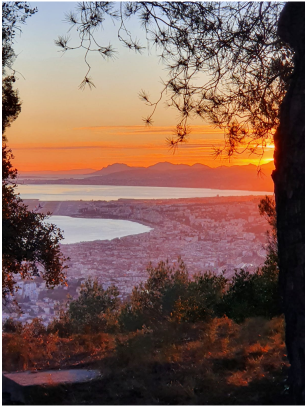
The Borel Map

Bernhard Lamel (Joint work with P. Cordaro & G. Della Sala)

TEXAS A&M UNIVERSITY AT QATAR
Science

SCV, CR, and Dynamics
Nice 2021

First of all, congratulations to Xiaojun–we miss you here!



A classical result

- $\mathcal{C}_p^\infty(\mathbb{R})$: germs of (complex valued) smooth functions at $p \in \mathbb{R}$
- $\mathbb{C}[[x-p]]$: formal power series ring in one variable x

Theorem

The Taylor expansion map $\mathcal{C}_p^\infty(\mathbb{R}) \to \mathbb{C}[[x-p]]$ is onto.

The typical proof utilizes a bump function φ and realizes the formal Taylor series $\sum_k A_j (x-p)^j$ as the Taylor series of

$$f(x) = \sum_k A_j \varphi(\lambda_j x)(x-p)^j$$

for a sufficiently fast increasing series $\lambda_j \to \infty$.

Freezing PDEs and formal solutions

If we have a PDO $P = P(x, \partial_x)$ of order k with $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$P(x, \partial_x) = \sum_{|\alpha|=1}^{k} P_\alpha(x)\partial_x^\alpha, \quad P_\alpha \in \mathcal{C}_0^\infty(\mathbb{R}^n)$$

then we can "freeze" its coefficients at a point (say 0):

$$\hat{P}(x, \partial_x) = \sum_{|\alpha|=1}^{k} \hat{P}_\alpha(x)\partial_x^\alpha, \quad \hat{P}_\alpha(x) \in \mathbb{C}[[x]]$$

where $\hat{P}_\alpha(x)$ is the Taylor series of P_α at the origin.
A formal power series $A(x)$ is said to be a *formal solution* of P if

$$PA = \sum_{|\alpha|=1}^{k} \hat{P}_\alpha(x)(\partial_x^\alpha A(x)) = 0 \in \mathbb{C}[[x]].$$

Formal and real solutions

Every solution u of P gives rise to the formal solution $\hat{u}$ given by the Taylor series of u at the origin.

General Borel Question

Given a formal solution A of P, does there exist a solution u of P such that $\hat{u} = A$?

We refer to the map $\mathfrak{b}$ associating to the solution u the formal solution $\hat{u}$ as the *Borel map* of P.

Borel property

We say that P has the Borel property if $\mathfrak{b}$ is onto.

Unique Continuation

P satisfies a (strong) unique continuation principle if and only if $\mathfrak{b}$ is injective.

Failure of the Borel property

Consider a Levi-flat hypersurface M, and as a partial differential operator $\bar{\partial}_b$. Then one sees that the image of the Borel map is contained in $\mathbb{C}\{z\}[[w]]$, where z is a variable in the leaf of the Levi-foliation through 0 and w is a transverse variable.

In general, *failure of the Borel property* occurs if solutions of P have non-formal restrictions, like maximum theorems.

On the other hand *unique continuation* often happens if we have maximum theorems.

Another general question

What properties of P are encoded in properties of its Borel map?

First order linear equations: Algebraic and functional analytic methods available!

Locally integrable structures

We say that $(M, \mathcal{V})$, where $\mathcal{V} \subset \mathbb{C}TM$, is a locally integrable structure if for every $p \in M$ there exists a neighbourhood U of p and N complex-valued functions $Z_1, \dots, Z_N$ such that the dZ_j are linearly independent and span $T'M = \mathcal{V}^\perp$.

Theorem (The Baouendi-Treves Approximation Theorem)

For every p in M there exists a compact neighbourhood K of p such that every solution f of $\mathcal{V}$ is a uniform limit of polynomials in basic solutions $Z = (Z_1, \dots, Z_N)$ on K: For every $\varepsilon > 0$ there exists $P_\varepsilon \in \mathbb{C}[Z]$ such that

$$|f(x) - P_\varepsilon(Z(x))| < \varepsilon \quad \forall x \in K.$$

Therefore, the sets $Z(K) \subset \mathbb{C}^N$ and their complex analytic properties have a strong relationship to solutions of locally integrable structures.

Hypocomplexity and unique continuation

We say that a locally integrable structure $(M, \mathcal{V})$ is hypocomplex at $p \in M$ if every germ of a solution u of $\mathcal{V}$ is of the form $u = h \circ Z$ where h is a germ of a holomorphic function at $Z(p)$.

Theorem (Characterization of Hypocomplexity)

TFAE:

1. *$(M, \mathcal{V})$ is hypocomplex at p;*
2. *For every (small) compact neighbourhood $K \subset\subset M$ of p, the polynomially convex hull of $Z(K)$ is a neigbourhood of $Z(p)$;*
3. *For every (small) compact neighbourhood $K \subset\subset M$ of p, the rationally convex hull of $Z(K)$ is a neigbourhood of $Z(p)$.*

Hypcomplexity $\Longrightarrow$ Unique continuation property.

In corank 1 structures, the two properties are in fact equivalent (Math. Ann. 2020)

The algebra $A^\infty(K)$

For $K \subset \mathbb{C}^N_Z$, the algebra $A^\infty(K)$ contains all "functions" which have the property that they are approximated in all of their derivatives by holomorphic polynomials. Formally:
Consider the set of formal power series $\mathcal{C}(K)[[Z]]$ with coefficients which are continuous functions on K.
Then $\mathcal{O}(\mathbb{C}^N)$ embeds into $\mathcal{C}(K)[[X]]$ by the algebra homomorphism

$$h(Z) \mapsto \sum_{\alpha \in \mathbb{N}^N} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} h}{\partial Z^\alpha}\bigg|_K X^\alpha.$$

$A^\infty(K)$ is defined to be the closure of the image of this homomorphism under the seminorm system

$$p_k\left(\sum A_\alpha(Z) X^\alpha\right) = \sum_{|\alpha| \le k} \sup_{Z \in K} |A_\alpha(Z)|, \quad k \in \mathbb{N}.$$

For flexibility

It is sometime necessary and interesting to work with "relative" versions of $A^\infty(K)$. We define $A^\infty(K;L)$ to be the closure of $\mathcal{O}(L)$ in $\mathcal{C}(K)[[X]]$ as before. Then by Oka-Weil:

$$A^\infty(K,\widehat{K}) = A^\infty(K) \quad \widehat{K} \text{ polynomially convex hull of } K;$$

$$A^\infty(K,\widehat{K}_R) = R^\infty(K) \quad \widehat{K}_R \text{ rationally convex hull of } K.$$

The spectrum

The spectrum of $A^\infty(K)$ is the polynomial hull of K. More generally, one can identify the spectrum of $A^\infty(K,L)$ with the limit of the spectra $\sigma(K,U)$ of the Banach algebras $A(K,U)$, where U ranges over the neighbourhoods of L. In particular, if L has a Stein neighbourhood basis, then the spectrum of $A^\infty(K,L)$ is

$$\widehat{K}_L = \bigcap_{\Omega\supset L} \widehat{K}_\Omega$$

The Borel map of K

We assume from now on that $0 \in K$.

Definition

The continuous algebra homomorphism

$$\mathfrak{b} \colon A^\infty(K, L) \to \mathbb{C}[[X]], \quad \sum A_\alpha(Z) X^\alpha \mapsto \sum A_\alpha(0) X^\alpha$$

is called the Borel map of K relative to L (at the origin).

An algebraic property

Since $\mathbb{C}[[X]]$ is a local ring and (unless K is a point) $A^\infty(K, L)$ is not local, the Borel map is *never* an isomorphism of algebras.

An example

Consider $K \subset \mathbb{C}^2_{(z,w)}$ defined by

$$K = \{|z| < 1, w \in [0,1]\}.$$

Then

$$\mathfrak{b}(A^\infty(K)) \cong \mathfrak{b}(A^\infty(\Delta)) \otimes \mathbb{C}[[w]].$$

(Non)surjectivity of the Borel map

Proposition (Non-surjectivity and Cauchy estimates)

$\mathfrak{b}$ *is* not surjective *if and only if there exists a sequence of polynomials* P_j *of degree at least* j*, and a constant* m *such that*

$$(*)\colon |P_j(\partial)f(0)| \leq p_m(f) = \sum_{|\alpha|\leq k} \sup_{Z\in K} \left| \frac{\partial^{|\alpha|} f}{\partial Z^\alpha} \right|, \quad \forall f \in \mathcal{O}(L).$$

Proof: $\mathfrak{b}$ is not a homomorphism, so $\mathfrak{b}^t$ is not weakly continuous.

Proposition (Complex Structure implies non-surjectivity)

Assume that there is a complex disc passing through 0 *contained in* $\widehat{K}_L$*. Then* $\mathfrak{b}$ *is not surjective.*

Proof: If $g(\zeta)$ is the disc, use

$$P_j(\partial)f(0) = \left.\frac{d^k}{d\zeta^k}\right|_{\zeta=0} f \circ g.$$

Peaking and surjectivity

If K has a suitable peak function then the Borel map of K *is* surjective. However, there is a weaker property which also implies surjectivity.

Peaking families

We say that $\left\{u_{\alpha,\varepsilon}\colon \alpha \in \mathbb{N}^N, \varepsilon \in [0,\varepsilon_\alpha)\right\} \subset \mathcal{O}(L)$ is a *peaking family* if either

$$p_{|\alpha|-1}(u_{\alpha,\varepsilon}) < C_\alpha, \quad \frac{\partial^{|\alpha|} u_{\alpha,\varepsilon}}{\partial Z^\alpha}(0) \to \infty \, (\varepsilon \to 0)$$

or

$$p_{|\alpha|-1}(u_{\alpha,\varepsilon}) \to 0 \, (\varepsilon \to 0), \quad \frac{\partial^{|\alpha|} u_{\alpha,\varepsilon}}{\partial Z^\alpha}(0) = 1.$$

One can define partial peaking families in an analogous way.

Proposition

Existence of a peaking family implies surjectivity of the Borel map.

Sufficient condition for surjectivity

Define

$$\mathcal{N}(K,L) = \left\{ f'(0) \colon f \in \mathcal{O}(L) \setminus \{0\}, \operatorname{Re} f|_K \le 0 \right\} \subset \mathbb{C}^N.$$

One checks that every family of polynomials P_j appearing in $(*)$ necessarily satisfies

$$\tilde{P}_j(v) = 0, \quad \forall v \in \mathcal{N}(K,L)$$

where $\tilde{P}_j$ is the highest degree homogeneous part of P_j. So:

Proposition

If $\mathcal{N}(K,L)$ is large enough (i.e. contains a maximally real set for example), then the Borel map is surjective.

Corollary

If K is a convex polyhedron, and 0 is a vertex (or the face through 0 does not contain a complex line), then the Borel map of K is surjective.

Excursion: One dimension

The one-dimensional case is quite clean, and one gets a good intuition of what to *avoid* when trying to answer questions about the injectivity of the Borel map in higher dimensions.

Theorem (JFA, 2020)

If $K \subset \mathbb{C}$ is subanalytic and $0 \in K$, then the following dichotomy holds:

1. 0 *is in the interior of K and the Borel map is injective;*
2. 0 *is on the boundary of K and the Borel map is surjective.*

Here the Borel map is with respect to the rationally convex hull (i.e. K).

One can explore a range of "milder" conditions on K, but one runs into (typical) topological problems. In particular, the one-dimensional theorem also holds if K is non-spiraling or if K is path-bounded.

Open Mapping Property

Another somewhat mysterious property of solutions of first order equations is when they happen to *all be open*. It's closely related to the maximum principle.

Theorem

Let $K \subset L \subset \mathbb{C}^N$, with K and L compact, $0 \in K$, and assume that L is connected and subanalytic. Assume also that $\mathfrak{b}$ is injective. If $h \in \mathcal{O}(L)$ vanishes at the origin and if L is not contained in the zero set of h then $h(L)$ is a neighborhood of the origin in the complex plane.

The proof utilizes the one-dimensional result to construct a peaking family from a non-open map.

Last words

Theorem

If $K \subset \mathbb{C}^N$ is convex and subanalytic, and $0 \in K$, then the following dichotomy holds:

1. 0 *is in the interior of* K *and the Borel map is injective;*
2. 0 *is on the boundary of* K *and the Borel map is surjective.*

Actually, we can say quite a bit more in the convex setting: If $\mathfrak{b}$ is neither surjective nor injective, then the image of the Borel map can be thought of as splitting into a tensor product of "holomorphic" factors corresponding to complex varieties in the boundary of K and "formal" factors corresponding to the other directions.

Conjecture

If K is subanalytic and polynomially/rationally convex, then the Borel map is injective if and only if K is a neighbourhood of 0.

Thank you for your attention!

Thanks to the organizers!

The local organizing committee has done a tremendous job accomodating different timezones, schedules, and weather restrictions.





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