probability space and* $(B_t)_{t\geq 0}$ *a (real) Brownian motion.*

Exercise 1. For λ and θ in $\mathbb{R}$, we consider the process

$$\forall t \geq 0,\ X_t = \exp(-\lambda t)\cos(\theta B_t).$$

(1) Compute dX_t for $t \geq 0$.
(2) What are the values of (λ, θ) for which the dt-term in dX_t vanishes?
(3) Deduce $\mathbb{E}[\cos(\theta B_t)]$ for $t \geq 0$.

Exercise 2. For r and σ in $\mathbb{R}$, we consider the process

$$\forall t \geq 0,\ X_t = \exp(rt + \sigma B_t).$$

(1) Compute dX_t for $t \geq 0$.
(2) What are the values of (r, σ) for which the dt-term vanishes?
(3) For the values of r and σ obtained above, show that, for all $0 \leq s < t$,

$$\mathbb{E}\big[X_t | \mathcal{F}_s\big] = X_s,$$

where $\mathcal{F}_s$ is the σ-field generated by $(B_u)_{0\leq u\leq s}$.

Exercise 3. Let n be an integer larger than 1.

(1) Show that

$$\forall t \geq 0,\ B_t^{2n} = 2n\int_0^t B_s^{2n-1} dB_s + n(2n-1)\int_0^t B_s^{2n-2} ds.$$

(2) Deduce that

$$\mathbb{E}(B_1^{2n}) = (2n-1)\mathbb{E}(B_1^{2n-2}).$$

(3) Let Z be an $\mathcal{N}(0,1)$ Gaussian variable. Deduce from the above expression that

$$\mathbb{E}(Z^{2n}) = [(2n)!]/[2^n \times n!].$$

Exercise 4. Show that the following processes are martingales w.r.t. the filtration generated by B:

(1) $\forall t \geq 0,\ X_t = \exp(t/2)\cos(B_t)$.
(2) $\forall t \geq 0,\ Y_t = \exp(t/2)\sin(B_t)$.
(3) $\forall t \geq 0,\ Z_t = (B_t + t)\exp(-B_t - t/2)$.
(4) $\forall t \geq 0,\ W_t = B_t^3 - 3tB_t$.

Exercise 5. Let $(B_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion. Show that $(B_t^4 - 6tB_t^2 + 3t^2)_{t\geq 0}$ is a martingale w.r.t. to the filtration $(\sigma(B_s, s \leq t))_{t\geq 0}$.

Exercise 6. Let $(B_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion and $(b_t)_{t\geq 0}$ be a continuous and $(\mathcal{F}_t)_{t\geq 0}$-adapted process. Set

$$\forall t \geq 0,\ X_t = \int_0^t b_s ds + B_t.$$

We assume that there exist two constants K and λ such that

$$\forall t \geq 0,\ \forall \omega \in \Omega,\ |b_t(\omega)| \leq K,\ b_t(\omega) X_t(\omega) \leq -(\lambda/2) X_t^2(\omega).$$

(1) Show that for all $T \geq 0$, $\sup_{0 \leq t \leq T} \mathbb{E}[X_t^2] < +\infty$.
(2) Applying Itô's formula to $(\exp(\lambda t) X_t^2)_{t \geq 0}$, show

$$\sup_{t \geq 0} \mathbb{E}[X_t^2] < +\infty.$$