Distribution of scattering resonances for generic Schrödinger operators

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- Consider the Schrödinger operator $-\Delta + V : L^2(\mathbb{R}^d, \mathbb{C}) \dashrightarrow L^2(\mathbb{R}^d, \mathbb{C})$ with $d \geqslant 3$ odd.
- The potential V is a bounded function supported by $\overline{\mathbb{B}}_a := \{\|x\| \leqslant a\}$.
- For $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda)$ large enough, $R_V(\lambda) := (-\Delta + V - \lambda^2)^{-1}$ is well-defined and is a bounded operator.
- If ρ is smooth with compact support and equal to 1 on $\overline{\mathbb{B}}_a$, then

$$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1-\rho)).$$

- We can extend $R_V(\lambda) : L^2_{\mathrm{compact}} \to L^2_{\mathrm{loc}}$ to a meromorphic family for $\lambda \in \mathbb{C}$.
- The poles are called the resonances of $-\Delta + V$.

Problem (main problem, Melrose, Zworski..., similar to Weyl's law)

Study the distribution of the resonances, in particular, the growth of the number $n_V(r)$ of resonances in $\overline{\mathbb{D}(0,r)}$ with $r \to \infty$.

Remark

Dimension $d = 1$: Melrose and Zworski.

Recall that $n_V(r)$ is the number of resonances in $\overline{\mathbb{D}(0,r)}$. Define

$$N_V(r) := \int_1^r \frac{n_V(t)}{t}\,dt.$$

Theorem (Zworski, Stefanov, D-Vu)

There is an explicit dimensional constant $c_d > 0$ such that

$$N_V(r) \leqslant \frac{c_d a^d r^d}{d} + O(r^{d-1}\log r) \qquad \textit{and} \qquad n_V(r) \leqslant c_d a^d r^d + O(r^{d-1/2+\epsilon}).$$

Define

$$\mathfrak{M}_a := \left\{ V \in L^\infty(\mathbb{B}_a, \mathbb{C}) :\ n_V(r) - c_d a^d r^d = o(r^d) \right\}$$

$$\mathfrak{M}_a^\delta := \left\{ V \in L^\infty(\mathbb{B}_a, \mathbb{C}) :\ n_V(r) - c_d a^d r^d = O(r^{d-\delta+\epsilon}) \text{ for every } \epsilon > 0 \right\}.$$

Theorem (Christiansen, complex potentials without resonances)

There is $V \notin \mathfrak{M}_a$ smooth, not real, non-vanishing on the boundary of $\mathbb{B}_a$.

Problem (self-adjoint case)

The question is open for potentials V with real values.

Theorem (Zworski, Stefanov, D-Vu, the only known explicit examples)

Assume that V *is smooth, real-valued, radial, non-vanishing on the boundary of* $\mathbb{B}_a$*. Then* $V \in \mathfrak{M}_a^{3/4}$*.*

Theorem (D-Vu)

We have $V \in \mathfrak{M}_a^{3/8}$ *for generic* V *in* $L^\infty(\mathbb{B}_a, \mathbb{C})$ *or* $L^\infty(\mathbb{B}_a, \mathbb{R})$*.*

Define

$$\mu_{V,r} := \frac{1}{c_d a^d r^d} \sum_{z \text{ resonance}} \delta_{z/r}.$$

Theorem (D-Nguyen)

There is an explicit positive measure μ_{MZ} *supported by* $\{\mathrm{Im}(z) \leqslant 0\}$*, independent of* a, V*, such that*

- *if* $V \in \mathfrak{M}_a$ *then* $\mu_{V,r} \to \mu_{MZ}$*;*
- *if* $V \in \mathfrak{M}_a^\delta$ *then the speed of convergence is* $O(r^{-\delta+\epsilon})$*.*

Remark

The mass on the unit disc is 1. On both $\mathbb{R}$ and $\{\mathrm{Im}(z) < 0\}$, the mass is positive and the density is smooth. Invariant by $\lambda^d (A_\lambda)_*$ for $A_\lambda(z) := \lambda z$.

- In infinite dimension, G_δ-density is often used as a notion of genericity. However, such a set may be very small. In dimension 1, such a set may have length 0.
- $L^\infty(\mathbb{B}_a, \mathbb{C})$ can be seen as a complex manifold of infinite dimension.
- A subset A is *almost full* if there is V_0 such that for any holomorphic map $f : D \to L^\infty(\mathbb{B}_a, \mathbb{C})$, with $f(0) = V_0$, we have $D \setminus f^{-1}(A)$ is pluripolar. Here, D is a neighbourhood of 0 in some $\mathbb{C}^N$.

Lemma (D-Nguyen-Sibony, Vu, Monge-Ampère measures of Hölder potentials)

Let A be as above with $V_0 \in L^\infty(\mathbb{B}_a, \mathbb{R})$. Then $A \cap L^\infty(\mathbb{B}_a, \mathbb{R})$ is almost full in the sense that if the above map f is real, then $D_\mathbb{R} \setminus f^{-1}(A)$ is of Lebesgue measure 0.

- Thus, we have a notion of genericity for $L^\infty(\mathbb{B}_a, \mathbb{C})$ and $L^\infty(\mathbb{B}_a, \mathbb{R})$.
- In the above definition, we can consider f linear.

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- Let $\theta \in \mathbb{S}^{d-1}$ and $x \in \mathbb{R}^d$.
- Consider the solution of the equation $(-\Delta + V - \lambda^2)\psi = 0$ with Sommerfeld condition

$$(\partial/\partial r - i\lambda)(\psi - e^{i\lambda\theta\cdot x}) = O(r^{-(d+1)/2}).$$

- One has

$$\psi = e^{i\lambda\theta\cdot x} + \frac{e^{i\lambda r}}{r^{(d-1)/2}} a(x/r, \theta, \lambda) + \text{h.o.t.}$$

- Let $A(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ be the operator with Schwartz's kernel $a(\omega, \theta, \lambda)$.
- Define the scattering operator and its determinant by

$$S_V(\lambda) := I + A(\lambda) \qquad \text{and} \qquad s_V(\lambda) := \det S_V(\lambda).$$

- We have

$$s_V(\lambda) s_V(-\lambda) = 1.$$

- The poles of s_V are exactly the resonances except for finitely many points.
- So the problem is reduced to study the zeros on s_V on the upper half-plane and we can use $dd^c \log |s_V|$.

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Theorem (Zworski, Stefanov, D-Vu, the only known explicit examples)

Assume that V is smooth, real-valued, radial, non-vanishing on the boundary of $\mathbb{B}_a$. Then $V \in \mathfrak{M}_a^{3/4}$ (good asymptotic number of resonances with error $O(r^{d-3/4+\epsilon})$).

- Assume that V is radial. Consider the space of spherical harmonics H_l of harmonic homogeneous polynomials of degree l restricted to $\mathbb{S}^{d-1}$.
- $S_V(\lambda)$ acts on each H_l as the multiplication by a number which depends meromorphically on λ.
- The problem is to estimate the numbers of zeros of these meromorphic functions in λ.
- We use some good change of variable, leading terms of those functions and Rouché's theorem.

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Theorem (Zworski, Stefanov, D-Vu)

There is an explicit dimensional constant $c_d > 0$ *such that*

$$N_V(r) \leqslant \frac{c_d a^d r^d}{d} + O(r^{d-1} \log r) \qquad \textit{and} \qquad n_V(r) \leqslant c_d a^d r^d + O(r^{d-1/2+\epsilon}).$$

- One can show that $\log |s_V(\lambda)|$ is small on $\mathbb{R}$ and deduce that

$$\left| N_V(r) - \frac{1}{2\pi} \int_0^\pi \log |s_V(re^{i\theta})| d\theta \right| = O(r^{d-1}).$$

- Then we show that for some explicit function $h(\theta)$ on $[0, \pi]$

$$\log |s_V(re^{i\theta})| \leqslant h(\theta) a^d r^d + O(r^{d-1} \log r).$$

- Denote by μ_j the singular values of an operator. Two main ingredients are

$$\log |\det(I + A_V)| \leqslant \sum \log(1 + \mu_j(A_V)) \qquad \text{and} \qquad \mu_j(BC) \leqslant \|B\| \, \mu_j(C).$$

- We also need properties of Bessel functions.

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- Recall that

$$\left| N_V(r) - \frac{1}{2\pi} \int_0^\pi \log |s_V(re^{i\theta})| d\theta \right| = O(r^{d-1}).$$

- We can replace N_V by the above integral which is p.s.h. in $V \in L^\infty(\mathbb{B}_a, \mathbb{C})$.
- Main ingredient (in finite dimension): if u_n p.s.h. such that $u_n \leqslant c + o(1)$ and $u_n(0) \to c$, then $u_n \to c$ in L^p_{loc} (maximum principle).
- Quantitative version: if the hypotheses are quantitative and strong enough, then $u_n \to c$ pointwise outside a pluripolar set with a speed control.
- Fix a good radial potential $V_0 \in L^\infty(\mathbb{B}_a, \mathbb{C})$ and apply the argument for analytic sets passing through V_0.

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- *if $V \in \mathfrak{M}_a$ then $\mu_{V,r} \to \mu_{MZ}$;*
- *if $V \in \mathfrak{M}_a^\delta$ then the speed of convergence is $O(r^{-\delta+\epsilon})$.*

Remark

The mass on the unit disc is 1. On both $\mathbb{R}$ and $\{\mathrm{Im}(z) < 0\}$, the mass is positive and the density is smooth. Invariant by $\lambda^d (A_\lambda)_*$ for $A_\lambda(z) := \lambda z$.

- Main ingredient: if v_n is subharmonic on the unit disc $\mathbb{D} \subset \mathbb{C}$ such that

$$v_n \leqslant v + o(1) \qquad \text{and} \qquad \Big| \int_0^{2\pi} v_n(te^{i\theta}) d\theta - \int_0^{2\pi} v(te^{i\theta}) d\theta \Big| = o(1),$$

  then $v_n \to v$ in L^p_{loc} and $dd^c v_n \to dd^c v$ weakly.
- There is a quantitative version. We apply this for the functions $r^{-d} \log |s_V(r^{-1}z)|$ with $r \to \infty$.
- Technical difficulty: these functions are subharmonic only on a half-plane. We need a version of Schwarz's reflection.