Notions of visibility with respect to the Kobayashi distance

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1 A framing device: The Wolff–Denjoy Theorem

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- Beardon's work among the earliest to clarify the purely metrical aspects of the above.

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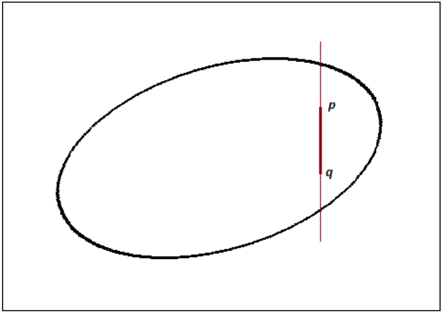

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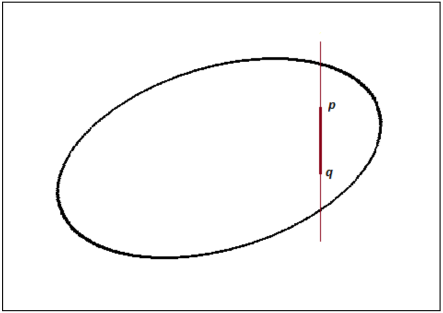

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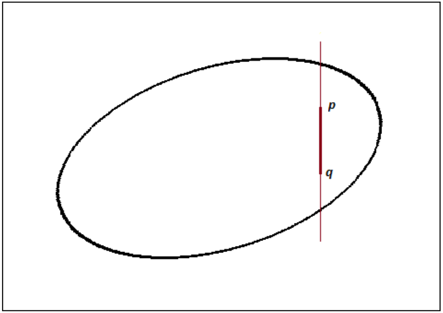

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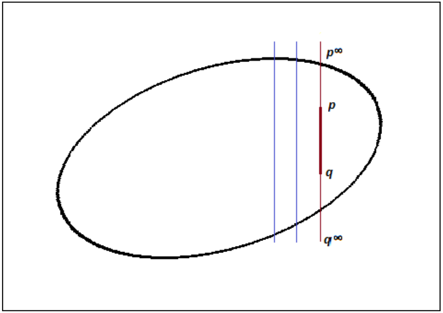

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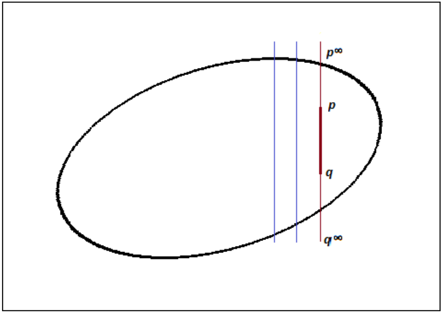

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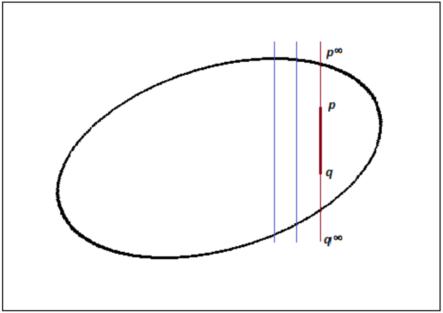

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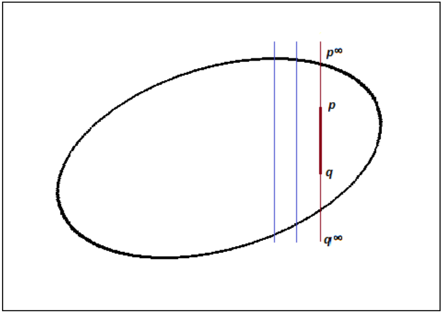

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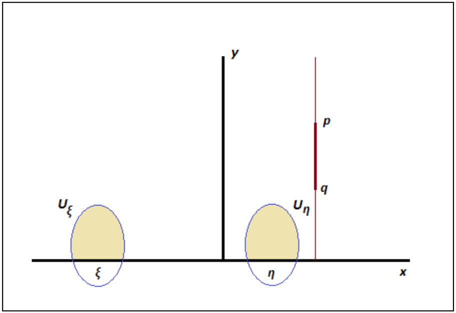

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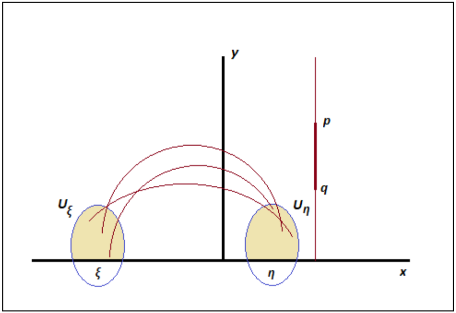
(∗) $\exists K \subset_{\text{cpt.}} \mathbb{H}^2$ s.t. **every** geodesic originating in U_ξ & ending in U_η intersects K.



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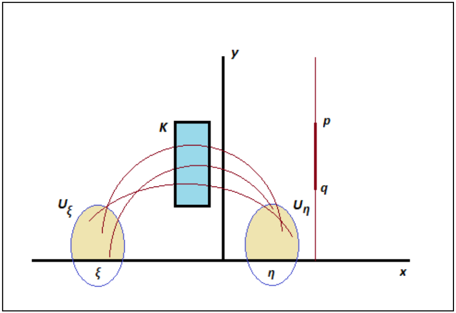

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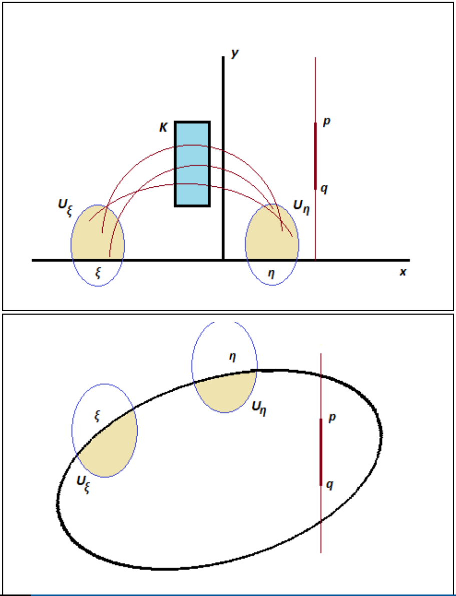

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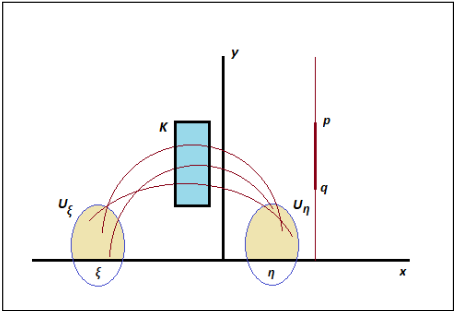

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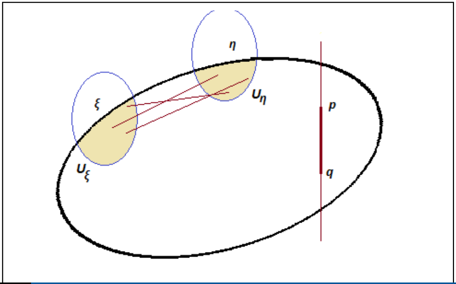

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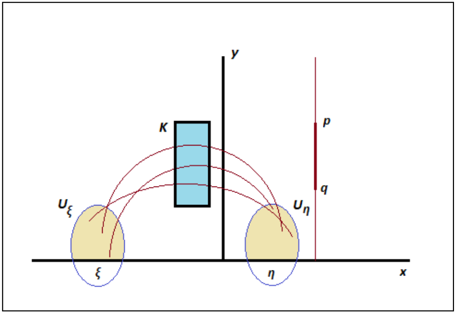

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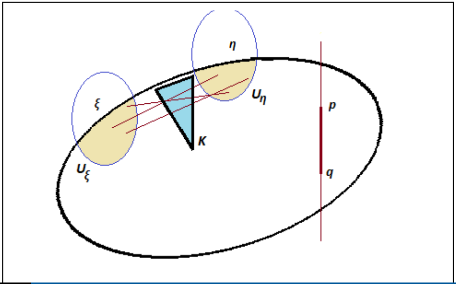
- Shape of $\partial\Omega$ gives existence of K!



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- Expectedly, $\sigma := \gamma \circ \big(\varphi^{-1}\big)$ works **[we need the Lebesgue Diff. Thm. to estimate σ' a.e.]**. ■

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- a *visibility domain w.r.t. the Kobayashi distance* if, for each $\lambda \geq 1$ and $\kappa \geq 0$, for each pair of points $\xi \neq \eta \in \partial\overline{\Omega}^{End}$, and for each pair of $\mathrm{clos}(\overline{\Omega}^{End})$-open nbhds. $U_\xi \ni \xi$ & $U_\eta \ni \eta$ s.t. $\mathrm{clos}(U_\xi) \cap \mathrm{clos}(U_\eta) = \varnothing$, there exists a compact $K \subsetneq \Omega$ with the following property:
  - $(\star)$ for any (λ, κ)-almost-geodesic $\sigma : [0, T] \longrightarrow \Omega$ with $\sigma(0) \in U_\xi$ & $\sigma(T) \in U_\eta$, $\langle\sigma\rangle \cap K \neq \varnothing$.
- a *weak visibility domain w.r.t. the Kobayashi distance* if we consider **just** $\lambda = 1$ in the last definition.

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Part II: The extra degree of regularity of quasi-geodesics is very useful for proofs. E.g., this degree of regularity produces a very useable sufficient cond'n. for a domain to be a v. domain.

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Examples of visibility domains

Recall the condition:

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$$Z(\overline{\Omega}) \cap B^n(0,R) \subset \{(Z', Z_n) : \mathrm{Im}(Z_n) \geq Ce^{-1/\|Z'\|^\alpha},\ \|Z'\| < R\}.$$

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- Assume (passing to subsequences and relabelling) $f(z_j) \to \xi$, $f(w_j) \to \eta$, and $\eta \neq \xi$. Let λ, κ, R be as given by our "shadowing lemma" and let σ_j be a (λ, κ)-almost geodesic joining $f(z_j)$ and $f(w_j)$ and shadowing $f \circ \tau_j$.

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- We have a type of *shadowing of quasi-geodesics in Ω:* given $\lambda_0 \geq 1$, $\kappa_0 \geq 0$, $\exists \lambda \geq 1$, $\kappa \geq 0$ & $R > 0$, depending only on (λ_0, κ_0), s.t. any (λ_0, κ_0)-quasi-geodesic γ in Ω admits a (λ, κ)-almost-geodesic with the same endpoints as γ and within Hausdorff distance R of γ.
- Let $(z_j), (w_j) \subset D$ s.t. $z_j \xrightarrow{G} x$ and $w_j \xrightarrow{G} x$. Let τ_j be a K_D-geodesic joining z_j to w_j. Then $f \circ \tau_j$ is a cont. (λ_0, κ_0)-quasi-geodesic for some $\lambda_0 \geq 1$ and $\kappa_0 \geq 0$.
- Assume (passing to subsequences and relabelling) $f(z_j) \to \xi$, $f(w_j) \to \eta$, and $\eta \neq \xi$. Let λ, κ, R be as given by our "shadowing lemma" and let σ_j be a (λ, κ)-almost geodesic joining $f(z_j)$ and $f(w_j)$ and shadowing $f \circ \tau_j$.
- By **visibility** $\sup K_\Omega(f(o), f \circ \tau_j) < +\infty$, whereby $\sup K_D(o, \tau_j) < +\infty$.

An application of visibility – II

Theorem (Zimmer–B.). *Let $D \subset \mathbb{C}^k$ and $\Omega \subset \mathbb{C}^n$. Suppose (D, K_D) is a Cauchy-complete Gromov hyperbolic space, and suppose Ω is a visibility domain. Then, any continuous quasi-isometric embedding $f : D \to \Omega$ w.r.t. the Kobayashi distances extends continuously to a map $\widetilde{f} : (D \cup \partial_G D) \to \overline{\Omega}^{End}$.*

Sketch of proof. Fix some $o \in D$ and $x \in \partial_G D$.

- We have a type of *shadowing of quasi-geodesics in Ω:* given $\lambda_0 \geq 1$, $\kappa_0 \geq 0$, $\exists \lambda \geq 1,\ \kappa \geq 0$ & $R > 0$, depending only on (λ_0, κ_0), s.t. any (λ_0, κ_0)-quasi-geodesic γ in Ω admits a (λ, κ)-almost-geodesic with the same endpoints as γ and within Hausdorff distance R of γ.
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- By **visibility** $\sup K_\Omega(f(o), f \circ \tau_j) < +\infty$, whereby $\sup K_D(o, \tau_j) < +\infty$. But the latter can't happen! So, we've got a candidate for $\widetilde{f}(x)$. ∎

THANK YOU!