Université Nice-Sophia Antipolis
SMEMP302 - ECUE Probabilitic computational methods, 2023-2024
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Answers for Home Project

(1) The increments of W are independant so

$$\mathbb{P}(A_N) = \mathbb{P}(|W_{t_1}| \geq \frac{3N}{T})\mathbb{P}(\sup_{1\leq k\leq N} |W_{t_{k+1}} - W_{t_k}| \leq 1)\,.$$

If $\sup_{t\in[0,T]} |W_t(\omega)| \leq 1/2$, then for all $t, s \in [0,T]$, $|W_t(\omega) - W_s(\omega)| \leq |W_t(\omega)| + |W_s(\omega)| \leq 1$. So

$$\{\sup_{t\in[0,T]} |W_t| \leq 1/2\} \subset \{\sup_{1\leq k\leq N-1\}} |W_{t_{k+1}} - W_{t_k}| \leq 1\}\,.$$

So

$$\mathbb{P}(A_N) = \mathbb{P}(|W_{t_1}| \geq \frac{3N}{T})\mathbb{P}(\sup_{t\in[0,T]} |W_t| \leq 1/2)\,.$$

(2) We have:

$$\begin{aligned}\int_x^{+\infty} \left(1 + \frac{1}{y^2}\right) e^{-y^2/2} dy &= \left[-\frac{e^{-y^2/2}}{y}\right]_x^{+\infty} \\ &= \frac{e^{-x^2/2}}{x}\,.\end{aligned}$$

We compute:

$$\begin{aligned}\frac{xe^{-x^2/2}}{1+x^2} &= \frac{x^2}{1+x^2}\int_x^{+\infty} \left(1 + \frac{1}{y^2}\right) e^{-y^2/2} dy \\ &\leq \frac{x^2}{1+x^2}\int_x^{+\infty} \left(1 + \frac{1}{x^2}\right) e^{-y^2/2} dy \\ &= \frac{x^2}{1+x^2}\int_x^{+\infty} \left(\frac{x^2+1}{x^2}\right) e^{-y^2/2} dy \\ &= \int_x^{+\infty} e^{-y^2/2} dy\,.\end{aligned}$$

(3) We compute (as $W_{t_1} \sim \mathcal{N}(0, T/N)$

$$\begin{aligned}\mathbb{P}(|W_{t_1}| \geq \frac{3N}{T}) &= 2\int_{3N/T}^{+\infty} \frac{e^{-\frac{u^2}{2(T/N)}}}{\sqrt{2\pi(T/N)}} du \\ \text{(change of variable } y = u/(\sqrt{T/N})) &= 2\int_{3(N/T)^{3/2}}^{+\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\ \text{(Question (2))} &\geq \frac{1}{\sqrt{2\pi}} \frac{6(N/T)^{3/2}}{1 + 9(N/T)^3} e^{-9(N/T)^3/2} \\ &= \frac{6(NT)^{3/4}}{T^3 + 9N^3} \frac{e^{-9(N/T)^3/2}}{\sqrt{2\pi}}\,.\end{aligned}$$

Using Question (1), we get the desired result.

(4) We suppose $w \in A_N$, $k \geq 1$ and $|\overline{X}^N_{t_k}(\omega)| \geq 1$. We then have (first equality coming from the Euler scheme)

$$\begin{array}{rcl} |\overline{X}^N_{t_{k+1}}(\omega)| & = & |\overline{X}^N_{t_k}(\omega) + W_{t_{k+1}}(\omega) - W_{t_k(\omega)} - (\overline{X}^N_{t_k}(\omega))^3 \frac{T}{N}| \\ & \geq & \frac{T}{N}|\overline{X}^N_{t_k}(\omega)|^3 - |\overline{X}^N_{t_k}(\omega) + W_{t_{k+1}}(\omega) - W_{t_k}(\omega)| \\ & \geq & \frac{T}{N}|\overline{X}^N_{t_k}(\omega)|^3 - |\overline{X}^N_{t_k}(\omega)| - |W_{t_{k+1}}(\omega) - W_{t_k}(\omega)| \\ (\omega \in A_N) & \geq & \frac{T}{N}|\overline{X}^N_{t_k}(\omega)|^3 - |\overline{X}^N_{t_k}(\omega)| - 1 \\ (|\overline{X}^N_{t_k}(\omega)| \geq 1) & \geq & \frac{T}{N}|\overline{X}^N_{t_k}(\omega)|^3 - 2|\overline{X}^N_{t_k}(\omega)|^2 \\ & = & |\overline{X}^N_{t_k}(\omega)|^2 \left(\frac{T}{N}|\overline{X}^N_{t_k}(\omega)| - 1 \right) . \end{array}$$

(5) We prove it by recurrence on k. We suppose that $\omega \in A_N$ and $N \geq T/3$.

- For $k = 1$, $|\overline{X}^N_{t_1}(\omega)| = |W_{t_1}(\omega)| \geq \frac{3N}{T} = \left(\frac{3N}{T}\right)^{2^{1-1}}$.
- If the property is true for k ($k \geq 1$). Then $|\overline{X}^N_{t_k}(\omega)| \geq 1$ and so, by Question (4),

$$\begin{array}{rcl} |\overline{X}^N_{t_{k+1}}(\omega)| & \geq & |\overline{X}^N_{t_k}(\omega)|^2 \left(\frac{T}{N}|\overline{X}^N_{t_k}(\omega)| - 1 \right) \\ \text{(recurrence)} & \geq & \left(\frac{3N}{T}\right)^{2^k} \left(\frac{T}{N}\left(\frac{3N}{T}\right)^{2^{k-1}} - 1 \right) \\ (k \geq 1) & \geq & \left(\frac{3N}{T}\right)^{2^k} \left(\frac{T}{N}\left(\frac{3N}{T}\right) - 1 \right) \\ & = & \left(\frac{3N}{T}\right)^{2^{k+1-1}} . \end{array}$$

(6) By Question (1), (3), (5), we have (if $N \geq T/3$)

$$\begin{array}{rcl} \mathbb{E}(|\overline{X}^N_T|) & \geq & \mathbb{E}(|\overline{X}^N_T|\mathbb{1}_{A_N}) \\ & \geq & \left(\frac{3N}{T}\right)^{2^{N-1}} \mathbb{P}(A_N) \\ & \geq & \left(\frac{3N}{T}\right)^{2^{N-1}} \mathbb{P}(\sup_{t\in[0,T]} |W_t| \leq 1/2) \frac{6(NT)^{3/4}}{T^3 + 9N^3} \frac{e^{-9(N/T)^3/2}}{\sqrt{2\pi}} . \end{array}$$

We have $\mathbb{P}(\sup_{t\in[0,T]} |W_t| \leq 1/2) > 0$ (see stochastic calculus course) and

$$\begin{array}{rcl} \left(\frac{3N}{T}\right)^{2^{N-1}} \frac{6(NT)^{3/4}}{T^3 + 9N^3} \frac{e^{-9(N/T)^3/2}}{\sqrt{2\pi}} & = & \frac{1}{\sqrt{2\pi}} \frac{6(NT)^{3/4}}{T^3 + 9N^3} \exp\left(-\frac{9}{2}\left(\frac{N}{T}\right)^3 + 2^{N-1}\ln\left(\frac{3N}{T}\right) \right) \\ & = & \underbrace{\frac{1}{\sqrt{2\pi}} \frac{6(NT)^{3/4}}{T^3 + 9N^3} \exp\left(2^{N/2}\ln\left(\frac{3N}{T}\right) \right)}_{\underset{N\to+\infty}{\longrightarrow} +\infty} \\ & & \underbrace{\times \exp\left(-\frac{9}{2}\left(\frac{N}{T}\right)^3 + 2^{N/2-1}\ln\left(\frac{3N}{T}\right) \right)}_{\underset{N\to+\infty}{\longrightarrow} +\infty} \end{array}$$

(the two convergences coming from comparison theorems).

(7) We have

$$\begin{aligned}\mathbb{E}(|\overline{X}_T^N - X_T|) &\geq \mathbb{E}(|\overline{X}_T^N|) - \mathbb{E}(|X_T|)\\ &\geq \mathbb{E}(|\overline{X}_T^N|) - \mathbb{E}(|X_T|^4)^{1/4}\\ \text{(Question (6) + Assumptions)} &\underset{N\to+\infty}{\longrightarrow} +\infty\,.\end{aligned}$$

And

$$\begin{aligned}\mathbb{E}(\sup_{t\in[0,T]} |\overline{X}_t^N - X_t|) &\geq \mathbb{E}(|\overline{X}_T^N - X_T|)\\ &\underset{N\to+\infty}{\longrightarrow} +\infty\,.\end{aligned}$$

(8) We compute (using: $\forall x$, $x_+ - K \leq (x-K)_+$)

$$\begin{aligned}\frac{1}{2}\mathbb{E}(|\overline{X}_T^N|) - K &= \mathbb{E}((\overline{X}_T^N)_+ - K)\\ &\leq \mathbb{E}((\overline{X}_T^N - K)_+)\,.\end{aligned}$$

(9) We have $\mathbb{E}((\overline{X}_T^N - K)_+) \underset{N\to+\infty}{\longrightarrow} +\infty$ by Questions (8), (6). And

$$\begin{aligned}\mathbb{E}((X_T - K)_+) &\leq \mathbb{E}(|X_T| + K)\\ \text{(Assumptions)} &< \infty\,.\end{aligned}$$

(10)
```
import numpy as np
import scipy.stats as sps
N=500; T=10
def euler(N,T):
        h=T/N
        h1=np.sqrt(h)
        x=0
        z=sps.norm.rvs(size=(N))
        for i in range(N):
                x=x+h1*z[i]-h*x**3
        return(x)

def mc(M):
        tab1=np.ones(M)*N
        tab1=tab1.astype(int)
        tab2=[np.abs(euler(n)) for n in tab1]
        return(np.mean(tab2))
```