On regularity of $\overline{\partial}$ solutions on a_q domains with C^2 boundary in complex manifolds

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Introduction

We study the regularity of $\overline{\partial}$ equation

$$\overline{\partial} u = f, \quad \text{on } D$$

where f is a $\overline{\partial}$-closed $(0,q)$-form and D is a relatively compact C^2 domain in a complex manifold X.

We can consider a slightly more general situation, where f is E-valued $(0,q)$-forms for a holomorphic vector bundle E on X.

Conditions

Main assumption.

∂D has a_q property: If D is defined by $\rho < 0$, its Levi-form $L\rho$ has at least $(q+1)$ negative eigenvalues, or at least $(n-q)$ positive eigenvalues on ∂D.

In general the equation may not be solvable if we don't impose further conditions on D.

A minimum requirement: $\overline{\partial} u = f$ has a solution $u \in L^2_{loc}(D)$.

Applications of a_q property on domains.

After work of Morrey (1958) and Kohn (1963, 64) on strictly pseudoconvex domains, Hörmander developed L^2 theory for a_q domains with C^3 boundary. a_q property is also called $Z(q)$ property for which the basic estimate hold for $(0, q)$ forms.

1962: Grauert used strict pseudoconvexity for neighborhood of negative normal line bundle of C to establish the formal principle for such neighborhoods of C.

1966: a_2 property was used by Griffiths to obtained “finite jet” determination of germs of neighborhoods of C in the normal direction.

1977: Hamilton used a_2 property to study the stability of small deformation of the complex structure on an a_2 domain D is a complex manifold.

Result

Our goal is to keep minimum smoothness requirement on ∂D while seeking higher order regularity of a solution u.

Here is a short version of our results:

$q = 1$: We show the existence of the solutions on the closure of the domain that gain 1/2 derivative when the given $(0, 1)$ form $f \in \Lambda^r(\overline{D})$ with $r > 1$.

$q > 1$: The same regularity result for the solutions is achieved for $(0, q)$ for f, when the boundary is either sufficiently smooth or has $(n - q)$ positive Levi eigenvalues.

Here $\Lambda^a = C^{k,\alpha}$ for $k \in \mathbf{N}$ and $0 < \alpha < 1$. By $u \in \Lambda^1(\mathbf{R}^N)$ we mean

$$|u(x+y) + u(x-y) - 2u(x)| \leq A|y|$$

for all $x, y \in \mathbf{R}^N$.

Theorems

We will prove the following.

Theorem ($q = 1$ or strictly $(n-q)$-convex boundary)

Let $a \in (1, \infty)$ and $1 \leq q \leq n$. Let D be an a_q domain with C^2 boundary in X. If $\overline{\partial} u = f$ has an L^2 solution u_0, there is a linear solution operator $u = H_q f$ so that

$$|H_q f|_{\Lambda^{a+1/2}(\overline{D})} \leq C_a(D) |f|_{\Lambda^a(\overline{D})}.$$

In our proof, it is important that $\overline{\partial} f = 0$.

Early results by G. and Shi-Yao.

What is special about $(0,1)$-forms?

Basic estimate: Morrey for $(0,1)$-forms; Kohn for $(0,q)$ forms.

$\overline{\partial}$ solutions of gaining $1/2$ derivative for C^k forms f was proved by Siu for $(0,1)$-forms. And the same results for $(0,q)$-forms was proved by Lieb-Range for all $q>0$, when D is in $\mathbf{C}^n$ and $\partial D \in C^{k+2}$.

We will first prove the following.

Theorem ($q \geq 1$ and a_q domains)

Let $a \in (1, \infty)$. Let D be an a_q domain with $\Lambda^{a+5/2}$ boundary in X. If $\overline{\partial} u = f$ has an L^2 solution u_0, there is a linear solution operator $u = H_q f$ so that

$$|H_q f|_{\Lambda^{a+1/2}(\overline{D})} \leq C |f|_{\Lambda^a(\overline{D})}.$$

Again, we require $\overline{\partial}$-closedness of f to construct H_q and estimate $H_q f$.

The above two theorems were conjectured by Henkin-Leiterer. They proved a local result for 1/2 estimate when $f \in C^0$ by using a local homotopy formula.

Results for strictly pseudoconvex domains

- Sup norm estimates (Grauert-Lieb and Lieb)
- $C^{1/2-0}$-estimate (Kerzman); $C^{1/2}$ estimate (Henkin-Romanov)
- $C^{k+1/2}$ estimate: $(0,1)$ forms (Siu); $(0,q)$ forms (Lieb-Range). Lieb-Range constructed a new solution operator $u = H_q f$.
- Gong: Strictly pseudoconvex C^2 domains in $\mathbf{C}^n$ for $\Lambda^{a+1/2}$ estimates for homotopy operators H_q, H_{q+1} in

  $$f = \overline{\partial} H_q f + H_{q+1} \overline{\partial} f.$$

  When f is a $\overline{\partial}$ closed $(0,1)$-form, $\partial D \in C^\infty$ and $a \in (0,\infty)$, the result for $\overline{\partial}$ solutions is due to Phong-Stein for $\mathbf{C}^2$ and Greiner-Stein for $\mathbf{C}^n$.
- G.-Lanzani: $\Lambda^{a+1/2}$ estimates for *strongly* $\mathbf{C}$-linear convex domain with $C^{1,1}$ boundary:

  $$|\rho_\zeta \cdot (\zeta - z)| \geq c|\zeta - z|^2, \quad \zeta \in \partial D, \quad z \in \overline{D}.$$
- Shi: Gain almost 1/2-derivative for weighted L^p Sobolev spaces for strictly pseudoconvex domains with C^2 boundary.

Ziming Shi and Liding Yao (2021):

If D is a strictly pseudoconvex domain with C^{k+3} boundary, then $\overline{\partial} u = f$ admits $H^{s+1/2,p}$ solutions u for $f \in H^{s,p}$, when

$$-k + 1/p < s < 0, \quad 1 < p < \infty, \ i.e. \ s \text{ is negative!}$$

This is the first result since L^2 theory for boundary regularity of solutions in the distribution sense.

An open problem

Open Problem for $q > 1$. Let $D \Subset X$ be an $(q+1)$ *concave* domain with C^2 boundary. Let $f = \overline{\partial} u_0$ be continuous on $\overline{D}$ with $u_0 \in L^2(D)$. Does there exist $u \in \Lambda_{1/2}(\overline{D})$ such that $\overline{\partial} u = f$ in D?

Recall that for strictly pseudoconvex domains with C^2 boundary in $\mathbf{C}^n$. The answer is affirmative (Henkin-Romanov). Here the issue is $\partial D \in C^2$. If $\partial D \in C^3$, an affirmative answer is obtained by Henkin-Leiterer for *local* solutions.

Main ingredients in our approach.

1. We will find local solutions with exact regularity near each boundary point of ∂D by constructing local homotopy formula

$$f = \overline{\partial} H_q f + H_{q+1} \overline{\partial} f.$$

2. Use these local solutions and Grauert's bumping method to achieve

$$f = \overline{\partial} u_1 + \tilde{f}$$

where $u_1 \in \Lambda^{r+1/2}(\overline{D})$ and $\tilde{f} \in \Lambda^a(\tilde{D})$ remains $\overline{\partial}$-closed on a larger domain $\tilde{D}$.

3. By the stability of L^2 solutions for a_q domains, which is due to Hörmander, we know that $\tilde{f}$ is still solvable in L^2 space.

4. We can show that Kohn's canonical solution $\tilde{u}$, $\overline{\partial}\tilde{u} = \tilde{f}$, is in $\Lambda^{r+1}_{loc}(\tilde{D})$, by adapting a method of Kerzman.

A local homotopy formula for strictly pseudoconvex C^2 domains in $\mathbf{C}^n$ constructed by using a method of Lieb-Range

The local homotopy formula has the form

$$H_q\varphi(z) = \int_{\mathbf{C}^n} \Omega^0_{0,q-1}(z,\zeta) \wedge E\varphi(\zeta) + \int_{\mathbf{C}^n\setminus D} \Omega^{01}_{0,q-1}(z,\zeta) \wedge [\overline{\partial}, E]\varphi(\zeta)$$

An important feature in the above homotopy operator is the commutator $[\overline{\partial}, E]$. We have

$$[\overline{\partial}, E]\varphi|_D = 0.$$

So the second term is C^∞ in D.

Estimate of the integral in the exterior domain $\mathbf{C}^n \setminus D$

We now deal with the boundary term illustrated for a simpler situation:

Proposition

Let $\partial D \in C^2$ be strictly convex. Let

$$Kf(z) = \int_{\xi \in \mathcal{U} \setminus D} \frac{A(\partial_\zeta^2 r, \zeta, z)(\overline{\zeta}_i - \overline{z}_i)}{(r_\zeta \cdot (z - \zeta))^{n-m} |\zeta - z|^{2m}} [\partial, E] f \, dV(\zeta), \quad \forall z \in D$$

where $0 \le m < n$. Let $a > 1$. Then

$$\|Kf\|_{\overline{D}; a+1/2} \le C \|f\|_{\mathcal{U}; a}.$$

Proof. We need a non-trivial fact: When $\partial D \in Lip$ and $f \in C^\infty(D)$ then $Kf \in \Lambda^b(\overline{D})$ if

$$|\partial^k Kf(z)| \le A \, dist(z, \partial D)^{b-k}, \quad k = [b] + 1.$$

The above argument can be modified near q convex boundary points via local homotopy formula and we can still introduce commutator $[\overline{\partial}, E]$ to get rid of boundary integral.

Notice that for the proof to work for $\rho \in C^2$. It is important that $\partial_\zeta^2 r$ does not depend on z, i.e. it is C^∞ in z.

This feature is *lost* when the boundary become concave. In the concave case, we must replace $\partial_\zeta^2 \rho$ by $\partial_z^2 \rho$. In fact, we can write the integral as

$$Kf(z) = \partial_z^2 \rho \int_{\xi \in \mathcal{U} \setminus D} \frac{\overline{\zeta}_i - \overline{z}_i}{(\rho_\zeta \cdot (z-\zeta))^{n-m} |\zeta - z|^{2m}} [\partial, E] f \, dV(\zeta), \quad \forall z \in D$$

Clearly $Kf(z)$ cannot have higher order regularity up to boundary when $\rho \in C^2$!

What is special about $(0,1)$ forms f? – An application of Hartogs' theorem to the $\overline{\partial}$ problem

When f is a $(0,1)$ form,

$$\overline{\partial}u_1 = f = \overline{\partial}u_2$$

implies that $u_2 - u_1$ is holomorphic. When ∂D is 1-concave, the $u_2 - u_1$ is holomorphic in a domain containing $\overline{D}$ by Hartogs' theorem.

Therefore, u_2, u_1 have the same regularity.

How to use concavity?

C^2 domains again

Applying Nash-Moser method, we can show the following.

Theorem ($q > 1$ and a_q domains)

Let D be an a_q domain with C^2 boundary in X. If $\overline{\partial} u = f$ has an L^2 solution u_0 and $f \in C^\infty(\overline{D})$, there exists a solution $u \in C^\infty(\overline{D})$ to $\overline{\partial} u = f$.

This time, we will not use support domains. We can use a sequence of smooth domains D_j approximate D from inside. On each D_j we find solution u_j for $\overline{\partial} u_j = f$. Then using Nash-Moser smoothing operator $S_{t_j} u_j$ and then apply cutoff functions $\chi_j S_{t_j} u_j$ and then iterate the procedure:

$$\overline{\partial} u_{j+1} = f - \partial(\chi_j S_{t_j} u_j).$$

We can construct a genuine solution u that is smooth (C^∞) on $\overline{D}$.

A counter-example (Sibony): There is a smooth pseudoconvex domain (that is not finite type) such that $\overline{\partial}u = f$ has no Λ^r solution u for some $f \in \Lambda^r$.

An open problem: If D is C^2 pseudoconvex domain in $\mathbf{C}^n$, can $\overline{\partial}u = f$ have $C^\infty(\overline{D})$ solution when $f \in C^\infty(\overline{D})$?

Thank you for your attention!