Intersection of entire curves with very generic hypersurfaces

Duc-Viet Vu

University of Cologne

Entire curves

• X smooth projective variety.

• $L \to X$ a very ample line bundle. D a divisor of L.

• $f : \mathbb{C} \to X$ a non-constant holomorphic curve with $f(\mathbb{C}) \not\subset \mathrm{supp} D$ (f is called an entire curve).

• **Nevanlinna's theory**: study the intersection of f with D (counting the number of points $f(C) \cap D$).

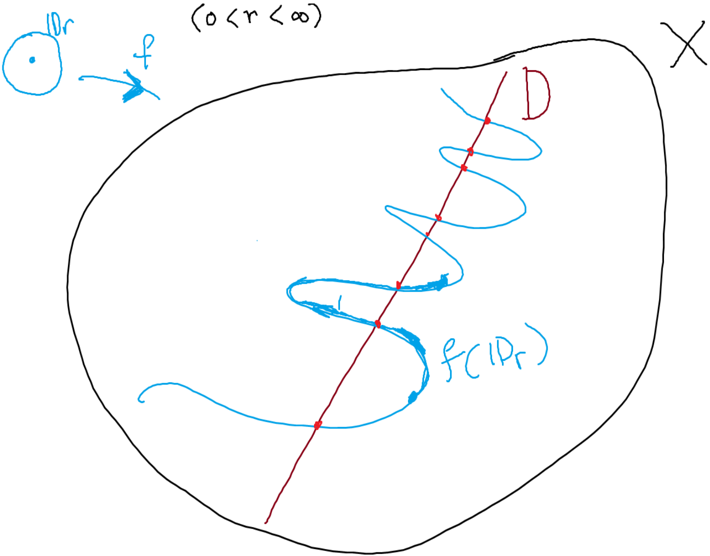
very related to the question of Kobayashi hyperbolicity.

• If f is algebraic, the number of intersection points (counted with multiplicity) of f and D is equal to the product of the degrees of f and D.

• Moreover, for D generic, the number of intersection points (without counting multiplicity) of f and D is also equal to the product of the degrees of f and D.

• We only consider from now on f non-algebraic: the cardinality of $f(\mathbb{C}) \cap D$ is infinite in general.

Counting functions



Counting function

- $\mathbb{D}_r$: the disk of radius r of center 0 in $\mathbb{C}$.

- We count the number of points in $f(\mathbb{D}_r) \cap D$. Let $k \in \mathbb{N} \cup \{\infty\}$:

$$n_f^{[k]}(t, D) := \sum_{z \in \mathbb{D}_t} \min\{k, ord_z f^* D\}.$$

- The counting function of truncation level k of f ($k \in \mathbb{N} \cup \{\infty\}$) with respect to D is

$$N_f^{[k]}(r, D) := \int_1^r \frac{n_f^{[k]}(t, D)}{t}\, dt.$$

- The most interesting case is $N_f^{[1]}(r, D)$ (without counting multiplicity). We denote by $N_f(r, D)$ for $N_f^{[\infty]}(r, D)$.

- ω a Chern form of L (Kähler). The characteristic function of f is

$$T_f(r, L) := \int_1^r \frac{dt}{t} \int_{\mathbb{D}_t} f^* \omega$$

- Why logarithmic average? By the Lelong-Jensen formula

$$N_f^{[1]}(r, D) \le N_f(r, D) \le T_f(r, L) + O(1).$$

Main conjecture

- $T_f(r, L) = O(\log r)$ if and only if f is algebraic ([Stoll]).
- The defect of f of truncation k with respect to D:

$$0 \leq \delta_f^{[k]}(D) := \liminf_{r \to \infty} \left(1 - \frac{N_f^{[k]}(r, D)}{T_f(r, D)}\right) \leq 1.$$

Casorati-Weierstrass Theorem. For almost everywhere D, we have $\delta_f(r, D) = 0$, in other words, there is a sequence $(r_k)_k \subset \mathbb{R}$ converging to ∞ such that $N_f(r_k, D) = T_f(r_k, D) + O(1)$ as $k \to \infty$.

Conjecture

(Griffiths, Noguchi-Winkelmann) Suppose that D is simple normal crossing and f is algebraically non-degenerate. Then

$$T_f(r, L \otimes K_X) \leq N_f^{[1]}(r, D) + o(T_f(r, L))$$

for $r \to \infty$ outside a Borel subset of $\mathbb{R}$ of finite Lebesgue measure.

- Known for X (semi-)abelian ([Noguchi-Winkelmann-Yamanoi]).
- Partial results: [Duval-Huynh], [McQuillan], [Eremenko-Sodin], [Ru], ...

A theorem of Cartan

Theorem

(Cartan) $X = \mathbb{P}^n$, $D = \cup_{j=1}^q H_j$ *in general position, and* f *is linearly non-degenerate. Then*

$$(q-n-1)T_f(r,\omega_{FS}) \leq \sum_{j=1}^q N_f^{[n]}(r,H_j) + o(T_f(r,\omega_{FS}))$$

for r *outside a subset of* $\mathbb{R}$ *of finite Lebesgue measure.*

Reformulation:

$$T_f(r, D \otimes K_X) \leq N_f^{[n]}(r,D) + o(T_f(r,L)).$$

Example of a linearly non-degenerate f (but algebraically degenerate) ([Duval-Huynh]): the truncation n cannot be replaced by 1 in general.

Corollary

$X = \mathbb{P}^n$, $D = \cup_{j=1}^q H_j$ *in general position, and* f *linearly non-degenerate. Then*

$$\sum_{j=1}^q \delta_f^{[n]}(r,H_j) \leq n+1.$$

In particular, for very generic hyperplane H, *we have* $\delta_f^{[n]}(r,H) = 0$.

Defect with respect to very generic hypersurfaces

Theorem

(Theorem A) ([Huynh-V]) Let $f: \mathbb{C} \to \mathbb{P}^n$ be a non-constant holomorphic curve. Then for a very generic hyperplane H, we have

$$\delta_f^{[1]}(r, H) = 0$$

(i.e., the number of intersection points without counting multiplicity of $f(\mathbb{D}_{r_k})$ and H is of the maximal growth)

• being very generic = outside the union of a countable number of proper subvarities (à priori depending on f) in the space of hyperplanes.

• The result is optimal for $n = 1$ (a classical example of Drasin: a curve f in $\mathbb{P}^1$ has defect > 0 for a given arbitrary countable number of points in $\mathbb{P}^1$).

For X, L as before and a very generic divisor D of L, by embedding X to $\mathbb{P}^N$ we also have $\delta_f^{[1]}(r, D) = 0$.

• A usual approach (construction of suitable jet differentials associated to the curve) does not work: it only gives sth like $\delta_f^{[1]}(r, D) \leq 1 - 1/degD$, for D generic (independent de f) of sufficiently large degree $degD$ ([Brotbek-Deng], [Huynh-V-Xie]).

Defect with respect to very generic hypersurfaces

Recall: (Theorem A) For every non-constant entire curve f, and for a very generic hyperplane H, we have

$$\delta_f^{[1]}(r, H) = 0$$

Question

Does there exist a proper subvariety $\mathcal{E}$ of the space of hypersurfaces of a given fixed degree in $\mathbb{P}^n$ such that for $H \notin \mathcal{E}$ and for every non-constant curve f in $\mathbb{P}^n$, we have

$$\delta_f^{[1]}(r, H) = 0.$$

Question

(Stronger version) Does there exist a proper subvariety $\mathcal{E}$ of the space of hypersurfaces of a given fixed degree in $\mathbb{P}^n$ such that for $H \notin \mathcal{E}$ and for every non-constant curve f in $\mathbb{P}^n$, we have

$$N_f^{[1]}(r, D) = T_f(r, D) + O(1)$$

as $r \to \infty$ outside a subset of $\mathbb{R}$ of finite Lebesgue measure.

Intersection of f with a subvariety of codimension at least 2

- V an algebraic subset in X: we can define the counting function $N_f(r, V)$ for $f(\mathbb{C}) \cap V$ even when $codim V \geq 2$ ([Noguchi]).

Theorem

(Theorem B) ([Huynh-V]) Let Z be a complex manifold. Let X be a compact Kähler manifold, and ω a Kähler form on X. Let $f : \mathbb{C} \to X$ be a non-constant holomorphic curve. Let $(\mathcal{V}_a)_{a \in Z}$ be a "family" of analytic subsets of codimension $s \geq 2$ in X. Then for very generic point $a \in Z$, we have

$$\liminf_{r \to \infty} \frac{N_f(r, \mathcal{V}_a)}{T_f(r, \omega)} = 0.$$

Example: $X = \mathbb{P}^n$, and $\mathcal{V}$ is the family of linear subspaces of codimension s in $\mathbb{P}^n$.

If X is (semi-)abelian, by [Noguchi-Winkelmann-Yamanoi], we know

$$\liminf_{r \to \infty} \frac{N_f(r, V)}{T_f(r, \omega)} = 0$$

for every V of codimension ≥ 2.

The strategy

• **Step 1.** To detect asymptotically the tangent points of f and the hyperplane H, Use an idea of Dinh and Dinh-Sibony in complex dynamics: look at the tangent bundles of f and H in a suitable bigger space $\widehat{X}$.

• **Step 2.** Lift f, H to the space $\widehat{X}$. Obtain $\widehat{f}, \widehat{H}$: $codim \widehat{H} = 2$ (Ex: $n = 2$, $\dim \widehat{X} = 3$, $\dim \widehat{H} = 1$).

• **Step 3.** Study the intersection of $\widehat{f}$ with $\widehat{H}$ (Ex: $n = 2$, two curves on a 3-fold).

• **Point of view.** There are closed positive currents of bi-dimension $(1,1)$ (the Nevanlinna or Ahlfors currens) associated to entire curves (there are curves $f, \widehat{f}$).

• Shift focus from the intersection of curves with hypersurfaces to that of currents with subvarieties.

• The classical theory of intersection of currents of bi-degree (1,1) ([Bedford-Taylor], [Demailly], [Fornaess-Sibony],...) does not work in this setting:

$$dd^c u \wedge T := dd^c(uT), \quad dd^c := i\partial\overline{\partial}$$

The strategy

Reason why the classical theory of intersection of $(1,1)$-currents fails here: our currents are of bi-degree $(\widehat{n}-1, \widehat{n}-1)$ and of bi-degree $(2,2)$, where $\widehat{n} = \dim \widehat{X}$.

• We use **the theory of tangent currents (density currents)** introduced by Dinh-Sibony.

• T a closed positive current, V a smooth complex submanifold.

• The tangent currents to T along V are limits of dilation of T along fibers of the normal bundle of V in X.

Example 1. V is a point, T an irreducible analytic subset: The only tangent current to T at V is the tangent cone of T at V.

• Formal intersection of currents T_1, T_2 are encoded on the tangent currents of $T_1 \otimes T_2$ along the diagonal of $X \times X$ (these currents are called density currents associated to T_1, T_2).

Example 2. T_1, T_2 are smooth submanifolds of complementary dimension: $T_1 \cap T_2 =$ the unique density currents associated to T_1, T_2.

Nevanlinna's currents

Let μ_r the Haar measure on the circle $\partial \mathbb{D}_r$ of $\mathbb{D}_r$.

Observation:

$$dd^c \int_1^r \frac{dt}{t}[\mathbb{D}_t] = \mu_r - \mu_1, \tag{1}$$

where $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$.

Put

$$c_r := T_f(r,\omega), \quad S_r := c_r^{-1} \int_1^r \frac{dt}{t} f_*[\mathbb{D}_t]$$

$$\|dd^c S_r\| \lesssim c_r^{-1}. \tag{2}$$

Every limit current of the family $(S_r)_{r\in\mathbb{R}^+}$ as $r \to \infty$ is dd^c-closed.

It is well-known that there exists $(r_k)_k \subset \mathbb{R} \to \infty$ such that $(S_k)_{k\in\mathbb{N}} \to S$ closed positive (a Nevanlinna's current).

Proof of Theorem B: Intersection of S and $\mathcal{V}_a$

Theorem B: For Z a complex manifold, $f : \mathbb{C} \to X$ a non-constant holomorphic curve, $(\mathcal{V}_a)_{a \in Z}$ a "family" of analytic subsets of codimension $s \geq 2$ in X, for very generic point $a \in Z$, we have

$$\liminf_{r \to \infty} \frac{N_f(r, \mathcal{V}_a)}{T_f(r, \omega)} = 0.$$

Proposition

There exists a countable union $\mathcal{A}$ of proper analytic subsets of Z such that for $a \in Z \backslash \mathcal{A}$, we have that $\mathcal{V}_a$ is smooth and

$$S \curlywedge [\mathcal{V}_a] = 0.$$

The equality means that the intersection of S and $\mathcal{V}_a$ is well-defined in the sens of theory of density currents, in which case, it must be zero by a bi-dimension reason ($\dim S + \dim V_a < \dim X$).

Remark: Siu's semi-continuity of Lelong numbers is essential here.

• [Continuity of the cohomology classes of density currents+ counting functions] gives Theorem B: f avoid asymptotically $\mathcal{V}_a$ for every $a \in Z \backslash \mathcal{A}$).

Proof of Theorem A: Lifting of Nevanlinna's currents

Let TX be the tangent bundle of $X = P^n$, and $\widehat{X} := \mathbb{P}(TX)$ the projectivisation of TX and $\pi : \widehat{X} \to X$ the natural projection.

The curve f lifts to an entire curve $\widehat{f}$ in $\widehat{X}$ defined by

$$\widehat{f}(z) := (f(z), [f'(z)]),$$

where f' is the differential of f and $z \in \mathbb{C}$.

For $a = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1} \backslash \{0\}$, put $\sigma_a(x) := \sum_{j=0}^n a_j x_j$, where $x = [x_0 : \ldots : x_n] \in X = \mathbb{P}^n$.

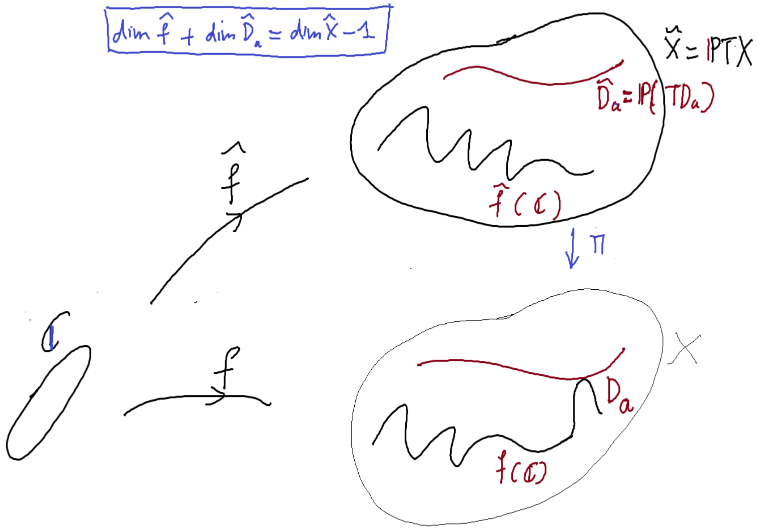
Denote by D_a the hyperplane defined by σ_a. Let $\widehat{D}_a := \mathbb{P}(TD_a)$.

We obtain a family $(\widehat{D}_a)_{a \in \mathbb{P}^n}$ of submanifolds of codimension 2 in $\widehat{X}$.

By Theorem B, $\widehat{f}$ avoid asymptotically $\widehat{D}_a$ for a very generic a:

$$\liminf_{r \to \infty} \frac{N_{\widehat{f}}(r, \widehat{D}_a)}{T_{\widehat{f}}(r, \widehat{\omega})} = 0.$$

Lifting of Nevanlinna's currents



Proof of Theorem A

Let $\omega_{\mathcal{O}_{\widehat{X}}(1)}$ a Chern form of $\mathcal{O}_{\widehat{X}}(1)$. Observe

$$\widehat{\omega} = \omega + c\,\omega_{\mathcal{O}_{\widehat{X}}(1)}$$

is a Kähler form on $\widehat{X}$ (for some constant $c > 0$ sufficiently small).

Lemma

(more or less the lemma on logarithmic derivative) We have

$$T_{\widehat{f}}(r, \widehat{\omega}) = T_f(r, \omega) + o\big(T_f(r, \omega)\big)$$

as $r \to \infty$ outside of subset of finite Lebesgue measure on $\mathbb{R}$.

Lemma

For a generic a, we have

$$N_{\widehat{f}}(r, \widehat{D}_a) = N_f(r, D_a) - N_f^{[1]}(r, D_a).$$

These two above results+ $\liminf_{r \to \infty} \frac{N_{\widehat{f}}(r, \widehat{D}_a)}{T_{\widehat{f}}(r, \widehat{\omega})} = 0$ gives Theorem A.