Hyperbolic Cauchy-Riemann singularities and KAM-like theory for holomorphic involutions

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joint work with L. Stolovitch
SCV, CR geometry and Dynamics

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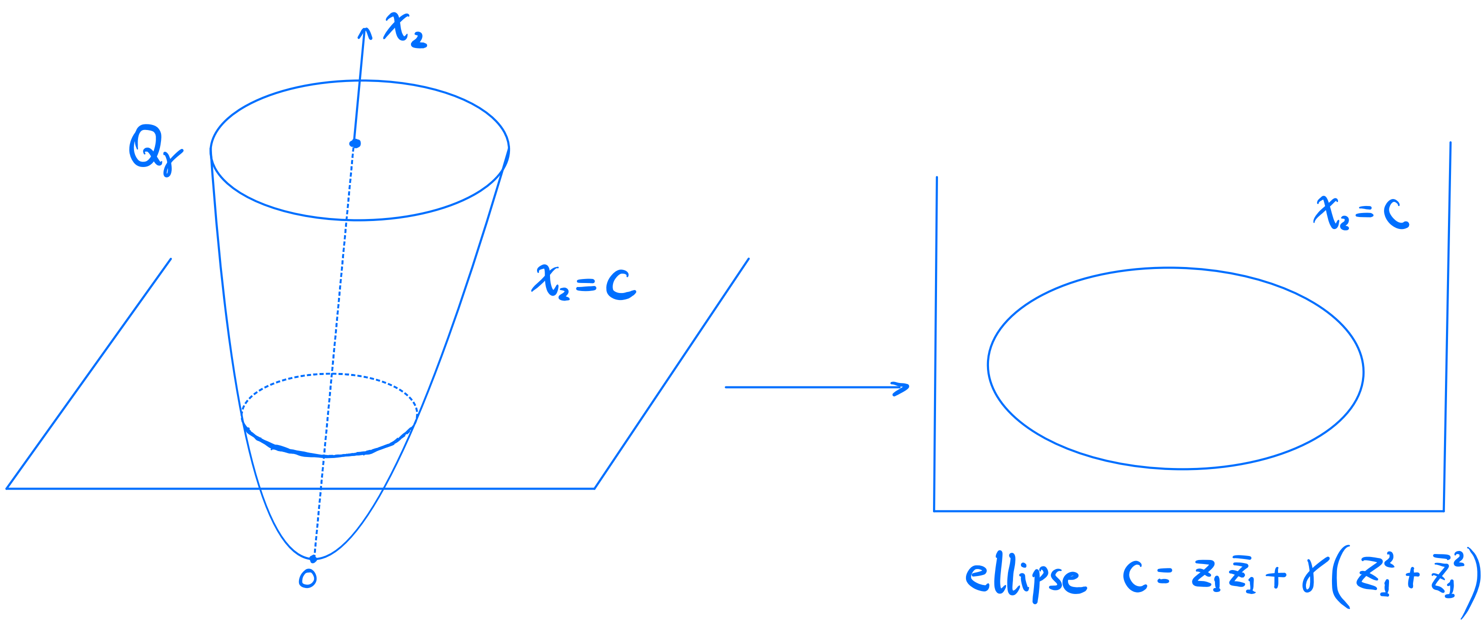
Bishop quadric

Bishop 1965: elliptic quadric

$$Q_\gamma \subset \mathbb{C}^2 : z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2), \quad 0 < \gamma < \frac{1}{2}$$



Surface with Cauchy-Riemann singularity

Surface with Cauchy-Riemann singularity: real-analytic surface $M \subset (\mathbb{C}^2, 0)$:

$$M : z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + O^3(z_1, \bar{z}_1), \quad \gamma \geq 0$$

- $Q_\gamma : z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)$ — *Bishop quadric*
- γ — *Bishop invariant*

The origin is a stable isolated *Cauchy-Riemann singularity* if $\gamma \neq \frac{1}{2}$:

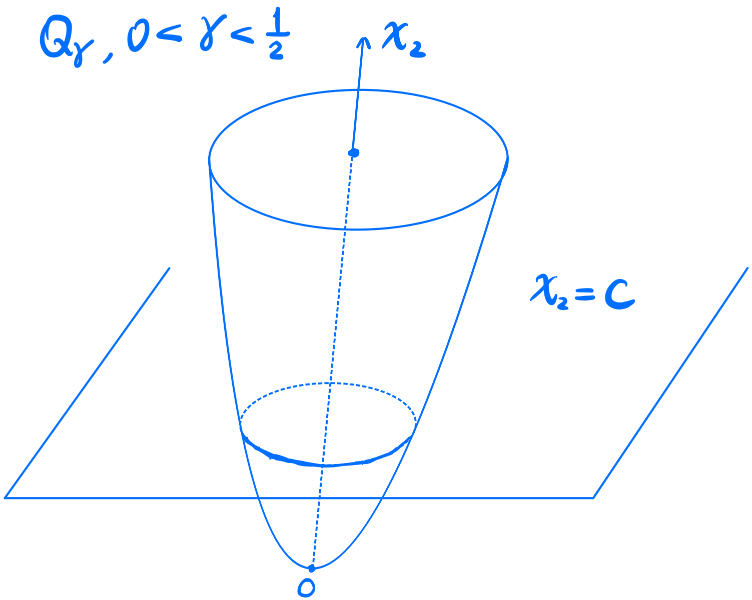
- the tangent space $T_0M = \{z_2 = 0\}$
- $\forall\ p \neq 0$, $\mathbb{C} \not\subset T_pM$ (i.e., totally real at $p \neq 0$)

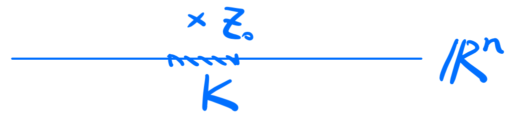
M (or the complex tangent T_0M) is called:

- *elliptic* if $0 \leq \gamma < \frac{1}{2}$
- *hyperbolic* if $\gamma > \frac{1}{2}$
- *parabolic* if $\gamma = \frac{1}{2}$

Question.

- Holomorphic flattening: $\phi(M) \subset \{\Im(z_2) = 0\}$ for some biholomorphic ϕ?
- Local hull of holomorphy





$\exists P$ polynomial s.t. $\sup_K |P(z)| < |P(z_0)|$

$$f(z) = \frac{1}{P(z) - P(z_0)}$$

Geometry near an elliptic CR singularity

Theorem (Moser-Webster 1983, normalization of elliptic surface)

For $0 < \gamma < \frac{1}{2}$, there exists a change of coordinates, holomorphic on a neighborhood of the origin, such that, in the new coordinates, M is presented by

$$\begin{cases} x_2 = z_1\bar{z}_1 + (\gamma + \delta x_2^s)(z_1^2 + \bar{z}_1^2) \\ y_2 = 0 \end{cases}, \quad z_2 = x_2 + \mathrm{i}y_2$$

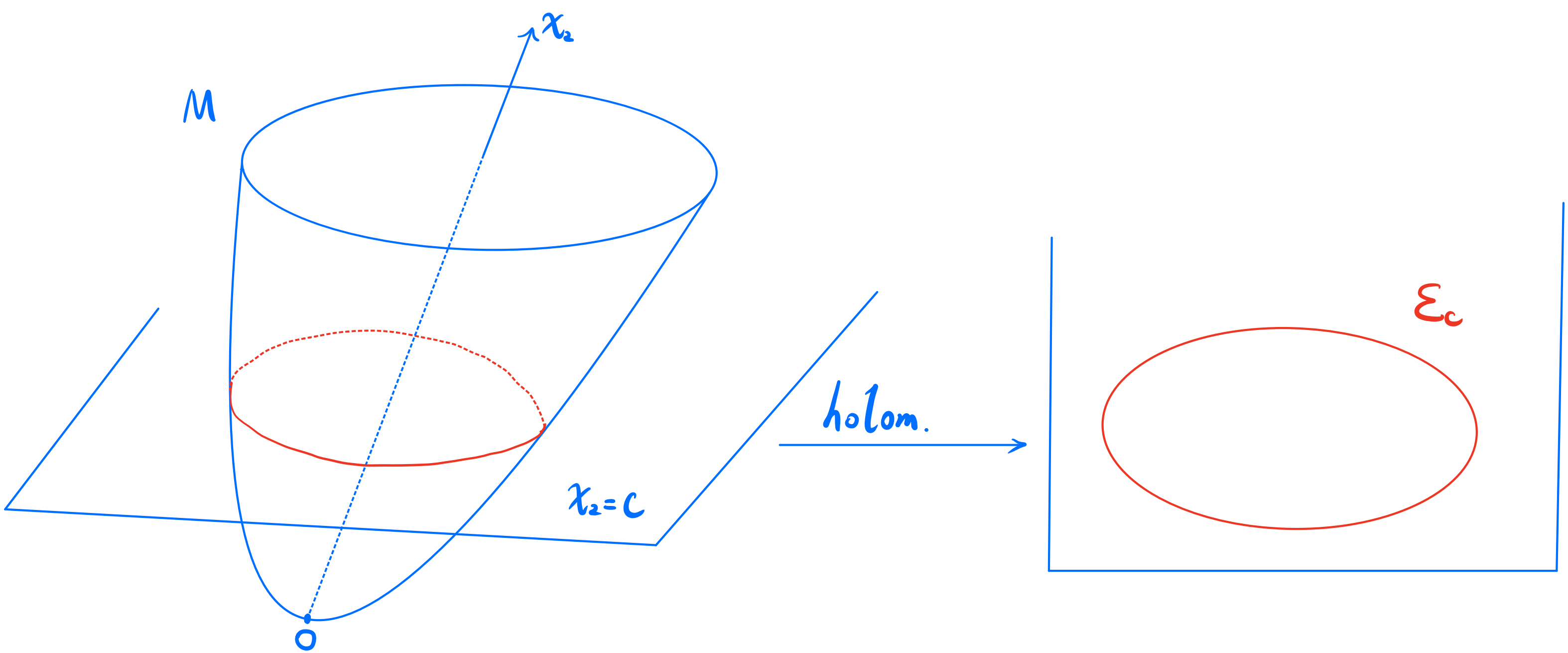
with $\delta = \pm 1$ ($s \in \mathbb{N}^*$) or $\delta = 0$ ($s = \infty$).

Remark. In "right" coordinates, M is "flat" and is still a collection of ellipes.

A real-analytic family of ellipses

An ellipse for $c > 0$

$$\mathcal{E}_c : c = z_1 \bar{z}_1 + (\gamma + c^s)(z_1^2 + \bar{z}_1^2)$$



$c \mapsto \mathcal{E}_c$ is real-analytic in $]0, r[$, with r sufficiently small

Complexification of M

$$M \subset \mathbb{C}^2 : \begin{cases} z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + H(z_1, \bar{z}_1) \\ \bar{z}_2 = \bar{z}_1 z_1 + \gamma(\bar{z}_1^2 + z_1^2) + \bar{H}(\bar{z}_1, z_1) \end{cases}$$

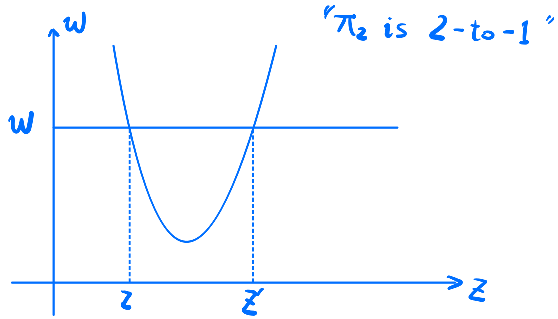
Complexification of M: $(z_1, z_2, \bar{z}_1, \bar{z}_2) \leftarrow (z_1, z_2, w_1, w_2) =: (z, w) \in \mathbb{C}^4$

$$\mathcal{M} \subset \mathbb{C}^4 : \begin{cases} z_2 = z_1 w_1 + \gamma(z_1^2 + w_1^2) + H(z_1, w_1) \\ w_2 = z_1 w_1 + \gamma(z_1^2 + w_1^2) + \bar{H}(w_1, z_1) \end{cases}$$

Invariant projections: $\pi_1(z, w) = z$ and $\pi_2(z, w) = w$ for $(z, w) \in \mathcal{M}$.

According to [Moser-Webster 1983], π_1 and π_2 are two-to-one branched covering maps:

- $\pi_2(z, w) = \pi_2(z', w)$, $(z, w), (z', w) \in \mathcal{M}$
  $\Longrightarrow$ a unique solution (z', w) with $z' \neq z$
- $\pi_1(z, w) = \pi_1(z, w')$, $(z, w), (z, w') \in \mathcal{M}$
  $\Longrightarrow$ a unique solution (z, w') with $w' \neq w$



Moser-Webster involutions

A pair of holomorphic involutions: for $\gamma > 0$

$$\tau_1 : \begin{cases} z_1' = -z_1 - \frac{1}{\gamma} w_1 + O^2(z_1, w_1) \\ w_1' = w_1 \end{cases} \quad - - - - - - - \quad \tau_1 \circ \tau_1 = Id$$

$$\tau_2 : \begin{cases} z_1' = z_1 \\ w_1' = -\frac{1}{\gamma} z_1 - w_1 + O^2(z_1, w_1) \end{cases} \quad - - - - - - - \quad \tau_2 \circ \tau_2 = Id$$

$$\tau_2 = \rho \circ \tau_1 \circ \rho, \quad \rho(z, w) := (\bar{w}, \bar{z})$$

Proposition (Moser-Webster 1983)

Holomorphic classification of $\mathcal{M} \in \mathbb{C}^4 \longleftrightarrow$ *Holomorphic classification of* (τ_1, τ_2)

Remark. Normal form of $M \subset \mathbb{C}^2 \sim$ Normal form of (τ_1, τ_2)

Appropriate change of coordinates

$$\tau_1 : \begin{cases} \xi' = \lambda\eta + O^2(\xi,\eta) \\ \eta' = \lambda^{-1}\xi + O^2(\xi,\eta) \end{cases}, \quad \tau_2 : \begin{cases} \xi' = \lambda^{-1}\eta + O^2(\xi,\eta) \\ \eta' = \lambda\xi + O^2(\xi,\eta) \end{cases}$$

$$\tau_1 \circ \tau_2 : \begin{cases} \xi' = \lambda^2\xi + O^2(\xi,\eta) \\ \eta' = \lambda^{-2}\eta + O^2(\xi,\eta) \end{cases}$$

λ is a root of $\gamma\lambda^2 - \lambda + \gamma = 0$

- For elliptic Bishop surface M, $0 < \gamma < \frac{1}{2} \Longrightarrow \lambda = \bar{\lambda}$ and $|\lambda| \neq 1$
  $\Longrightarrow$ The origin is a hyperbolic fixed point of $\tau_1 \circ \tau_2$
- For hyperbolic Bishop surface M, $\gamma > \frac{1}{2} \Longrightarrow |\lambda| = 1$
  $\Longrightarrow$ The origin is an elliptic fixed point of $\tau_1 \circ \tau_2$

Formal normal form and convergence in elliptic case

Theorem (Moser-Webster 1983, Formal normal form)

For holomorphic involutions τ_1, τ_2, if λ is not a root of unity, then there exists a formal transformation ψ such that

$$\psi^{-1} \circ \tau_1 \circ \psi : \begin{cases} \xi' = \Lambda(\xi\eta)\,\eta \\ \eta' = \Lambda^{-1}(\xi\eta)\,\xi \end{cases}, \quad \psi^{-1} \circ \tau_2 \circ \psi : \begin{cases} \xi' = \Lambda^{-1}(\xi\eta)\,\eta \\ \eta' = \Lambda(\xi\eta)\,\xi \end{cases},$$

where $\Lambda(t) \in \mathbb{C}[\![t]\!]$, and

- $\Lambda(t) = \bar{\Lambda}(t)$ *in the elliptic case*
- $\Lambda(t)\bar{\Lambda}(t) = 1$ *in the hyperbolic case*

Theorem (Moser-Webster 1983, convergence in elliptic surface)

If $\lambda = \bar{\lambda}$ and $|\lambda| \neq 1$, then Λ and ψ are holomorphic in a neighborhood of the origin.

$\Longrightarrow$ Holomorphic equivalence of initial manifold M and NF manifold.

Non-exceptional degenerate hyperbolic surface

λ is not a root of unity and $|\lambda| = 1$ (non-exceptional hyperbolic case)
Moser-Webster: the formal normalizing transformation ψ might not converge in any neighborhood of the origin.

- no holomorphic equivalence
- no holomorphic flattening

Theorem (Gong 1994, Non-exceptional degenerate hyperbolic case)

If M (equipped with τ_1, τ_2) satisfies that

1. $|\lambda| = 1$ *and* λ *verifies the Diophantine condition:*

$$|\lambda^n - 1| > \frac{c}{n^\delta}, \qquad \delta > 0, \quad c > 1,$$

2. M *is formally equivalent to the quadratic* Q_γ *(i.e.,* τ_1, τ_2 *are formally linearisable with* $\Lambda(\xi\eta) = \lambda$*)*

then M is holomorphically equivalent to Q_γ (i.e., ψ is a transformation holomorphic in a neighborhood of the origin).

Non-degenerate hyperbolic surface

$$\tau_1 : \begin{cases} \xi' = e^{\frac{i}{2}\alpha}\eta + O^2(\xi,\eta) \\ \eta' = e^{-\frac{i}{2}\alpha}\xi + O^2(\xi,\eta) \end{cases}, \quad \tau_2 : \begin{cases} \xi' = e^{-\frac{i}{2}\alpha}\eta + O^2(\xi,\eta) \\ \eta' = e^{\frac{i}{2}\alpha}\xi + O^2(\xi,\eta) \end{cases}, \quad \frac{\alpha}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$$

Non-degeneracy assumption: $\Lambda(\xi\eta) = e^{\frac{i}{2}\alpha} + \sum_{n\geq 1} c_n(\xi\eta)^n$, c_n not all vanishing

Theorem (Stolovitch-Z. 2020)

For $r > 0$ small enough, there exists an "asymptotically full-measure" $\mathcal{O}_r \subset]-r^2, r^2[$ such that $\forall\ \omega \in \mathcal{O}_r$, one has $\mu_\omega \in \mathbb{R}$ and a holomorphic transformation Ψ_ω, Whitney smooth in ω, on $\mathcal{C}_\omega^r := \{\xi\eta = \omega,\ |\xi|, |\eta| < r\}$, such that*

$$\Psi_\omega^{-1} \circ \tau_1 \circ \Psi_\omega : \begin{cases} \xi' = e^{\frac{i}{2}\mu_\omega}\eta \\ \eta' = e^{-\frac{i}{2}\mu_\omega}\xi \end{cases}, \quad \Psi_\omega^{-1} \circ \tau_2 \circ \Psi_\omega : \begin{cases} \xi' = e^{-\frac{i}{2}\mu_\omega}\eta \\ \eta' = e^{\frac{i}{2}\mu_\omega}\xi \end{cases}.$$

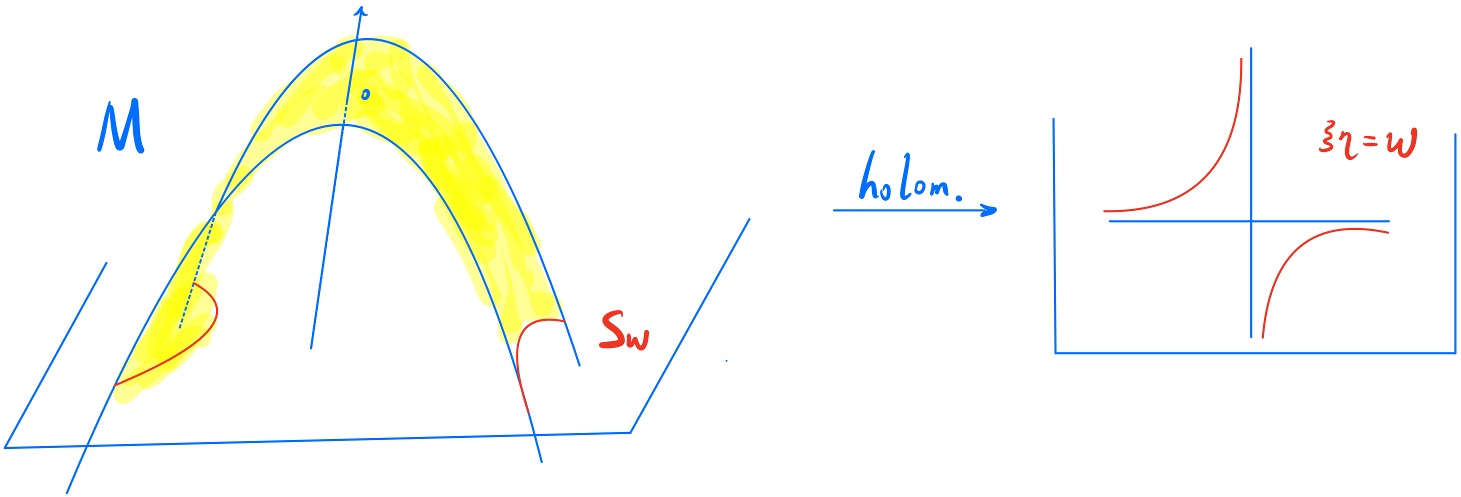
Remark. $\Psi_\omega(\mathcal{C}_\omega^r)$ is a homomorphic invariant set of τ_1 and τ_2 and their restrictions are conjugated to a linear map.

*asymptotically full-measure: $\frac{|\mathcal{O}_r|}{2r^2} \longrightarrow 1$, $r \to 0$

Signification in CR geometry

Theorem (Stolovitch-Z. 2020)

For the hyberbolic surface $M \subset \mathbb{C}^2$, non-exceptional and not formally equivalent to a quadric, there exists a neighborhood of the origin (of radius r) and a Whitney smooth family of holomorphic surfaces $\{\mathcal{S}_\omega\}_{\omega \in \mathcal{O}_r}$, which intersect M along holomorphic hyperbolas, i.e., two real curves which can be holomorphically conjugated to two branches of hyperbolas $\xi\eta = \omega$, $\omega \neq 0$.



$\omega \mapsto \mathcal{S}_\omega$ is Whitney smooth on $\mathcal{O}_r$

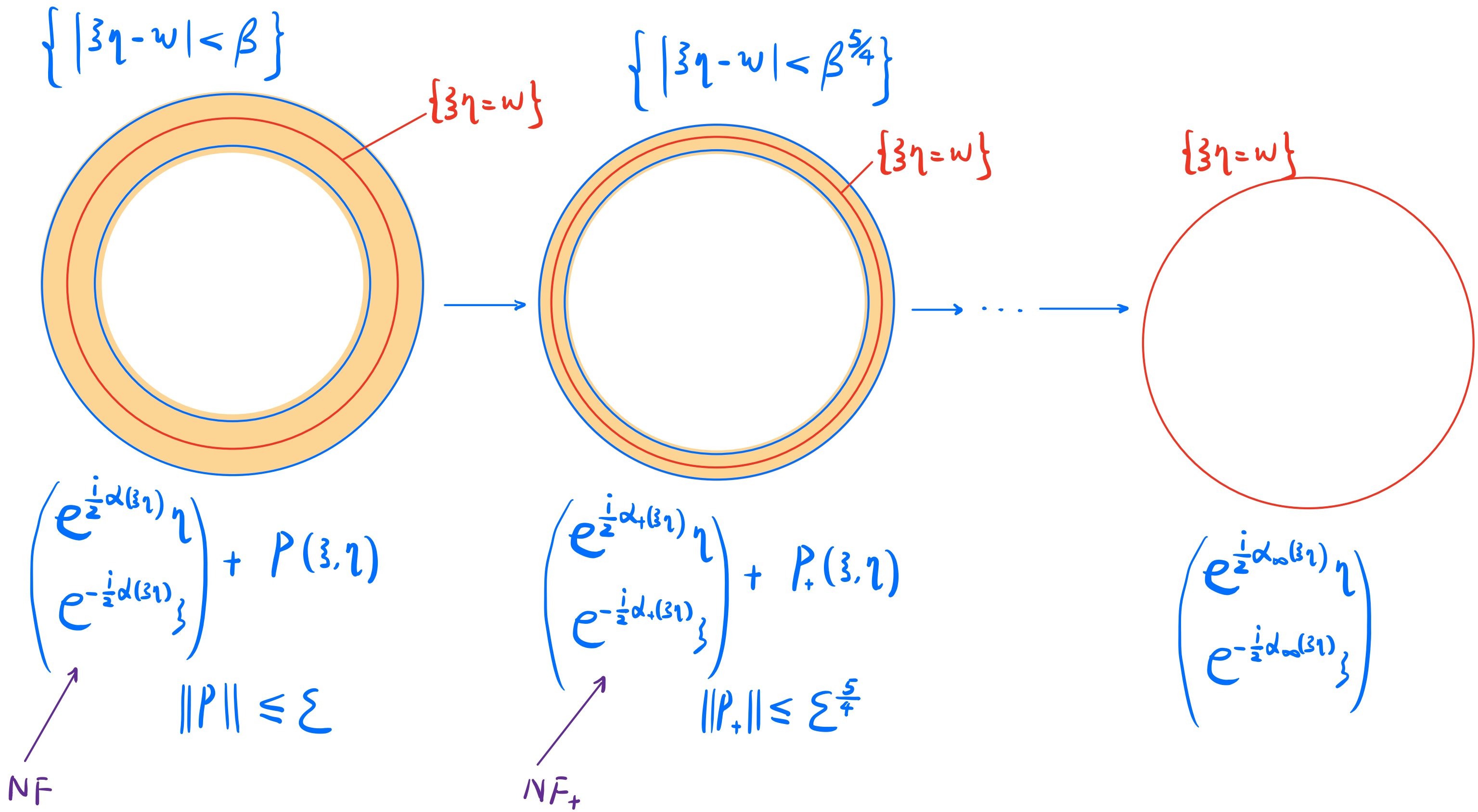
Intersection of M by two straight lines at the origin

Theorem (Klingenberg 1985)

For the hyberbolic Bishop surface M, if $\lambda = e^{\frac{i}{2}\alpha}$ verifies the Diophantine condition, then there exists a unique surface, holomorphic on a neighborhood of the origin, intersecting M in two real curves crossing transversely the origin. These two curves can be conjugated holomorphically to two straight lines $\xi\eta = 0$.

Idea of proof

Kolmogorov-Arnold-Moser (KAM) scheme for holomorphic involutions:



Thanks!