On the Ohsawa-Takegoshi extension theorem

Junyan CAO, Université Côte d'Azur, Nice

Conference "Dynamics, SCV and CR geometry", Nice, 06/12/2021

Setting :

- X is a projective manifold (or more generally weakly pseudoconvex Kähler manifolds). $K_X := \det \Omega_X$ canonical bundle.
- $Y \subset X$ be a simple normal crossing ("SNC" for short) divisor. Let $\mathcal{O}_X(Y)$ be the natural holomorphic line bundle associated to Y. Fix a smooth metric h_Y on $\mathcal{O}_X(Y)$.
- (L, h_L) be a holomorphic line bundle on X satisfying certain curvature conditions.

Let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$.

- Question 1 : Could we find a $F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $F|_Y = f$.
- Question 2 : If Question 1 holds, could we control the norm of F ?

Setting :

- X is a projective manifold (or more generally weakly pseudoconvex Kähler manifolds). $K_X := \det \Omega_X$ canonical bundle.
- $Y \subset X$ be a simple normal crossing ("SNC" for short) divisor. Let $\mathcal{O}_X(Y)$ be the natural holomorphic line bundle associated to Y. Fix a smooth metric h_Y on $\mathcal{O}_X(Y)$.
- (L, h_L) be a holomorphic line bundle on X satisfying certain curvature conditions.

Let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$.

- Question 1 : Could we find a $F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $F|_Y = f$.
- Question 2 : If Question 1 holds, could we control the norm of F ?

I. Motivation

Setting :

- X is a projective manifold (or more generally weakly pseudoconvex Kähler manifolds). $K_X := \det \Omega_X$ canonical bundle.
- $Y \subset X$ be a simple normal crossing ("SNC" for short) divisor. Let $\mathcal{O}_X(Y)$ be the natural holomorphic line bundle associated to Y. Fix a smooth metric h_Y on $\mathcal{O}_X(Y)$.
- (L, h_L) be a holomorphic line bundle on X satisfying certain curvature conditions.

Let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$.

- Question 1 : Could we find a $F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $F|_Y = f$.
- Question 2 : If Question 1 holds, could we control the norm of F ?

I. Motivation

Setting :

- X is a projective manifold (or more generally weakly pseudoconvex Kähler manifolds). $K_X := \det \Omega_X$ canonical bundle.
- $Y \subset X$ be a simple normal crossing ("SNC" for short) divisor. Let $\mathcal{O}_X(Y)$ be the natural holomorphic line bundle associated to Y. Fix a smooth metric h_Y on $\mathcal{O}_X(Y)$.
- (L, h_L) be a holomorphic line bundle on X satisfying certain curvature conditions.

Let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$.

- Question 1 : Could we find a $F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $F|_Y = f$.
- Question 2 : If Question 1 holds, could we control the norm of F ?

I. Motivation

If Y is smooth, there are affirmative (practically optimal) answers to the above questions :

Theorem (Ohsawa-Takegoshi 87)

Let Ω be a weakly pseudoconvex bounded domain in $\mathbb{C}^n$. Let H be an affine subspace in $\mathbb{C}^n$ and φ be a plurisubharmonic function on Ω, i.e., φ is upper semi-continuous and the restriction on any complex line is subharmonic.

Then $\forall f \in H^0(H \cap \Omega, \mathcal{O}_{H\cap\Omega})$ with $\int_{H\cap\Omega} |f|^2 e^{-\varphi} < +\infty$, $\exists F \in H^0(\Omega, \mathcal{O}_\Omega)$ such that $F|_{H\cap\Omega} = f$ and

$$\int_\Omega |F|^2 e^{-\varphi} \leq C(\Omega, H) \int_{H\cap\Omega} |f|^2 e^{-\varphi}.$$

Here $C(\Omega, H)$ is a constant independent of φ and f.

If Y is smooth, there are affirmative (practically optimal) answers to the above questions :

Theorem (Ohsawa-Takegoshi 87)

Let Ω be a weakly pseudoconvex bounded domain in $\mathbb{C}^n$. Let H be an affine subspace in $\mathbb{C}^n$ and φ be a plurisubharmonic function on Ω, i.e., φ is upper semi-continuous and the restriction on any complex line is subharmonic.

Then $\forall f \in H^0(H \cap \Omega, \mathcal{O}_{H\cap\Omega})$ with $\int_{H\cap\Omega} |f|^2 e^{-\varphi} < +\infty$,

$\exists F \in H^0(\Omega, \mathcal{O}_\Omega)$ such that $F|_{H\cap\Omega} = f$ and

$$\int_\Omega |F|^2 e^{-\varphi} \leq C(\Omega, H) \int_{H\cap\Omega} |f|^2 e^{-\varphi}.$$

Here $C(\Omega, H)$ is a constant independent of φ and f.

If Y is smooth, there are affirmative (practically optimal) answers to the above questions :

Theorem (Ohsawa-Takegoshi 87)

Let Ω be a weakly pseudoconvex bounded domain in $\mathbb{C}^n$. Let H be an affine subspace in $\mathbb{C}^n$ and φ be a plurisubharmonic function on Ω, i.e., φ is upper semi-continuous and the restriction on any complex line is subharmonic.
Then $\forall f \in H^0(H \cap \Omega, \mathcal{O}_{H\cap\Omega})$ with $\int_{H\cap\Omega} |f|^2 e^{-\varphi} < +\infty$, $\exists F \in H^0(\Omega, \mathcal{O}_\Omega)$ such that $F|_{H\cap\Omega} = f$ and

$$\int_\Omega |F|^2 e^{-\varphi} \leq C(\Omega, H) \int_{H\cap\Omega} |f|^2 e^{-\varphi}.$$

Here $C(\Omega, H)$ is a constant independent of φ and f.

Global version : X projective manifold, $Y \subset X$ a smooth divisor. Let (L, h_L) be a holomorphic line bundle. We assume that

$$\Theta_{h_L}(L) \geq 0 \text{ and } \Theta_{h_L}(L) \geq \delta \Theta_{h_Y}(\mathcal{O}_X(Y)) \quad (1)$$

for some constant $\delta > 0$.

Theorem (Manivel 93)

In the above setting, let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\int_Y |f|^2_{\omega_X, h_L, h_Y} dV_{\omega_Y} < +\infty$.
Then $\exists F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $F|_Y = f$ and

$$\int_X |F|^2_{h_L, h_Y} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L},$$

where s_Y is the canonical section of $\mathcal{O}_X(Y)$ such that $|s_Y|_{h_Y} \leq e^{-\frac{1}{\delta}}$ and C is a constant independent of h_L and δ.

Global version : X projective manifold, $Y \subset X$ a smooth divisor. Let (L, h_L) be a holomorphic line bundle. We assume that

$$\Theta_{h_L}(L) \geq 0 \text{ and } \Theta_{h_L}(L) \geq \delta \Theta_{h_Y}(\mathcal{O}_X(Y)) \quad (1)$$

for some constant $\delta > 0$.

Theorem (Manivel 93)

In the above setting, let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\int_Y |f|^2_{\omega_X, h_L, h_Y} dV_{\omega_Y} < +\infty$.
Then $\exists F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $F|_Y = f$ and

$$\int_X |F|^2_{h_L, h_Y} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L},$$

where s_Y is the canonical section of $\mathcal{O}_X(Y)$ such that $|s_Y|_{h_Y} \leq e^{-\frac{1}{\delta}}$ and C is a constant independent of h_L and δ.

I. Motivation

Global version : X projective manifold, $Y \subset X$ a smooth divisor. Let (L, h_L) be a holomorphic line bundle. We assume that

$$\Theta_{h_L}(L) \geq 0 \text{ and } \Theta_{h_L}(L) \geq \delta \Theta_{h_Y}(\mathcal{O}_X(Y)) \tag{1}$$

for some constant $\delta > 0$.

Theorem (Manivel 93)

In the above setting, let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\int_Y |f|^2_{\omega_X, h_L, h_Y} dV_{\omega_Y} < +\infty$.
Then $\exists F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $F|_Y = f$ and

$$\int_X |F|^2_{h_L, h_Y} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L},$$

where s_Y is the canonical section of $\mathcal{O}_X(Y)$ such that $|s_Y|_{h_Y} \leq e^{-\frac{1}{\delta}}$ and C is a constant independent of h_L and δ.

I. Motivation

Many applications in complex geometry :

- Analytic appriximation of closed positive currents (Demailly) ;
- Invariance of plurigenera (Siu, Păun) ;
- Positivity of direct images (Guan-Zhou, Deng-Wang-Zhang-Zhou)

We are now interested in the case when Y is not smooth. It is closely related to the abundance conjecture.

Abundance conjecture : let X be a projective manifold and K_X is numerically effective, i.e., $c_1(K_X) \cdot \mathcal{C} \geq 0$ for any projective curve $\mathcal{C} \subset X$. Then K_X is semi-ample.

I. Motivation

Many applications in complex geometry :

- Analytic appriximation of closed positive currents (Demailly) ;
- Invariance of plurigenera (Siu, Păun) ;
- Positivity of direct images (Guan-Zhou, Deng-Wang-Zhang-Zhou)

We are now interested in the case when Y is not smooth. It is closely related to the abundance conjecture.

Abundance conjecture : let X be a projective manifold and K_X is numerically effective, i.e., $c_1(K_X) \cdot \mathcal{C} \geq 0$ for any projective curve $\mathcal{C} \subset X$. Then K_X is semi-ample.

I. Motivation

Many applications in complex geometry :

- Analytic apprixımation of closed positive currents (Demailly) ;
- Invariance of plurigenera (Siu, Păun) ;
- Positivity of direct images (Guan-Zhou, Deng-Wang-Zhang-Zhou)

We are now interested in the case when Y is not smooth. It is closely related to the abundance conjecture.

Abundance conjecture : let X be a projective manifold and K_X is numerically effective, i.e., $c_1(K_X) \cdot \mathcal{C} \geq 0$ for any projective curve $\mathcal{C} \subset X$. Then K_X is semi-ample.

The abundance conjecture can be split into two sub-conjectures : a non-vanishing conjecture and an extension conjecture.

Conjecture (Non vanishing conjecture)

Let X be a projective manifold and K_X is numerically effective. Then for $m \in \mathbb{N}^*$ sufficiently divisible, we have

$$H^0(X, mK_X) \neq 0.$$

Conjecture (Extension conjecture (simplified version))

Let X be a projective manifold with K_X numerically effective. Let $s \in H^0(X, mK_X)$. We suppose for simplicity that $Y := Div(s)$ is SNC. Then the restriction map

$$H^0(X, m_1 K_X) \to H^0(Y, m_1 K_X).$$

is surjective for m_1 sufficiently divisible.

I. Motivation

The abundance conjecture can be split into two sub-conjectures : a non-vanishing conjecture and an extension conjecture.

Conjecture (Non vanishing conjecture)

Let X be a projective manifold and K_X is numerically effective. Then for $m \in \mathbb{N}^*$ sufficiently divisible, we have

$$H^0(X, mK_X) \neq 0.$$

Conjecture (Extension conjecture (simplified version))

Let X be a projective manifold with K_X numerically effective. Let $s \in H^0(X, mK_X)$. We suppose for simplicity that $Y := Div(s)$ is SNC. Then the restriction map

$$H^0(X, m_1 K_X) \to H^0(Y, m_1 K_X).$$

is surjective for m_1 sufficiently divisible.

I. Motivation

The abundance conjecture can be split into two sub-conjectures : a non-vanishing conjecture and an extension conjecture.

Conjecture (Non vanishing conjecture)

Let X be a projective manifold and K_X is numerically effective. Then for $m \in \mathbb{N}^*$ sufficiently divisible, we have

$$H^0(X, mK_X) \neq 0.$$

Conjecture (Extension conjecture (simplified version))

Let X be a projective manifold with K_X numerically effective. Let $s \in H^0(X, mK_X)$. We suppose for simplicity that $Y := Div(s)$ is SNC. Then the restriction map

$$H^0(X, m_1 K_X) \to H^0(Y, m_1 K_X).$$

is surjective for m_1 sufficiently divisible.

I. Motivation

The abundance conjecture can be split into two sub-conjectures : a non-vanishing conjecture and an extension conjecture.

Conjecture (Non vanishing conjecture)

Let X be a projective manifold and K_X is numerically effective. Then for $m \in \mathbb{N}^*$ sufficiently divisible, we have

$$H^0(X, mK_X) \neq 0.$$

Conjecture (Extension conjecture (simplified version))

Let X be a projective manifold with K_X numerically effective. Let $s \in H^0(X, mK_X)$. We suppose for simplicity that $Y := Div(s)$ is SNC.Then the restriction map

$$H^0(X, m_1 K_X) \to H^0(Y, m_1 K_X).$$

is surjective for m_1 sufficiently divisible.

I. Motivation

- Demailly-Hacon-Păun : the extension conjecture holds when Y is smooth.
  Methods : Ohsawa-Takegoshi extension, the techniques of invariance of plurigenera.
- If we can generalize the Ohsawa-Takegoshi extension theorem to the case when Y is SNC, it will imply the full extension conjecture.

I. Motivation

- Demailly-Hacon-Păun : the extension conjecture holds when Y is smooth.
  Methods : Ohsawa-Takegoshi extension, the techniques of invariance of plurigenera.
- If we can generalize the Ohsawa-Takegoshi extension theorem to the case when Y is SNC, it will imply the full extension conjecture.

Theorem (Demailly, Demailly-C. -Matsumura, Zhou-Zhu)

Let X be a projective manifold and $Y \subset X$ be a SNC divisor. Let (L, h_L) be a holomorphic line bundle such that

$$\Theta_{h_L}(L) \geq 0 \text{ and } \Theta_{h_L}(L) \geq \delta \Theta_{h_Y}(\mathcal{O}_X(Y)) \quad (2)$$

for some $\delta > 0$. Let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\int_Y |f|^2_{\omega_X, h_L, h_Y} dV_{\omega_Y} < +\infty$ and admits a local L^2-holomorphic extension.

Then $\exists F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $F|_Y = f$ and

$$\int_X |F|^2_{h_L, h_Y} < +\infty.$$

Local L^2-holomorphic extension : $\exists$ covering $\cup U_i = X$ such that $f|_{U_i \cap Y}$ admits an extension $F_i \in H^0(U_i, K_X \otimes L \otimes \mathcal{O}_X(Y))$ with $\int_{U_i} |F_i|^2_{h_L, h_Y} < +\infty$.

II. Main results

Theorem (Demailly, Demailly-C. -Matsumura, Zhou-Zhu)

Let X be a projective manifold and $Y \subset X$ be a SNC divisor. Let (L, h_L) be a holomorphic line bundle such that

$$\Theta_{h_L}(L) \geq 0 \text{ and } \Theta_{h_L}(L) \geq \delta \Theta_{h_Y}(\mathcal{O}_X(Y)) \tag{2}$$

for some $\delta > 0$. Let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\int_Y |f|^2_{\omega_X, h_L, h_Y} dV_{\omega_Y} < +\infty$ and admits a local L^2-holomorphic extension.

Then $\exists F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $F|_Y = f$ and

$$\int_X |F|^2_{h_L, h_Y} < +\infty.$$

Local L^2-holomorphic extension : $\exists$ covering $\cup U_i = X$ such that $f|_{U_i \cap Y}$ admits an extension $F_i \in H^0(U_i, K_X \otimes L \otimes \mathcal{O}_X(Y))$ with $\int_{U_i} |F_i|^2_{h_L, h_Y} < +\infty$.

Theorem (Demailly, Demailly-C. -Matsumura, Zhou-Zhu)

Let X be a projective manifold and $Y \subset X$ be a SNC divisor. Let (L, h_L) be a holomorphic line bundle such that

$$\Theta_{h_L}(L) \geq 0 \text{ and } \Theta_{h_L}(L) \geq \delta \Theta_{h_Y}(\mathcal{O}_X(Y)) \tag{2}$$

for some $\delta > 0$. Let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $\int_Y |f|^2_{\omega_X, h_L, h_Y} dV_{\omega_Y} < +\infty$ and admits a local L^2-holomorphic extension.

Then $\exists F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ such that $F|_Y = f$ and

$$\int_X |F|^2_{h_L, h_Y} < +\infty.$$

Local L^2-holomorphic extension : $\exists$ covering $\cup U_i = X$ such that $f|_{U_i \cap Y}$ admits an extension $F_i \in H^0(U_i, K_X \otimes L \otimes \mathcal{O}_X(Y))$ with $\int_{U_i} |F_i|^2_{h_L, h_Y} < +\infty$.

Remark :

- If $\int_Y |\frac{f}{ds_Y}|^2_{h_L} < +\infty$, by Manivel's theorem, we can find a holomorphic extension F such that

$$\int_X |F|^2_{h_L, h_Y} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L}.$$

- If Y is not smooth, $\int_Y |f|^2_{\omega_X, h_L, h_Y} dV_{\omega_Y} < +\infty$ will not imply that $\int_Y |\frac{f}{ds_Y}|^2_{h_L} < +\infty$!

  For example, if $Y = \{z_1 \cdot z_2 = 0\}$ and $|f|_{h_L, h_Y}((0,0))$ is non-zero :

$$\int_Y |\frac{f}{ds_Y}|^2_{h_Y} \sim \int_Y \frac{|f|^2_{\omega_X, h_L, h_Y}}{|z_1|^2 + |z_2|^2} dV_{\omega_Y} = +\infty.$$

Remark :

- If $\int_Y |\frac{f}{ds_Y}|^2_{h_L} < +\infty$, by Manivel's theorem, we can find a holomorphic extension F such that

$$\int_X |F|^2_{h_L,h_Y} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L}.$$

- If Y is not smooth, $\int_Y |f|^2_{\omega_X,h_L,h_Y} dV_{\omega_Y} < +\infty$ will not imply that $\int_Y |\frac{f}{ds_Y}|^2_{h_L} < +\infty$!

  For example, if $Y = \{z_1 \cdot z_2 = 0\}$ and $|f|_{h_L,h_Y}((0,0))$ is non-zero :

$$\int_Y |\frac{f}{ds_Y}|^2_{h_Y} \sim \int_Y \frac{|f|^2_{\omega_X,h_L,h_Y}}{|z_1|^2 + |z_2|^2} dV_{\omega_Y} = +\infty.$$

Remark :

- If $\int_Y |\frac{f}{ds_Y}|^2_{h_L} < +\infty$, by Manivel's theorem, we can find a holomorphic extension F such that

$$\int_X |F|^2_{h_L,h_Y} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L}.$$

- If Y is not smooth, $\int_Y |f|^2_{\omega_X,h_L,h_Y} dV_{\omega_Y} < +\infty$ will not imply that $\int_Y |\frac{f}{ds_Y}|^2_{h_L} < +\infty$!

  For example, if $Y = \{z_1 \cdot z_2 = 0\}$ and $|f|_{h_L,h_Y}((0,0))$ is non-zero :

$$\int_Y |\frac{f}{ds_Y}|^2_{h_Y} \sim \int_Y \frac{|f|^2_{\omega_X,h_L,h_Y}}{|z_1|^2+|z_2|^2} dV_{\omega_Y} = +\infty.$$

Important Example (Ohsawa) : Let Δ be a small neighboord of $(0,0) \in \mathbb{C}^2$ and $Y = \{z_1 \cdot z_2 = 0\}$. $\varphi_L := \log |z_1 - z_2|^2$ and $h_L := e^{-\varphi_L}$.

Let $f \in H^0(Y, \mathcal{O}_Y)$:

$$f \equiv 0 \text{ on } \{z_1 = 0\} \qquad \text{and} \qquad f = z_1 \text{ on } \{z_2 = 0\}.$$

Then $\int_Y |f|^2 e^{-\varphi_L} < +\infty$.

However, if $\exists F \in H^0(\Delta, \mathcal{O}_\Delta)$ such that $F|_Y = f$ and

$$\int_\Delta |F|^2 e^{-\varphi_L} = \int_\Delta \frac{|F|^2}{|z_1 - z_2|^2} < +\infty,$$

Then $\frac{F}{z_1 - z_2} \in H^0(\Delta, \mathcal{O}_\Delta)$. We have $\frac{F}{z_1 - z_2} \equiv 0$ on $\{z_1 = 0\}$ and $\frac{F}{z_1 - z_2} \equiv 1$ on $\{z_2 = 0\}$. Impossible !

Important Example (Ohsawa) : Let Δ be a small neighboord of $(0,0) \in \mathbb{C}^2$ and $Y = \{z_1 \cdot z_2 = 0\}$. $\varphi_L := \log |z_1 - z_2|^2$ and $h_L := e^{-\varphi_L}$.
Let $f \in H^0(Y, \mathcal{O}_Y)$:

$$f \equiv 0 \text{ on } \{z_1 = 0\} \qquad \text{and} \qquad f = z_1 \text{ on } \{z_2 = 0\}.$$

Then $\int_Y |f|^2 e^{-\varphi_L} < +\infty$.

However, if $\exists F \in H^0(\Delta, \mathcal{O}_\Delta)$ such that $F|_Y = f$ and

$$\int_\Delta |F|^2 e^{-\varphi_L} = \int_\Delta \frac{|F|^2}{|z_1 - z_2|^2} < +\infty,$$

Then $\frac{F}{z_1 - z_2} \in H^0(\Delta, \mathcal{O}_\Delta)$. We have $\frac{F}{z_1 - z_2} \equiv 0$ on $\{z_1 = 0\}$ and $\frac{F}{z_1 - z_2} \equiv 1$ on $\{z_2 = 0\}$. Impossible !

Important Example (Ohsawa) : Let Δ be a small neighboord of $(0,0) \in \mathbb{C}^2$ and $Y = \{z_1 \cdot z_2 = 0\}$. $\varphi_L := \log |z_1 - z_2|^2$ and $h_L := e^{-\varphi_L}$.
Let $f \in H^0(Y, \mathcal{O}_Y)$:

$$f \equiv 0 \text{ on } \{z_1 = 0\} \qquad \text{and} \qquad f = z_1 \text{ on } \{z_2 = 0\}.$$

Then $\int_Y |f|^2 e^{-\varphi_L} < +\infty$.

However, if $\exists F \in H^0(\Delta, \mathcal{O}_\Delta)$ such that $F|_Y = f$ and

$$\int_\Delta |F|^2 e^{-\varphi_L} = \int_\Delta \frac{|F|^2}{|z_1 - z_2|^2} < +\infty,$$

Then $\frac{F}{z_1 - z_2} \in H^0(\Delta, \mathcal{O}_\Delta)$. We have $\frac{F}{z_1 - z_2} \equiv 0$ on $\{z_1 = 0\}$ and $\frac{F}{z_1 - z_2} \equiv 1$ on $\{z_2 = 0\}$. Impossible !

Important Example (Ohsawa) : Let Δ be a small neighboord of $(0,0) \in \mathbb{C}^2$ and $Y = \{z_1 \cdot z_2 = 0\}$. $\varphi_L := \log |z_1 - z_2|^2$ and $h_L := e^{-\varphi_L}$.
Let $f \in H^0(Y, \mathcal{O}_Y)$:

$$f \equiv 0 \text{ on } \{z_1 = 0\} \qquad \text{and} \qquad f = z_1 \text{ on } \{z_2 = 0\}.$$

Then $\int_Y |f|^2 e^{-\varphi_L} < +\infty$.

However, if $\exists F \in H^0(\Delta, \mathcal{O}_\Delta)$ such that $F|_Y = f$ and

$$\int_\Delta |F|^2 e^{-\varphi_L} = \int_\Delta \frac{|F|^2}{|z_1 - z_2|^2} < +\infty,$$

Then $\frac{F}{z_1 - z_2} \in H^0(\Delta, \mathcal{O}_\Delta)$. We have $\frac{F}{z_1 - z_2} \equiv 0$ on $\{z_1 = 0\}$ and $\frac{F}{z_1 - z_2} \equiv 1$ on $\{z_2 = 0\}$. Impossible !

Let $\varphi_\epsilon := \log(|z_1 - z_2|^2 + \epsilon)$. Then $\varphi_\epsilon \to \varphi_L = \log |z_1 - z_2|^2$.

If $\exists$ an extension $F_\epsilon \in H^0(\Delta, \mathcal{O}_\Delta)$ with a uniform control

$$\int_\Delta |F_\epsilon|^2 e^{-\varphi_\epsilon} \leq C \int_Y |f|^2 e^{-\varphi_\epsilon},$$

by letting $\epsilon \to 0$, we get an extension $F \in H^0(\Delta, \mathcal{O}_\Delta)$ with $\int_\Delta |F|^2 e^{-\varphi_L} \leq C \int_Y |f|^2 e^{-\varphi_L}$. Contradiction.

Remark : It is easy to construct a compact analogue of Ohsawa's example.

Let $\varphi_\epsilon := \log(|z_1 - z_2|^2 + \epsilon)$. Then $\varphi_\epsilon \to \varphi_L = \log |z_1 - z_2|^2$.

If $\exists$ an extension $F_\epsilon \in H^0(\Delta, \mathcal{O}_\Delta)$ with a uniform control

$$\int_\Delta |F_\epsilon|^2 e^{-\varphi_\epsilon} \leq C \int_Y |f|^2 e^{-\varphi_\epsilon},$$

by letting $\epsilon \to 0$, we get an extension $F \in H^0(\Delta, \mathcal{O}_\Delta)$ with $\int_\Delta |F|^2 e^{-\varphi_L} \leq C \int_Y |f|^2 e^{-\varphi_L}$. Contradiction.

Remark : It is easy to construct a compact analogue of Ohsawa's example.

Let $\varphi_\epsilon := \log(|z_1 - z_2|^2 + \epsilon)$. Then $\varphi_\epsilon \to \varphi_L = \log|z_1 - z_2|^2$.

If $\exists$ an extension $F_\epsilon \in H^0(\Delta, \mathcal{O}_\Delta)$ with a uniform control

$$\int_\Delta |F_\epsilon|^2 e^{-\varphi_\epsilon} \leq C \int_Y |f|^2 e^{-\varphi_\epsilon},$$

by letting $\epsilon \to 0$, we get an extension $F \in H^0(\Delta, \mathcal{O}_\Delta)$ with $\int_\Delta |F|^2 e^{-\varphi_L} \leq C \int_Y |f|^2 e^{-\varphi_L}$. Contradiction.

Remark : It is easy to construct a compact analogue of Ohsawa's example.

Let $\varphi_\epsilon := \log(|z_1 - z_2|^2 + \epsilon)$. Then $\varphi_\epsilon \to \varphi_L = \log |z_1 - z_2|^2$.

If $\exists$ an extension $F_\epsilon \in H^0(\Delta, \mathcal{O}_\Delta)$ with a uniform control

$$\int_\Delta |F_\epsilon|^2 e^{-\varphi_\epsilon} \leq C \int_Y |f|^2 e^{-\varphi_\epsilon},$$

by letting $\epsilon \to 0$, we get an extension $F \in H^0(\Delta, \mathcal{O}_\Delta)$ with $\int_\Delta |F|^2 e^{-\varphi_L} \leq C \int_Y |f|^2 e^{-\varphi_L}$. Contradiction.

Remark : It is easy to construct a compact analogue of Ohsawa's example.

Conjecture (C. -Păun)

Let X be a projective manifold and $Y = \cup Y_i$ be a SNC divisor in X. Let L be a holomorphic line bundle on X such that

$$i\Theta_{h_L}(L) \geq 0 \qquad \text{and} \qquad i\Theta_{h_L}(L) \geq \delta\Theta_{h_Y}(\mathcal{O}_X(Y))$$

for some $\delta > 0$. Let V_{sing} be a neighborhood of the singular locus of Y.

Let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$ s.t. $\int_Y |f|^2_{\omega_X, h_L, h_Y} dV_{\omega_Y} < +\infty$ and admits a local L^2-holomorphic extension. There exists an extension $F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ s.t

$$\frac{1}{C}\int_{X \setminus V_{\mathrm{sing}}} |F|^2_{h_Y, h_L} \leq \int_{Y \setminus V_{\mathrm{sing}}} \left|\frac{f}{ds_Y}\right|^2_{h_L} + \left(\int_{Y \cap V_{\mathrm{sing}}} \left|\frac{f}{ds_Y}\right| e^{-\varphi_L} dV_{\omega_Y}\right)^2 .$$

where $h_Y = e^{-\varphi_L}$. Here C depends only on X and V_{sing}.

Conjecture (C. -Păun)

Let X be a projective manifold and $Y = \cup Y_i$ be a SNC divisor in X. Let L be a holomorphic line bundle on X such that

$$i\Theta_{h_L}(L) \geq 0 \qquad \text{and} \qquad i\Theta_{h_L}(L) \geq \delta \Theta_{h_Y}(\mathcal{O}_X(Y))$$

for some $\delta > 0$. Let $V_{\rm sing}$ be a neighborhood of the singular locus of Y.

Let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$ s.t. $\int_Y |f|^2_{\omega_X, h_L, h_Y} dV_{\omega_Y} < +\infty$ and admits a local L^2-holomorphic extension. There exists an extension $F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ s.t

$$\frac{1}{C}\int_{X \setminus V_{\rm sing}} |F|^2_{h_Y, h_L} \leq \int_{Y \setminus V_{\rm sing}} \left|\frac{f}{ds_Y}\right|^2_{h_L} + \left(\int_{Y \cap V_{\rm sing}} \left|\frac{f}{ds_Y}\right| e^{-\varphi_L} dV_{\omega_Y}\right)^2 .$$

where $h_Y = e^{-\varphi_L}$. Here C depends only on X and $V_{\rm sing}$.

Conjecture (C. -Păun)

Let X be a projective manifold and $Y = \cup Y_i$ be a SNC divisor in X. Let L be a holomorphic line bundle on X such that

$$i\Theta_{h_L}(L) \geq 0 \qquad \text{and} \qquad i\Theta_{h_L}(L) \geq \delta\Theta_{h_Y}(\mathcal{O}_X(Y))$$

for some $\delta > 0$. Let V_{sing} be a neighborhood of the singular locus of Y.
Let $f \in H^0(Y, K_X \otimes L \otimes \mathcal{O}_X(Y))$ s.t. $\int_Y |f|^2_{\omega_X,h_L,h_Y} dV_{\omega_Y} < +\infty$ and admits a local L^2-holomorphic extension. There exists an extension $F \in H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ s.t

$$\frac{1}{C}\int_{X \setminus V_{\text{sing}}} |F|^2_{h_Y,h_L} \leq \int_{Y \setminus V_{\text{sing}}} \left|\frac{f}{ds_Y}\right|^2_{h_L} + \left(\int_{Y \cap V_{\text{sing}}} \left|\frac{f}{ds_Y}\right| e^{-\varphi_L} dV_{\omega_Y}\right)^2 .$$

where $h_Y = e^{-\varphi_L}$. Here C depends only on X and V_{sing}.

- This conjecture implies the DLT-extension conjecture.
- For applications, we can suppose that φ_L is of analytic singularity.

Theorem (C. -Păun)

The conjecture holds if φ_L on V_{sing} is of type

$$\varphi_L = \log |s|^2 + \sum_j (1 - \frac{1}{k_j}) \log |z_j|^2 + \tau_L \qquad \text{on } V_{\mathrm{sing}},$$

for some holomorphic function s and a bounded function τ_L, and $i\Theta_{h_L}(L) \geq C\omega_{\mathcal{C}}$ on V_{sing} for some $C > 0$. Here $\omega_{\mathcal{C}}$ is a metric on V_{sing} of conic singularity with respect to $\sum_j (1 - \frac{1}{k_j}) Div(z_j)$.

- This conjecture implies the DLT-extension conjecture.
- For applications, we can suppose that φ_L is of analytic singularity.

Theorem (C. -Păun)

The conjecture holds if φ_L on $V_{\rm sing}$ is of type

$$\varphi_L = \log|s|^2 + \sum_j (1 - \frac{1}{k_j}) \log|z_j|^2 + \tau_L \qquad \text{on } V_{\rm sing},$$

for some holomorphic function s and a bounded function τ_L, and $i\Theta_{h_L}(L) \geq C\omega_{\mathcal{C}}$ on $V_{\rm sing}$ for some $C > 0$. Here $\omega_{\mathcal{C}}$ is a metric on $V_{\rm sing}$ of conic singularity with respect to $\sum_j (1 - \frac{1}{k_j}) Div(z_j)$.

II. Mains results

- This conjecture implies the DLT-extension conjecture.
- For applications, we can suppose that φ_L is of analytic singularity.

Theorem (C. -Păun)

The conjecture holds if φ_L on $V_{\rm sing}$ is of type

$$\varphi_L = \log|s|^2 + \sum_j (1 - \frac{1}{k_j}) \log|z_j|^2 + \tau_L \qquad \text{on } V_{\rm sing},$$

for some holomorphic function s and a bounded function τ_L, and $i\Theta_{h_L}(L) \geq C\omega_{\mathcal{C}}$ on $V_{\rm sing}$ for some $C > 0$. Here $\omega_{\mathcal{C}}$ is a metric on $V_{\rm sing}$ of conic singularity with respect to $\sum_j (1 - \frac{1}{k_j}) Div(z_j)$.

III. Sketch of the proof

Lemma (Siu, Berndtsson, C.- Păun)

Let $n = \dim X$ and let ξ be a smooth $(n,1)$-form with values in L. $h_L = e^{-\varphi_L}$ be a smooth hermitian metric on L satisfies our usual curvature conditions.

Let $\star$ be the Hodge star operator which sends $(n,1)$-forms to $(n-1,0)$-forms. Then

$$\int_Y \star\xi \wedge \overline{\star\xi} e^{-\varphi_L} \leq C \int_X \log^2 |s_Y| (|\overline{\partial}^{\star}\xi|^2_{h_L} + |\overline{\partial}\xi|^2_{h_L}) dV_{\omega_X}.$$

Proof : Set $T_\xi := \star\xi \wedge \overline{\star\xi} e^{-\varphi_L}$, is a $(n-1,n-1)$-form. We have

$$i\partial\overline{\partial} T_\xi = (|\overline{\partial}(\star\xi)|^2_{h_L} + |\overline{\partial}^{\star}\xi|^2_{h_L} - |\overline{\partial}\xi|^2_{h_L} - 2Re < \overline{\partial}\overline{\partial}^{\star}\xi, \xi >) dV_\omega$$
$$+ i\Theta_{h_L}(L) \wedge T_\xi.$$

We calculate $\int_X T_\xi \wedge i\partial\overline{\partial} \log |s_Y|^2_{h_Y}$. The residue part gives the LHS. Stokes, curvature conditions and the above equality imply the RHS.

Lemma (Siu, Berndtsson, C.- Păun)

Let $n = \dim X$ *and let* ξ *be a smooth* $(n,1)$*-form with values in* L. $h_L = e^{-\varphi_L}$ *be a smooth hermitian metric on* L *satisfies our usual curvature conditions.*
Let $\star$ *be the Hodge star operator which sends* $(n,1)$*-forms to* $(n-1,0)$*-forms.* *Then*

$$\int_Y \star\xi \wedge \overline{\star\xi} e^{-\varphi_L} \leq C \int_X \log^2 |s_Y| (|\overline{\partial}^\star \xi|^2_{h_L} + |\overline{\partial}\xi|^2_{h_L}) dV_{\omega_X}.$$

Proof : Set $T_\xi := \star\xi \wedge \overline{\star\xi} e^{-\varphi_L}$, is a $(n-1,n-1)$-form. We have

$$i\partial\overline{\partial} T_\xi = (|\overline{\partial}(\star\xi)|^2_{h_L} + |\overline{\partial}^\star \xi|^2_{h_L} - |\overline{\partial}\xi|^2_{h_L} - 2Re < \overline{\partial}\overline{\partial}^\star \xi, \xi >) dV_\omega$$
$$+ i\Theta_{h_L}(L) \wedge T_\xi.$$

We calculate $\int_X T_\xi \wedge i\partial\overline{\partial} \log |s_Y|^2_{h_Y}$. The residue part gives the LHS. Stokes, curvature conditions and the above equality imply the RHS.

III. Sketch of the proof

Lemma (Siu, Berndtsson, C.- Păun)

Let $n = \dim X$ and let ξ be a smooth $(n,1)$-form with values in L. $h_L = e^{-\varphi_L}$ be a smooth hermitian metric on L satisfies our usual curvature conditions.
Let $\star$ be the Hodge star operator which sends $(n,1)$-forms to $(n-1,0)$-forms. Then

$$\int_Y \star\xi \wedge \overline{\star\xi} e^{-\varphi_L} \leq C \int_X \log^2 |s_Y| (|\overline{\partial}^\star \xi|^2_{h_L} + |\overline{\partial}\xi|^2_{h_L}) dV_{\omega_X}.$$

Proof : Set $T_\xi := \star\xi \wedge \overline{\star\xi} e^{-\varphi_L}$, is a $(n-1,n-1)$-form. We have

$$i\partial\overline{\partial} T_\xi = (|\overline{\partial}(\star\xi)|^2_{h_L} + |\overline{\partial}^\star \xi|^2_{h_L} - |\overline{\partial}\xi|^2_{h_L} - 2Re < \overline{\partial}\overline{\partial}^\star \xi, \xi >) dV_\omega$$
$$+ i\Theta_{h_L}(L) \wedge T_\xi.$$

We calculate $\int_X T_\xi \wedge i\partial\overline{\partial} \log |s_Y|^2_{h_Y}$. The residue part gives the LHS. Stokes, curvature conditions and the above equality imply the RHS.

Lemma (Siu, Berndtsson, C.- Păun)

Let $n = \dim X$ and let ξ be a smooth $(n,1)$-form with values in L. $h_L = e^{-\varphi_L}$ be a smooth hermitian metric on L satisfies our usual curvature conditions.
Let $\star$ be the Hodge star operator which sends $(n,1)$-forms to $(n-1,0)$-forms. Then

$$\int_Y \star\xi \wedge \overline{\star\xi} e^{-\varphi_L} \leq C \int_X \log^2 |s_Y| (|\overline{\partial}^\star \xi|^2_{h_L} + |\overline{\partial}\xi|^2_{h_L}) dV_{\omega_X}.$$

Proof : Set $T_\xi := \star\xi \wedge \overline{\star\xi} e^{-\varphi_L}$, is a $(n-1,n-1)$-form. We have

$$i\partial\overline{\partial} T_\xi = (|\overline{\partial}(\star\xi)|^2_{h_L} + |\overline{\partial}^\star \xi|^2_{h_L} - |\overline{\partial}\xi|^2_{h_L} - 2Re < \overline{\partial}\overline{\partial}^\star \xi, \xi >) dV_\omega$$
$$+ i\Theta_{h_L}(L) \wedge T_\xi.$$

We calculate $\int_X T_\xi \wedge i\partial\overline{\partial} \log |s_Y|^2_{h_Y}$. The residue part gives the LHS. Stokes, curvature conditions and the above equality imply the RHS.

Lemma (Siu, Berndtsson, C.- Păun)

Let $n = \dim X$ and let ξ be a smooth $(n,1)$-form with values in L. $h_L = e^{-\varphi_L}$ be a smooth hermitian metric on L satisfies our usual curvature conditions.
Let $\star$ be the Hodge star operator which sends $(n,1)$-forms to $(n-1,0)$-forms. Then

$$\int_Y \star\xi \wedge \overline{\star\xi} e^{-\varphi_L} \leq C \int_X \log^2 |s_Y| (|\overline{\partial}^\star \xi|^2_{h_L} + |\overline{\partial}\xi|^2_{h_L}) dV_{\omega_X}.$$

Proof : Set $T_\xi := \star\xi \wedge \overline{\star\xi} e^{-\varphi_L}$, is a $(n-1,n-1)$-form. We have

$$i\partial\overline{\partial} T_\xi = (|\overline{\partial}(\star\xi)|^2_{h_L} + |\overline{\partial}^\star \xi|^2_{h_L} - |\overline{\partial}\xi|^2_{h_L} - 2Re < \overline{\partial}\overline{\partial}^\star \xi, \xi >) dV_\omega$$
$$+ i\Theta_{h_L}(L) \wedge T_\xi.$$

We calculate $\int_X T_\xi \wedge i\partial\overline{\partial} \log |s_Y|^2_{h_Y}$. The residue part gives the LHS. Stokes, curvature conditions and the above equality imply the RHS.

III. Sketch of the proof

Lemma (Siu, Berndtsson, C.- Păun)

Let $n = \dim X$ and let ξ be a smooth $(n,1)$-form with values in L. $h_L = e^{-\varphi_L}$ be a smooth hermitian metric on L satisfies our usual curvature conditions.
Let $\star$ be the Hodge star operator which sends $(n,1)$-forms to $(n-1,0)$-forms. Then

$$\int_Y \star\xi \wedge \overline{\star\xi} e^{-\varphi_L} \leq C \int_X \log^2 |s_Y| (|\overline{\partial}^\star \xi|^2_{h_L} + |\overline{\partial}\xi|^2_{h_L}) dV_{\omega_X}.$$

Proof : Set $T_\xi := \star\xi \wedge \overline{\star\xi} e^{-\varphi_L}$, is a $(n-1,n-1)$-form. We have

$$i\partial\overline{\partial} T_\xi = (|\overline{\partial}(\star\xi)|^2_{h_L} + |\overline{\partial}^\star \xi|^2_{h_L} - |\overline{\partial}\xi|^2_{h_L} - 2Re < \overline{\partial}\overline{\partial}^\star \xi, \xi >) dV_\omega$$
$$+ i\Theta_{h_L}(L) \wedge T_\xi .$$

We calculate $\int_X T_\xi \wedge i\partial\overline{\partial} \log |s_Y|^2_{h_Y}$. The residue part gives the LHS. Stokes, curvature conditions and the above equality imply the RHS.

III. Sketch of the proof

- Let ξ be a smooth $(n,1)$-form with values in L. Let $\xi = \xi_1 + \xi_2$ be the Hodge decomposition, where ξ_1 is $\overline{\partial}$-closed and ξ_2 is $\overline{\partial}^\star$-closed.
- By Demailly-C.-Matsumura, there exists a holomorphic extension F of f. Then $\overline{\partial}(\frac{F}{s_Y})$ is a current supported in Y.
- $\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi >= \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi_1 >= \int_Y \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}$.
- $$| \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > |^2 \leq \int_Y |\frac{f}{ds_Y}|^2_{h_L} \int_Y \star\xi_1 \wedge \overline{\star\xi_1} e^{-\varphi_L}$$
$$\leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} (|\overline{\partial}\xi_1|^2_{h_L} + |\overline{\partial}^\star \xi_1|^2_{h_L})$$
$$= C \int_Y |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} |\overline{\partial}^\star \xi|^2_{h_L}.$$

If Y is smooth, then $C(f) := C \int_Y |\frac{f}{ds_Y}|^2_{h_L} < +\infty$.

III. Sketch of the proof

- Let ξ be a smooth $(n,1)$-form with values in L. Let $\xi = \xi_1 + \xi_2$ be the Hodge decomposition, where ξ_1 is $\overline{\partial}$-closed and ξ_2 is $\overline{\partial}^\star$-closed.
- By Demailly-C.-Matsumura, there exists a holomorphic extension F of f. Then $\overline{\partial}(\frac{F}{s_Y})$ is a current supported in Y.
- $\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > = \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi_1 > = \int_Y \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}$.
- $$| \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > |^2 \leq \int_Y |\frac{f}{ds_Y}|^2_{h_L} \int_Y \star\xi_1 \wedge \overline{\star\xi_1} e^{-\varphi_L}$$
$$\leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} (|\overline{\partial}\xi_1|^2_{h_L} + |\overline{\partial}^\star \xi_1|^2_{h_L})$$
$$= C \int_Y |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} |\overline{\partial}^\star \xi|^2_{h_L}.$$

If Y is smooth, then $C(f) := C \int_Y |\frac{f}{ds_Y}|^2_{h_L} < +\infty$.

- Let ξ be a smooth $(n,1)$-form with values in L. Let $\xi = \xi_1 + \xi_2$ be the Hodge decomposition, where ξ_1 is $\overline{\partial}$-closed and ξ_2 is $\overline{\partial}^\star$-closed.
- By Demailly-C.-Matsumura, there exists a holomorphic extension F of f. Then $\overline{\partial}(\frac{F}{s_Y})$ is a current supported in Y.
- $\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > = \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi_1 > = \int_Y \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L}$.
-

$$|\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > |^2 \leq \int_Y |\frac{f}{ds_Y}|^2_{h_L} \int_Y \star\xi_1 \wedge \overline{\star\xi_1} e^{-\varphi_L}$$

$$\leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} (|\overline{\partial}\xi_1|^2_{h_L} + |\overline{\partial}^\star \xi_1|^2_{h_L})$$

$$= C \int_Y |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} |\overline{\partial}^\star \xi|^2_{h_L}.$$

If Y is smooth, then $C(f) := C \int_Y |\frac{f}{ds_Y}|^2_{h_L} < +\infty$.

III. Sketch of the proof

- Let ξ be a smooth $(n,1)$-form with values in L. Let $\xi = \xi_1 + \xi_2$ be the Hodge decomposition, where ξ_1 is $\overline{\partial}$-closed and ξ_2 is $\overline{\partial}^\star$-closed.
- By Demailly-C.-Matsumura, there exists a holomorphic extension F of f. Then $\overline{\partial}(\frac{F}{s_Y})$ is a current supported in Y.
- $\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > = \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi_1 > = \int_Y \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L}$.
-
$$| \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > |^2 \leq \int_Y |\frac{f}{ds_Y}|^2_{h_L} \int_Y \star\xi_1 \wedge \overline{\star\xi_1} e^{-\varphi_L}$$
$$\leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} (|\overline{\partial}\xi_1|^2_{h_L} + |\overline{\partial}^\star \xi_1|^2_{h_L})$$
$$= C \int_Y |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} |\overline{\partial}^\star \xi|^2_{h_L}.$$

If Y is smooth, then $C(f) := C \int_Y |\frac{f}{ds_Y}|^2_{h_L} < +\infty$.

III. Sketch of the proof

- If $C(f) < +\infty$, $\exists \mu$ such that $< \overline{\partial}(\frac{F}{s_Y}), \xi > = < \mu, \overline{\partial}^\star \xi >$ and

$$\int_X \frac{|\mu|^2_{h_L}}{\log^2 |s_Y|^2_{h_L}} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L}.$$

- $\overline{\partial}(\mu) = \overline{\partial}(\frac{F}{s_Y})$. Then $\widetilde{F} := \mu \cdot s_Y$ is holomorphic and $\widetilde{F}|_Y = f$.
- Main difficulty : If Y is not smooth, $C(f) = \int_Y |\frac{f}{ds_Y}|^2_{h_L}$ might not be finite !
- Idea : Let ξ be a smooth $(n,1)$-form with values in L and supported in $X \setminus V_{sing}$. Let $\xi = \xi_1 + \xi_2$ be the Hodge decomposition. If we could prove that

$$|\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > |^2 \leq C(f) \int_X \log^2 |s_Y|_{h_Y} \cdot |\overline{\partial}^\star \xi|^2_{h_Y} \qquad (3)$$

for some constant $C(f) < +\infty$ depending on f.

III. Sketch of the proof

- If $C(f) < +\infty$, $\exists \mu$ such that $< \overline{\partial}(\frac{F}{s_Y}), \xi > = < \mu, \overline{\partial}^\star \xi >$ and

$$\int_X \frac{|\mu|^2_{h_L}}{\log^2 |s_Y|^2_{h_L}} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L}.$$

- $\overline{\partial}(\mu) = \overline{\partial}(\frac{F}{s_Y})$. Then $\widetilde{F} := \mu \cdot s_Y$ is holomorphic and $\widetilde{F}|_Y = f$.

- Main difficulty : If Y is not smooth, $C(f) = \int_Y |\frac{f}{ds_Y}|^2_{h_L}$ might not be finite !

- Idea : Let ξ be a smooth $(n,1)$-form with values in L and supported in $X \setminus V_{sing}$. Let $\xi = \xi_1 + \xi_2$ be the Hodge decomposition. If we could prove that

$$|\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > |^2 \leq C(f) \int_X \log^2 |s_Y|_{h_Y} \cdot |\overline{\partial}^\star \xi|^2_{h_Y} \qquad (3)$$

for some constant $C(f) < +\infty$ depending on f.

III. Sketch of the proof

- If $C(f) < +\infty$, $\exists \mu$ such that $< \overline{\partial}(\frac{F}{s_Y}), \xi > = < \mu, \overline{\partial}^\star \xi >$ and

$$\int_X \frac{|\mu|^2_{h_L}}{\log^2 |s_Y|^2_{h_L}} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L}.$$

- $\overline{\partial}(\mu) = \overline{\partial}(\frac{F}{s_Y})$. Then $\widetilde{F} := \mu \cdot s_Y$ is holomorphic and $\widetilde{F}|_Y = f$.

- Main difficulty : If Y is not smooth, $C(f) = \int_Y |\frac{f}{ds_Y}|^2_{h_L}$ might not be finite !

- Idea : Let ξ be a smooth $(n,1)$-form with values in L and supported in $X \setminus V_{sing}$. Let $\xi = \xi_1 + \xi_2$ be the Hodge decomposition. If we could prove that

$$|\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > |^2 \leq C(f) \int_X \log^2 |s_Y|_{h_Y} \cdot |\overline{\partial}^\star \xi|^2_{h_Y} \qquad (3)$$

for some constant $C(f) < +\infty$ depending on f.

III. Sketch of the proof

- If $C(f) < +\infty$, $\exists\mu$ such that $< \overline{\partial}(\frac{F}{s_Y}), \xi > = < \mu, \overline{\partial}^{\star}\xi >$ and

$$\int_X \frac{|\mu|^2_{h_L}}{\log^2 |s_Y|^2_{h_L}} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L}.$$

- $\overline{\partial}(\mu) = \overline{\partial}(\frac{F}{s_Y})$. Then $\widetilde{F} := \mu \cdot s_Y$ is holomorphic and $\widetilde{F}|_Y = f$.
- Main difficulty : If Y is not smooth, $C(f) = \int_Y |\frac{f}{ds_Y}|^2_{h_L}$ might not be finite !
- Idea : Let ξ be a smooth $(n,1)$-form with values in L and supported in $X \setminus V_{sing}$. Let $\xi = \xi_1 + \xi_2$ be the Hodge decomposition. If we could prove that

$$|\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > |^2 \leq C(f) \int_X \log^2 |s_Y|_{h_Y} \cdot |\overline{\partial}^{\star}\xi|^2_{h_Y} \qquad (3)$$

for some constant $C(f) < +\infty$ depending on f.

III. Sketch of the proof

- If $C(f) < +\infty$, $\exists \mu$ such that $< \overline{\partial}(\frac{F}{s_Y}), \xi > = < \mu, \overline{\partial}^\star \xi >$ and

$$\int_X \frac{|\mu|^2_{h_L}}{\log^2 |s_Y|^2_{h_L}} \leq C \int_Y |\frac{f}{ds_Y}|^2_{h_L}.$$

- $\overline{\partial}(\mu) = \overline{\partial}(\frac{F}{s_Y})$. Then $\widetilde{F} := \mu \cdot s_Y$ is holomorphic and $\widetilde{F}|_Y = f$.

- Main difficulty : If Y is not smooth, $C(f) = \int_Y |\frac{f}{ds_Y}|^2_{h_L}$ might not be finite !

- Idea : Let ξ be a smooth $(n,1)$-form with values in L and supported in $X \setminus V_{sing}$. Let $\xi = \xi_1 + \xi_2$ be the Hodge decomposition. If we could prove that

$$|\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi > |^2 \leq C(f) \int_X \log^2 |s_Y|_{h_Y} \cdot |\overline{\partial}^\star \xi|^2_{h_Y} \qquad (3)$$

for some constant $C(f) < +\infty$ depending on f.

- Then we have $< \overline{\partial}(\frac{F}{s_Y}), \xi > = < \mu, \overline{\partial}^\star \xi >$ and the L^2-norm of μ is controlled by $C(f)$.
- As ξ is supported in $X \setminus V_{sing}$, we have thus

$$\overline{\partial}(\mu) = \overline{\partial}(\frac{F}{s_Y}) \qquad \text{on } X \setminus V_{sing}.$$

Then $\mu \cdot s_Y \in H^0(X \setminus V_{sing}, K_X \otimes L \otimes \mathcal{O}_X(Y))$ extends f.

- By Hartogs, $\mu \cdot s_Y|_{X \setminus V_{sing}}$ extends to be a holomorphic section $H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ with a control on its L^2-norm over $X \setminus V_{sing}$. It finishes the proof of the theorem.

- Then we have $<\overline{\partial}(\frac{F}{s_Y}), \xi> = <\mu, \overline{\partial}^\star \xi>$ and the L^2-norm of μ is controlled by $C(f)$.
- As ξ is supported in $X \setminus V_{sing}$, we have thus

$$\overline{\partial}(\mu) = \overline{\partial}(\frac{F}{s_Y}) \qquad \text{on } X \setminus V_{sing}.$$

Then $\mu \cdot s_Y \in H^0(X \setminus V_{sing}, K_X \otimes L \otimes \mathcal{O}_X(Y))$ extends f.

- By Hartogs, $\mu \cdot s_Y|_{X \setminus V_{sing}}$ extends to be a holomorphic section $H^0(X, K_X \otimes L \otimes \mathcal{O}_X(Y))$ with a control on its L^2-norm over $X \setminus V_{sing}$. It finishes the proof of the theorem.

- Then we have $<\overline{\partial}(\frac{F}{s_Y}),\xi>=<\mu,\overline{\partial}^\star\xi>$ and the L^2-norm of μ is controlled by $C(f)$.
- As ξ is supported in $X\setminus V_{sing}$, we have thus

$$\overline{\partial}(\mu)=\overline{\partial}(\frac{F}{s_Y})\qquad \text{on } X\setminus V_{sing}.$$

  Then $\mu\cdot s_Y\in H^0(X\setminus V_{sing},K_X\otimes L\otimes\mathcal{O}_X(Y))$ extends f.
- By Hartogs, $\mu\cdot s_Y|_{X\setminus V_{sing}}$ extends to be a holomorphic section $H^0(X,K_X\otimes L\otimes\mathcal{O}_X(Y))$ with a control on its L^2-norm over $X\setminus V_{sing}$. It finishes the proof of the theorem.

It remains to prove (3).

- As before, we have

$$\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi >= \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi_1 >= \int_Y \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L}$$

$$= \int_{Y \setminus V_{\mathrm{sing}}} \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L} + \int_{Y \cap V_{\mathrm{sing}}} \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L}.$$

- $|\int_{Y\setminus V_{\mathrm{sing}}} \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L}|^2 \leq \int_{Y\setminus V_{\mathrm{sing}}} |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_Y \star\xi_1 \wedge \overline{\star\xi_1} e^{-\varphi_L}$

$$\leq \int_{Y \setminus V_{\mathrm{sing}}} |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} \cdot |\overline{\partial}^\star \xi|^2_{h_L}.$$

- To control $\int_{Y \cap V_{\mathrm{sing}}} \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L}$, we recall that ξ is supported in $X \setminus V_{\mathrm{sing}}$, then $\xi_1 = -\xi_2$ on V_{sing}. Therefore $\overline{\partial}\xi_1 = 0$ and $\overline{\partial}^\star \xi_1 = -\overline{\partial}^\star \xi_2 = 0$ on V_{sing}.

It remains to prove (3).

- As before, we have

$$\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi >= \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi_1 >= \int_Y \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}$$

$$= \int_{Y \setminus V_{\text{sing}}} \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L} + \int_{Y \cap V_{\text{sing}}} \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}.$$

- $|\int_{Y \setminus V_{\text{sing}}} \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}|^2 \leq \int_{Y \setminus V_{\text{sing}}} |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_Y \star\xi_1 \wedge \overline{\star\xi_1} e^{-\varphi_L}$

$$\leq \int_{Y \setminus V_{\text{sing}}} |\frac{f}{ds_Y}|^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} \cdot |\overline{\partial}^\star \xi|^2_{h_L}.$$

- To control $\int_{Y \cap V_{\text{sing}}} \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}$, we recall that ξ is supported in $X \setminus V_{\text{sing}}$, then $\xi_1 = -\xi_2$ on V_{sing}. Therefore $\overline{\partial}\xi_1 = 0$ and $\overline{\partial}^\star \xi_1 = -\overline{\partial}^\star \xi_2 = 0$ on V_{sing}.

It remains to prove (3).

- As before, we have

$$\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi >= \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi_1 >= \int_Y \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}$$

$$= \int_{Y \setminus V_{\text{sing}}} \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L} + \int_{Y \cap V_{\text{sing}}} \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}.$$

- $| \int_{Y \setminus V_{\text{sing}}} \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L} |^2 \leq \int_{Y \setminus V_{\text{sing}}} | \frac{f}{ds_Y} |^2_{h_L} \cdot \int_Y \star\xi_1 \wedge \overline{\star\xi_1} e^{-\varphi_L}$

$$\leq \int_{Y \setminus V_{\text{sing}}} | \frac{f}{ds_Y} |^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} \cdot |\overline{\partial}^\star \xi|^2_{h_L}.$$

- To control $\int_{Y \cap V_{\text{sing}}} \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}$, we recall that ξ is supported in $X \setminus V_{\text{sing}}$, then $\xi_1 = -\xi_2$ on V_{sing}. Therefore $\overline{\partial}\xi_1 = 0$ and $\overline{\partial}^\star \xi_1 = -\overline{\partial}^\star \xi_2 = 0$ on V_{sing}.

It remains to prove (3).

- As before, we have

$$\int_X < \overline{\partial}(\frac{F}{s_Y}), \xi >= \int_X < \overline{\partial}(\frac{F}{s_Y}), \xi_1 >= \int_Y \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L}$$

$$= \int_{Y \setminus V_{\rm sing}} \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L} + \int_{Y \cap V_{\rm sing}} \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L}.$$

- $| \int_{Y \setminus V_{\rm sing}} \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L} |^2 \leq \int_{Y \setminus V_{\rm sing}} | \frac{f}{ds_Y} |^2_{h_L} \cdot \int_Y \star \xi_1 \wedge \overline{\star \xi_1} e^{-\varphi_L}$

$$\leq \int_{Y \setminus V_{\rm sing}} | \frac{f}{ds_Y} |^2_{h_L} \cdot \int_X \log^2 |s_Y|_{h_L} \cdot |\overline{\partial}^\star \xi|^2_{h_L}.$$

- To control $\int_{Y \cap V_{\rm sing}} \frac{f}{ds_Y} \wedge \overline{\star \xi_1} e^{-\varphi_L}$, we recall that ξ is supported in $X \setminus V_{\rm sing}$, then $\xi_1 = -\xi_2$ on $V_{\rm sing}$. Therefore $\overline{\partial} \xi_1 = 0$ and $\overline{\partial}^\star \xi_1 = -\overline{\partial}^\star \xi_2 = 0$ on $V_{\rm sing}$.

- By using the harmonicity of ξ_1, we can prove that

$$\sup_{\frac{1}{2}V_{\mathrm{sing}}} |\xi_1|^2_{h_L} \leq C \int_{V_{\mathrm{sing}}} |\xi_1|^2_{h_L} \leq C' \int_X |\overline{\partial}^\star \xi|^2_{h_L}.$$

The last inequality comes from Bochner equality and the curvature conditions.

- $|\int_{Y \cap \frac{1}{2}V_{\mathrm{sing}}} \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}| \leq \int_{Y \cap \frac{1}{2}V_{\mathrm{sing}}} |\frac{f}{ds_Y}|_{h_L} \cdot (\int_X |\overline{\partial}^\star \xi|^2_{h_L})^{\frac{1}{2}}$

- By using the harmonicity of ξ_1, we can prove that

$$\sup_{\frac{1}{2}V_{\mathrm{sing}}} |\xi_1|^2_{h_L} \leq C \int_{V_{\mathrm{sing}}} |\xi_1|^2_{h_L} \leq C' \int_X |\overline{\partial}^\star \xi|^2_{h_L}.$$

  The last inequality comes from Bochner equality and the curvature conditions.

- $|\int_{Y\cap\frac{1}{2}V_{\mathrm{sing}}} \frac{f}{ds_Y} \wedge \overline{\star\xi_1} e^{-\varphi_L}| \leq \int_{Y\cap\frac{1}{2}V_{\mathrm{sing}}} |\frac{f}{ds_Y}|_{h_L} \cdot (\int_X |\overline{\partial}^\star \xi|^2_{h_L})^{\frac{1}{2}}$

Thanks for your attention !