Obstruction flat rigidity of the standard CR 3-sphere

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Credits.

- Joint work with Sean Curry, Oklahoma State University.

Introduction.

- Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$.
- Assume Ω has a smooth (C^∞) boundary $M = M^{2n-1} := \partial\Omega$.
- Consider the complex Monge–Ampère operator

$$J(u) := (-1)^n \det \begin{pmatrix} u & u_{\bar{z}_k} \\ u_{z_j} & u_{z_j \bar{z}_k} \end{pmatrix},$$

and the Dirichlet problem

$$J(u) = 1, \quad u|_M = 0.$$

- Solution produces a complete Kähler–Einstein metric in Ω:

$$g_{j\bar{k}} := -\partial^2/\partial z_j \partial \bar{z}_k \log H, \quad H = u^{-(n+1)}.$$

- Cheng–Yau: $\exists$ unique solution $u \geq 0$ in $C^\infty(\Omega) \cap C^{n+2-\epsilon}(\overline{\Omega})$.

Approximate Solutions and the Obstruction Function.

- Fefferman: $\exists r \in C^\infty(\overline{\Omega})$ such that

$$J(r) = 1 + O(r^{n+1}), \quad r|_M = 0,$$

  and r is unique mod $O(r^{n+2})$.

- For any such *Fefferman defining function* r, we have

$$J(r) = 1 + \mathcal{O} r^{n+1} + O(r^{n+2}), \quad \mathcal{O} \in C^\infty(M).$$

- $\mathcal{O}$ is called the *obstruction function/density*.
- Graham: $\mathcal{O}$ is a *local* CR invariant. The Cheng-Yau solution is C^∞-smooth up to $U \subset M \iff \mathcal{O} = 0$ in $U \iff \exists$ a Fefferman defining function r with $J(r) = 1$ near U.
- A strictly pseudoconvex CR manifold M is *obstruction flat* if $\mathcal{O} = 0$.

The unit ball in $\mathbb{C}^n$.

- In the unit ball $\mathbb{B}^n$, the Cheng–Yau solution is given by $u(z) = 1 - |z|^2$.
- Since $u \in C^\infty(\overline{\mathbb{B}^n})$, it follows that the unit sphere S^{2n-1} is obstruction flat (i.e., $\mathcal{O} = 0$).
- Since $\mathcal{O}$ is a local CR invariant, any CR manifold M that is (locally) spherical is also obstruction flat.
- Converse is not true. A strictly pseudoconvex CR manifold can be obstruction flat on an open set U without being locally spherical.
- **Q:** Are there smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with obstruction flat but not spherical boundaries? **Q':** Compact strictly pseudoconvex CR manifolds more generally?
- In $\mathbb{C}^2$, the obstruction function $\mathcal{O}$ coincides with the boundary trace of the log-term in the Bergman kernel.
- **Strong Ramadanov Conjecture.** No to **Q** (in $\mathbb{C}^2$).

Obstruction flat CR manifolds of higher (≥ 5) dimension.

- Let (M^n, g) be a compact Kähler manifold.
- Let $(L, h) \to M$ be a Hermitian line bundle whose curvature form $=$ the Kähler form ω_g. Let (L^*, h^*) be its dual.
- The unit circle bundle $S(L^*) = \{v \in L^* \colon h^*(v, \bar{v}) = 1\}$ is a compact strictly pseudoconvex CR manifold of dimension $2n + 1$.
- $\mathrm{Ric}(g)$ induces an endomorphism $\mathrm{RicOp}(g) \colon TM \to TM$.

Theorem 0 (E.–M. Xiao–H. Xu, in prep.)

If $\mathrm{RicOp}(g)$ has constant eigenvalues, then $S(L^*)$ is obstruction flat.

Remarks.

- The condition on $\mathrm{RicOp}(g)$ holds if, e.g., (M, g) is Kähler-Einstein.
- If $n \geq 2$, then $S(L^*)$ is spherical $\iff$ M is Bochner flat. These are classified in a explicit list of compact quotients of Hermitian symmetric spaces. $\implies$ Yes to **Q'**.
- If $n = 1$, then $S(L^*)$ obstruction flat $\implies$ $S(L^*)$ spherical.

Obstruction flat rigidity of the unit sphere in $\mathbb{C}^2$.

Theorem 1 (Curry-E., 2019)

Let $\Omega \subset \mathbb{C}^2$ be a bounded domain with a smooth strictly pseudoconvex boundary $M = \partial\Omega$ and assume that M has an approximate* infinitesimal symmetry. If M is obstruction flat, then M is spherical.

As a result, we obtain obstruction flat rigidity of the unit sphere in $\mathbb{C}^2$ with respect to deformations inside $\mathbb{C}^2$. Note that most deformations of the standard CR 3-sphere are *not* embeddable.

Corollary 1 (Curry-E., 2019)

Let $\Omega \subset \mathbb{C}^2$ be a bounded domain with a smooth strictly pseudoconvex boundary $M = \partial\Omega$ and assume that M is a small* deformation of the unit sphere. If M is obstruction flat, then Ω is biholomorphic to $\mathbb{B}^2$.

- The analogous result in $\mathbb{C}^n$ with $n \geq 3$ is due to Hirachi (2019). Note that, in this case, there is no difference between embeddable and non-embeddable deformations.

Main topic of this talk.

- Obstruction flat rigidity of the standard CR 3-sphere under *general* deformations of the CR structure?
- When the CR structure is non-embeddable, the interpretation as the obstruction problem for the Cheng-Yau solution is no longer valid but the question can be thought of as an analogous obstruction problem in conformal geometry on the 4-dimensional Fefferman space.
- The obstruction flat equation corresponds to the Bach flat equation (equations of motion) in conformal gravity on the Fefferman space.

CR structures on the 3-sphere S^3.

- Let (S^3, H, J_0) denote the standard CR 3-sphere, identified with $\partial\mathbb{B}^2 \subset \mathbb{C}^2$ and equipped with the $(1,0)$-vector field $Z_1 = \bar{w}\partial/\partial z - \bar{z}\partial/\partial w$ tangent to $\partial\mathbb{B}^2$.
- We shall keep the contact structure (S^3, H) with contact form $\theta = -i(\bar{z}dz + \bar{w}dw)$ fixed and deform the CR structure on (S^3, H).
- Any* CR structure on S^3 is equivalent to (S^3, H, J_φ) given ${\varphi_1}^{\bar{1}} \in C^\infty(S^3)$ via

$$Z_1^\varphi = Z_1 + {\varphi_1}^{\bar{1}} Z_{\bar{1}} \qquad (Z_{\bar{1}} = \overline{Z_1}).$$

- The equation $\mathcal{O} = 0$ is a 6th order nonlinear PDE for $\varphi = {\varphi_1}^{\bar{1}}$.

Spherical harmonics, a slice theorem, and embeddability.

- Denote the space of spherical harmonics of bidegree (p, q) by $H_{p,q}$. These form eigenspaces for the sub-Laplacian $\Delta_b = Z_1 Z_{\bar{1}} + Z_{\bar{1}} Z_1$ and give an orthogonal decomposition of $L^2(S^3)$.
- Given a deformation tensor $\varphi = \varphi_1{}^{\bar{1}}$, consider its spherical harmonic decomposition $\varphi = \sum_{p,q} \varphi_{p,q}$ and define
  $\mathfrak{D}_0^{\perp} = \{\varphi \colon \varphi_{p,q} = 0 \text{ if } q \geq 2\}$;
  $\mathfrak{D}'_{BE} = \{\varphi \colon \varphi_{p,q} = 0 \text{ if } q < p + 4, \operatorname{Im}\,(Z_{\bar{1}})^2 \varphi_{p,p+4} = 0\}$.

Theorem 2 (Slice and Embeddability Theorem; Curry–E., 2020)

Any CR structure on S^3 sufficiently close to (S^3, H, J_0) is equivalent to (S^3, H, J_φ) with $\varphi \in \mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^{\perp}$ and φ is unique up to the action of $\operatorname{Aut}_{CR}(S^3, H, J_0)$. The structure is embeddable $\iff$ $\varphi \in \mathfrak{D}'_{BE}$.

- Credits: Cheng–Lee, Burns–Epstein, Bland, Lempert.
- Separate slide.

Obstruction flat rigidity of the standard CR 3-sphere under general deformations.

Theorem 3 (Obstruction Flat Rigity; Curry–E., 2020)

The standard CR 3-sphere is rigid under obstruction flat deformations. I.e., $\exists$ an open neighborhood $\mathcal{U}$ of CR structures on S^3 near the standard CR 3-sphere (S^3, H, J_0) such that any obstruction flat CR structure in $\mathcal{U}$ is equivalent to (S^3, H, J_0).

By Theorem 2, it suffices to prove the following PDE result.

Theorem 4 (Curry–E., 2020)

There is an open neighborhood $\mathcal{V}$ of 0 in $\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^{\perp}$ such that

$$\mathcal{O}(\varphi) = 0 \iff \varphi = 0.$$

$\mathcal{V}$ given by norm in the Folland–Stein space H^3_{FS}.

Main idea and outline of proof of Theorem 4.

Fix the standard contact form θ on $(S^3, H) \to$ pseudohermitian structures on (S^3, H, J_φ). The map (6th order PDO) $\varphi \mapsto \mathcal{O}$ can be broken up as

$$\varphi \longrightarrow Q_{11}, A_{11}, \nabla \longrightarrow \mathcal{O} = \nabla^1\nabla^1 Q_{11} - iA^{11}Q_{11},$$

where $\varphi \mapsto Q = Q_{11}$ is a 4th order PDO. The proof hinges on a carefully analysis of the linearized operators DQ and $D\mathcal{O}$ as well as exploiting certain homogeneities of the nonlinear parts. Assume $\exists \varphi^{(k)} = \epsilon_k \hat{\varphi}^{(k)}$ such that $\|\hat{\varphi}^{(k)}\|_{H^3_{FS}} = 1$, $\epsilon_k \to 0$, and $\mathcal{O}^{(k)} = 0$. Write $\hat{\varphi}^{(k)} = \hat{\varphi}^{(k)}_{\mathfrak{D}_{BE}} + \hat{\varphi}^{(k)}_{\mathfrak{D}_0^\perp}$.

(1) Show $\|\hat{\varphi}^{(k)}_{\mathfrak{D}_{BE}}\|_{H^3_{FS}} = O(\epsilon_k^{1/2}) \to 0$. This is tricky. Even though we understand the linear part of $\mathcal{O}^{(k)} = 0$, the nonlinear terms involve up to 6 derivatives and we are only controlling the H^3_{FS}-norm.

(2) Integrate the equation $\mathcal{O}^{(k)} = 0$ and use (1):

$$0 = \int_{S^3} \mathcal{O}^{(k)}\, \theta \wedge d\theta = \ldots = \epsilon_k^2 \int_{S^3} DQ(\hat{\varphi}^{(k)}_{\mathfrak{D}_0^\perp}) \cdot i\overline{\nabla_0 \hat{\varphi}^{(k)}_{\mathfrak{D}_0^\perp}}\, \theta \wedge d\theta + O(\epsilon_k^{5/2}).$$

Main idea of proof continued.

(2) It follows that

$$\lim_{k\to\infty}\int_{S^3} DQ(\hat{\varphi}^{(k)}_{\mathfrak{D}_0^\perp})\cdot \overline{i\nabla_0\hat{\varphi}^{(k)}_{\mathfrak{D}_0^\perp}}\,\theta\wedge d\theta = \lim_{k\to\infty}\frac{1}{\epsilon_k^2}\int_{S^3}\mathcal{O}^{(k)}\,\theta\wedge d\theta = 0. \tag{1}$$

(3) Next, one shows that

$$\int_{S^3} DQ(\hat{\varphi}^{(k)}_{\mathfrak{D}_0^\perp})\cdot \overline{i\nabla_0\hat{\varphi}^{(k)}_{\mathfrak{D}_0^\perp}}\,\theta\wedge d\theta \geq C\|\hat{\varphi}^{(k)}_{\mathfrak{D}_0^\perp}\|_{H^3_{FS}}. \tag{2}$$

(4) Part (1) implies that $\|\hat{\varphi}^{(k)}_{\mathfrak{D}_0^\perp}\|_{H^3_{FS}} \to 1$ and, hence, (2) contradicts (1). The proof is complete.

Thank You!