Dec. 6, 2021, 11:00 ~ 11:50.

Semipositive line bdls and hol. foliatns.

§0. Introduction

◎ X : compact Kähler manifold.

◎ $\alpha \in H^{1,1}(X,\mathbb{R}) \left(= H^{1,1}(X) \cap H^2(X,\mathbb{R}) \subset H^2(X,\mathbb{C}) \right)$

◎ $SP(\alpha) := \left\{ \theta \;;\; \begin{array}{c} C^\infty \text{ (1,1)-form} \\ \text{on } X \end{array} \;\middle|\; \begin{array}{l} d\theta = 0,\ \{\theta\} = \alpha \\ \overline{\theta} = \theta \\ \theta \geq 0 \end{array} \right\}$

[$\sqrt{-1} \sum a_{jk}\, dz^j \wedge d\bar{z}^k \geq 0$
$\iff (a_{jk}) \geq 0$ posi. semi-def.

e.g. 1 L : hol. line bdl on X.

$\rightsquigarrow$ L : semi-positive $\iff SP(c_1(L)) \neq \emptyset$ //

$\Updownarrow$ def

$\exists h$: C^∞ Herm. metric on L s.t. $\sqrt{-1}\,\Theta_h \geq 0$

Obs L : positive $\Longrightarrow$ L : semi-pos. $\iff SP(c_1(L)) \neq \emptyset$

$\Updownarrow$

L : ample. "$\iff$" $L^{\otimes m}$ has enough sect'ns so that they reflect "the geometry" of X //

Question

What is "the geometry" of X
which "corresponds" to α with $sp(\alpha) \neq \phi$?"

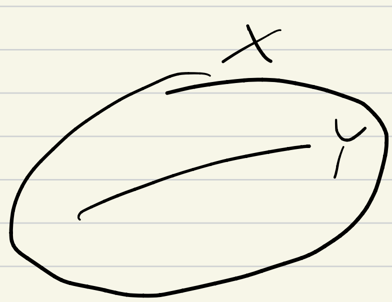
We expect (as an answer to this Q)

that $\exists\ \mathcal{F}_\alpha$: hol. foliation
on (a domain of) X for such α
if $\alpha \neq 0$, $\alpha \wedge \alpha = 0$ in $H^{2,2}(X, \mathbb{C})$,

e.g. 2 X: cpt cpx surface (non-sing)
$\cup$
Y: hol'ly embedded cpt Riemann Surface
(conn, non-sing)
$L := [Y]$; the l.b which corresp. to the divisor Y

$\alpha := c_1(L)$ $(\to \alpha \neq 0)$

$\odot$ $\alpha \wedge \alpha = \deg \underbrace{N_{Y/X}}_{\text{normal bdl.}}$



◎ $\alpha \wedge \alpha < 0 \iff \int_Y c_1(L) < 0$

$\implies L$: not semi-positive.

◎ $\alpha \wedge \alpha > 0 \xRightarrow{\text{alg. geom. argument}} L$: semi-ample

at least when X: proj. $\implies L$: semi-positive

◎ $\alpha \wedge \alpha = 0$ ---- the most interesting case!

$\exists$ both of the cases w/ L: semi-positive and L: NOT semi-positive.

Thm 0 (K-'20 Math. Ann + Ohsawa '21 Abhadlungen)

(Kä-ness assumpt'n is NOT needed in this thm [Ohsawa])

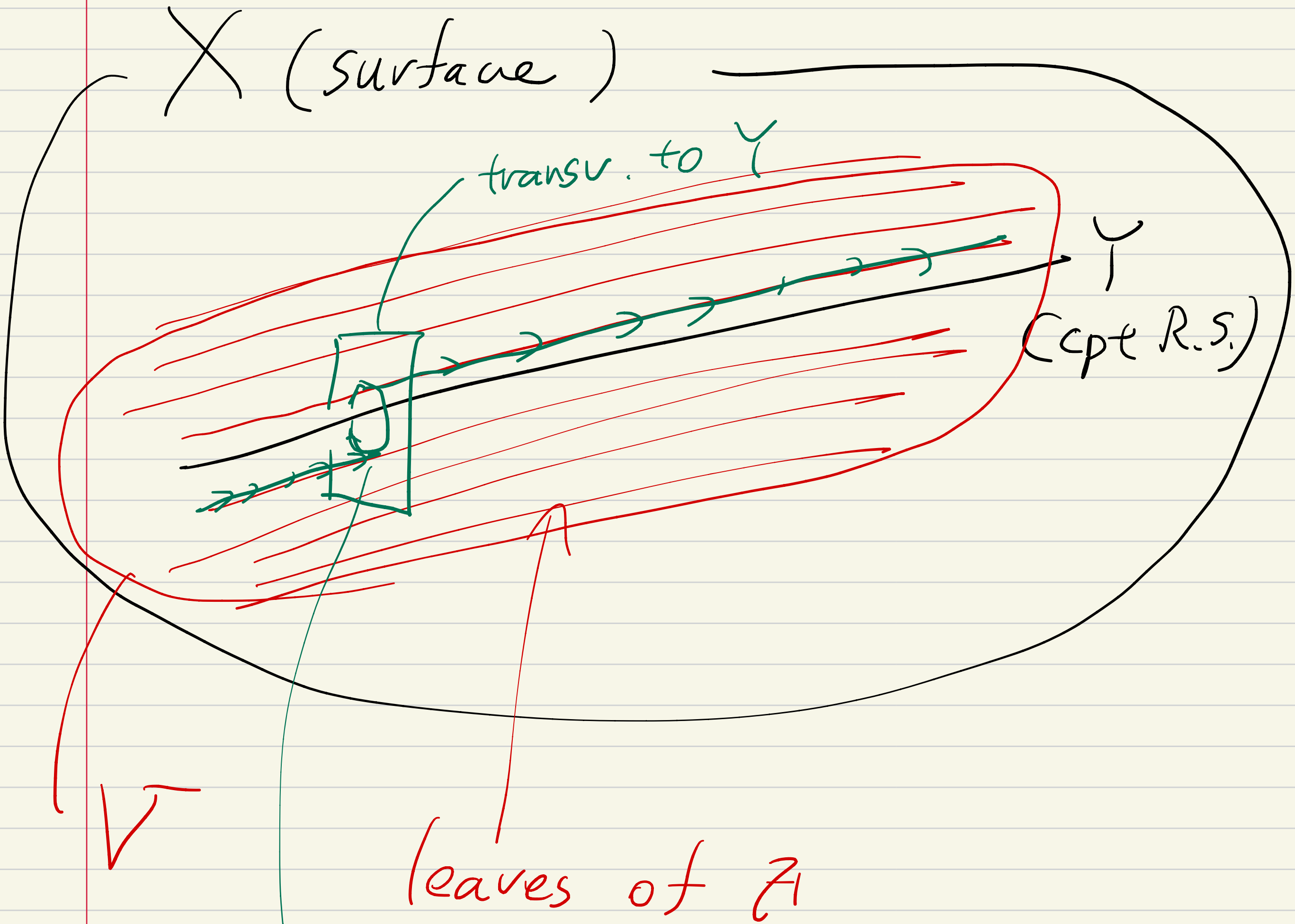
X, Y, α: as in e.g. 2

Assume $\alpha \wedge \alpha = 0$ ($\underset{\text{known}}{\iff} N_{Y/X}$: unitary flat)

Then $L = [Y]$: Semi-positive

$\iff \exists V$: nbhd of Y, $\exists \mathcal{F}$: hol. foliat'n on V

s.t. $\begin{cases} Y \text{ is a leaf of } \mathcal{F} \\ \text{The holonomy of } \mathcal{F} \text{ along } Y \text{ is } U(1)\text{-“linear”} \end{cases}$



the holonomy

$$\mathrm{Hol}_{\mathcal{F},Y} : \pi_1(Y,*) \longrightarrow \mathrm{Diff}(\mathbb{C},0)$$

coinsides

$\exists$ repr. $\pi_1(Y,*) \longrightarrow U(1)$

//

Q How large is V?

(when L: semi-positive)

$\boxed{\text{Schedule}}$ Main results $\rightarrow$ Examples $\rightarrow$ Outline of prf
(§1) (§2) (§3)

§1. Main results

X: cpt kä, connected.

$\alpha \in H^{1,1}(X,\mathbb{R})$ s.t. $\begin{cases} \alpha \neq 0 \\ \alpha \wedge \alpha = 0 \\ \# SP(\alpha) > 1 \end{cases}$

Def For $\theta \in SP(\alpha)$,

$$PSH^\infty(X,\theta) := \{ \varphi : X \to \mathbb{R} : C^\infty \mid \theta + \sqrt{-1}\partial\bar\partial\varphi \geq 0 \},$$

Def

$$K_\alpha := \bigcap_{\theta \in SP(\alpha)} \bigcap_{\varphi \in PSH^\infty(X,\theta)} \{ x \in X \mid (d\varphi)_x = 0 \}$$ //

Rmk

$(\# SP(\alpha) > 1$ and $\partial\bar\partial$-lem. and Sard's thm $\Rightarrow)$

$$K_\alpha \subsetneqq X$$ //

Thm 1 ← [K-'21], Arxiv: 2110.04864

$\exists ! \mathcal{F}_\alpha$: non-sing. hol. foliat'n on $X \setminus K_\alpha$ of $\operatorname{codim}_{\mathbb{C}} = 1$

s.t. $i_{\mathcal{L}}^* \theta \equiv 0$ for $\begin{cases} {}^\forall \theta \in SP(\alpha) \\ {}^\forall \mathcal{L} : \text{leaf of } \mathcal{F}_\alpha \end{cases}$ $\left(i_{\mathcal{L}} : \mathcal{L} \hookrightarrow X : \text{immersion} \right)$ //

Def K': a connected component of K_α

$\rightsquigarrow$ K': not an essential comp of K_α

$\underset{\text{def}}{\iff}$ $\exists W$: a conn. open nbhd of K' in X

$\exists h : \overline{W} \longrightarrow [-\infty, \infty]$: conti

s.t. $h|_W$: pluriharmonic

$h|_{W \setminus K_\alpha}$: $\mathcal{F}_\alpha$-leafwise constant

$h^{-1}(\{\min_{\overline{W}} h, \max_{\overline{W}} h\}) = \partial W$

$W \cap K_\alpha \underset{\text{rel.cpt}}{\Subset} W$

//

Def $K_\alpha^{ess} := \bigcup_{K' : \text{ess.comp.of } K_\alpha} K'$

//

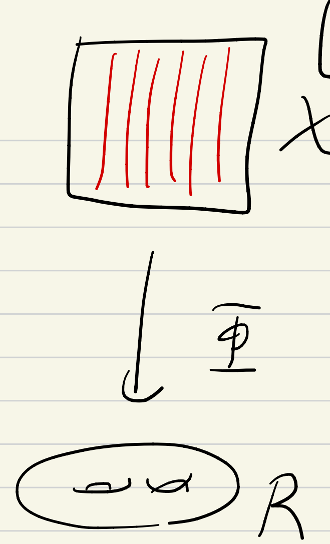
Thm 2 ([K-'21])

$\mathcal{F}_\alpha$ can be extended to $X \setminus K_\alpha^{ess}$

as a (maybe singular) hol. foliation.

Moreover, one of the following holds;

$\boxed{\text{Case I}}$



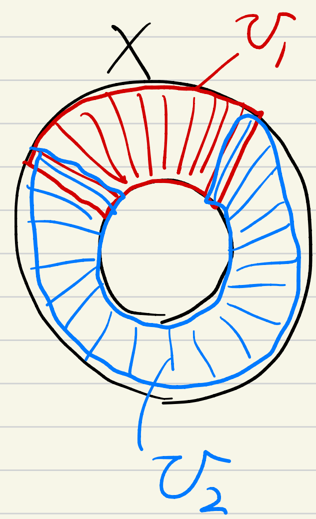
$\exists\, \Phi : X \xrightarrow{\text{hol}} {}^{\exists} R$ ~ cpt Riemann surf.

$\exists\, \alpha_R$: a Kä class of R

s.t. $\alpha = \Phi^* \alpha_R$.

($\rightsquigarrow$ $K_\alpha^{ess} = \emptyset$, $\mathcal{F}_\alpha$ = "the fibrat'n Φ")

$\boxed{\text{Case II}}$ Not in case I, and $K_\alpha^{ess} = \emptyset$



$\rightsquigarrow$ $\exists\, \{U_1, U_2\}$: open cov. of X

$\exists\, h_j : \overline{U_j} \longrightarrow [-\infty, \infty]$

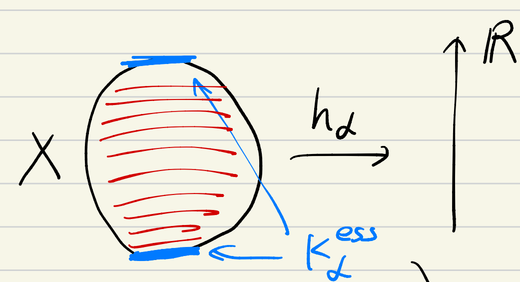
; satisfying ⊛

s.t. $h_2 = {}^{\exists}(\text{affine}) \circ h_1$

on each conn. comp. of $U_1 \cap U_2$

$T_{\mathcal{F}_\alpha|U_j} = (\partial h_j)^\perp$

$\boxed{\text{Case III}}$ $K_\alpha^{ess} \neq \emptyset$



$\rightsquigarrow$ $\exists\, h_\alpha : \overline{X \setminus K_\alpha^{ess}} \to [-\infty, \infty]$

; satisfying ⊛,

$T_{\mathcal{F}_\alpha} = (\partial h_\alpha)^\perp$, $K_\alpha \setminus K_\alpha^{ess} = \{\text{crit. points of } h_\alpha\}$

As an application of Thm 1, 2,
one can generalize Thm 0

Thm 3 ([K-'21])

X : connected cpt Kä mfd
$Y \subset X$: connected non-sing. hypersurface
s.t. $c_1(N_{Y/X}) = 0$

Then $[Y]$: semi-positive
$\iff$ $\exists V$: a nbhd of Y
$\exists \mathcal{F}$: a hol. foliation on V
s.t. Y is a leaf of $\mathcal{F}$
$\mathrm{Hol}_{\mathcal{F}, Y}$ is $U(1)$-linear

Moreover, when $N_{Y/X}^{\otimes m}$: hol'ly trivial for $\exists m \geq 1$,
[K-'20]
$[Y]$: semi-positive
$\iff$
$\exists \Phi : X \xrightarrow{\text{hol}} \exists R$ cpt Riemann surf
s.t. Y is a fiber of Φ

§2. Examples

e.g. 3 $X = \mathbb{C}_w \times \mathbb{C}_z \Big/ \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\sqrt{-1}\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}a+b\sqrt{-1}\\ \tau\end{pmatrix} \right\rangle$

: cpx torus $\left(\begin{array}{l} a, b \in \mathbb{R} \\ \tau \in \mathbb{H} \text{ (upper half plane)} \end{array} \right)$

$\alpha := \{\sqrt{-1}\, dw \wedge d\bar{w}\} \in H^{1,1}(X, \mathbb{R})$

$(\rightsquigarrow \alpha \neq 0, \quad \alpha \wedge \alpha = 0)$

In this case,

$X \longrightarrow \mathbb{C}_z / \langle 1, \tau \rangle$: $\mathbb{C}_w / \langle 1, \sqrt{-1} \rangle$: bdl str.

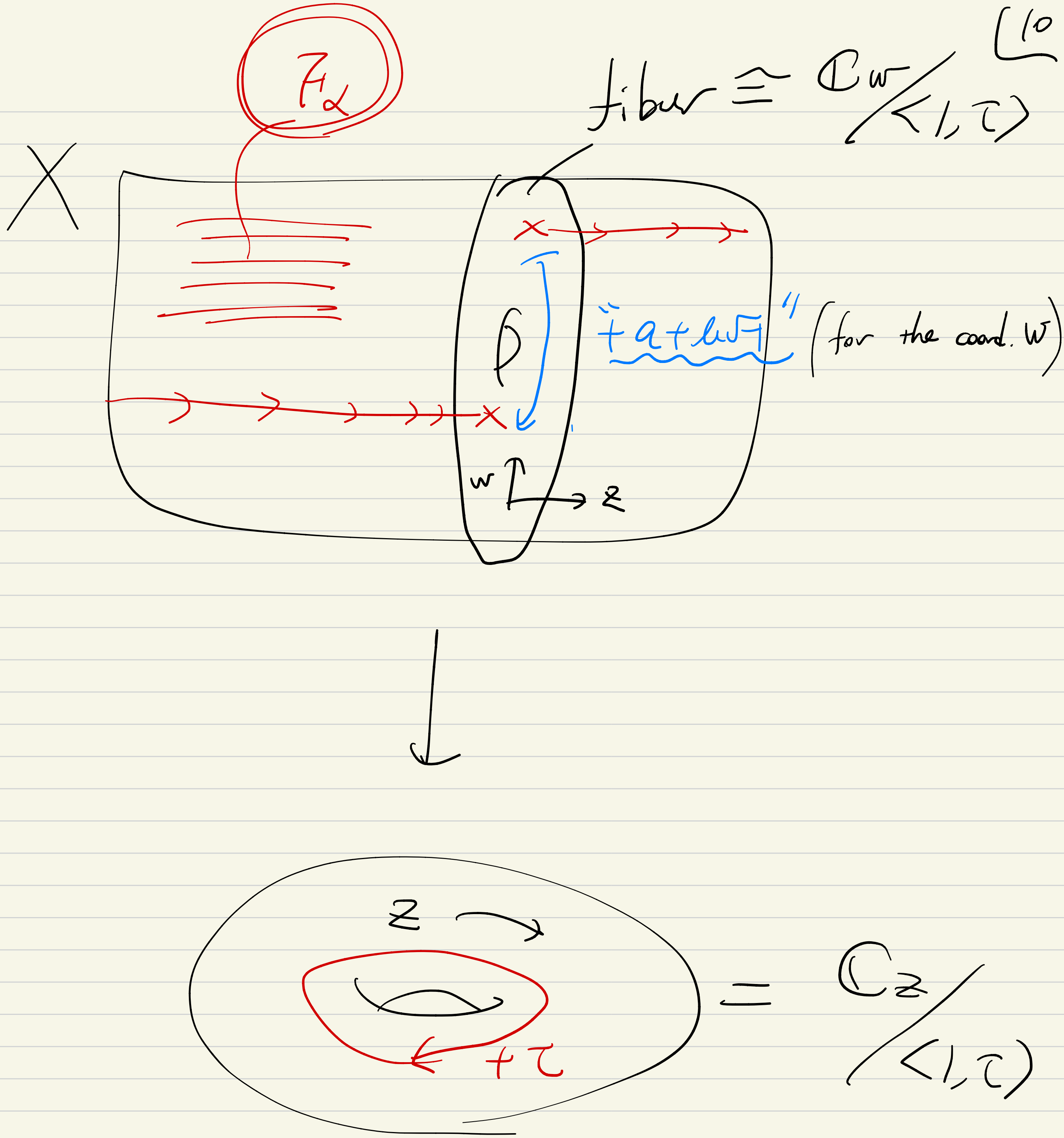
with monodromy $= \begin{cases} +1 \longmapsto \text{id} \\ +\tau \longmapsto \text{“} + a + b\sqrt{-1} \text{”} \end{cases}$

When $a, b \in \mathbb{Q}$, $\#SP(\alpha) > 1$,
Case I, $\mathcal{F}_\alpha =$ “$w = \text{const}$”.

When $\begin{cases} a \text{ or } b \in \mathbb{R} \setminus \mathbb{Q}, \\ \frac{b}{a} \text{ or } \frac{a}{b} \in \mathbb{Q} \end{cases}$, $\#SP(\alpha) > 1$
Case II,
$\mathcal{F}_\alpha =$ “$w = \text{const}$”.

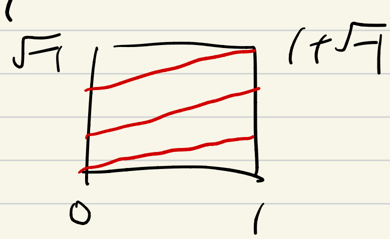
Otherwise, $\#SP(\alpha) = 1$

//



Obs When [a or $b \in \mathbb{R} \setminus \mathbb{Q}$, $\frac{b}{a}$ or $\frac{a}{b} \in \mathbb{Q}$],

$\exists$ many Levi-flat hypersurfaces



invariant by $\mathcal{F}_\alpha$.

e.g. 4 $\tau \in \mathbb{H}$.

$X \longrightarrow \mathbb{C}_z / \langle 1, \tau \rangle$; $\mathbb{P}^1$ - bdl

with coord. w

with monodromy $= \begin{cases} 1 \mapsto \mathrm{id} \\ \tau \mapsto [w \mapsto \lambda \cdot w] \end{cases}$

$(\lambda \in U(1))$

$\rightsquigarrow \#\mathrm{SP}(\alpha) > 1$

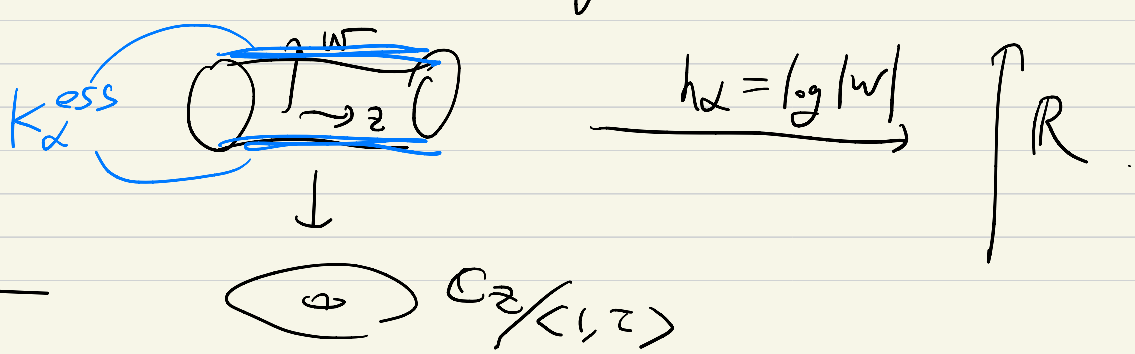
$\mathcal{F}_\alpha =$ "$w = \mathrm{const}$".

① $\lambda^m = 1$ for $\exists m > 0 \Longrightarrow$ Case I.

$\left(\Phi : X \longrightarrow \mathbb{P}^1, \ (z, w) \longmapsto w^m \right)$

② Otherwise $\Longrightarrow$ Case III.

$\left(\begin{aligned} K_\alpha^{ess} &= \{w = 0\} \cup \{w = \infty\} \\ h_\alpha &= \log|w| \end{aligned} \right)$



Rmk

$\exists$ examples of Case III

s.t. $\mathcal{F}_\alpha$ never can be extended to X

(The blow-up of $\mathbb{P}^2$ at (general) nine pts

[Brunella '10]

K3 surfaces constructed by "gluing" ...)

[K–' Uehara]

ArXiv: 1903.01444

§3. Outline of prf

$\boxed{\text{prf of Thm 1, 2}}$

Fix $\theta_0 \in SP(\alpha)$.

$$\# SP(\alpha) > 1 \xrightarrow[\partial\bar\partial\text{-lem}]{} \exists\, \varphi \in PSH^\infty(X, \theta_0) \setminus \mathbb{R}.$$

U: a conn. comp. of $\varphi^{-1}(\{\text{regular values of } \varphi\})$

$\rightsquigarrow \varphi|_U : U \to J \; (:= \varphi(U))$

: proper submersion.

⊙ $\mathcal{F}(\theta_0, \varphi, U) :=$ the Monge–Ampère foliation on U.

i.e.

$T_{\mathcal{F}(\theta_0, \varphi, U)} =$ "the eigenvectors belonging the eigenvalue 0 of $\theta_0 + \sqrt{-1}\,\partial\bar\partial \log(1 + e^{\varphi})$"

$\xrightarrow{\quad}$ $T_{\mathcal{F}(\theta_0, \varphi, U)} = \ker(\partial\varphi)$ (in T_U)

$\alpha \wedge \alpha = 0$,
+
linear algebraic arguments

~~A~~ proof by cases!

(X, α)

$^\exists\theta_0, {}^\exists\psi, {}^\exists r,$
$^\exists g : \{x \in U \mid \psi(x) = r\} \to \mathbb{R}$
$;\ C^\infty$, $\mathcal{F}(\theta_0, \psi, U)$-l.w. const,
and non-constant

Apply Sard's thm to such g.

$\Downarrow$

$\exists$ many cpt leaves of $\mathcal{F}(\theta_0, \psi, U)$

$\boxed{\text{Case I}}$ $\leftarrow$ Kähler geometric arguments.

$^\forall\theta_0, {}^\forall\psi, {}^\forall r,$
$^\forall g : \{x \in U \mid \psi(x) = r\} \to \mathbb{R}$
$;\ C^\infty$, $\mathcal{F}(\theta_0, \psi, U)$-l.w. const
$\Rightarrow g$; const.

by a lin. algebraic argument,

$$\sqrt{-1}\partial\bar\partial\psi = {}^\exists g_\psi \cdot \sqrt{-1}\partial\psi \wedge \bar\partial\psi.$$

$\mathcal{F}(\theta_0, \psi, U)$-l.w. const

$\Downarrow$

$$g_\psi = {}^\exists G \circ \psi.$$

solve $\chi'' = -G \cdot \chi'$

$^\exists h_U : U \to \mathbb{R}$; pluriharmonic.
s.t. $h_U = \chi \circ \psi|_U$ for $^\exists\chi$ w/ $\chi' > 0$

Case III: $\exists$ an analytic prolongation of h_U to $X \setminus K_\alpha^{ess}$

Case II: $\nexists$ such an analytic prolongation

$\boxed{\text{prf of Thm3}}$

① V: a (small) nbhd of Y,

$U := V \setminus Y$

$\rightsquigarrow$ Run the arguments in the prf of Thm 1, 2 to this U.

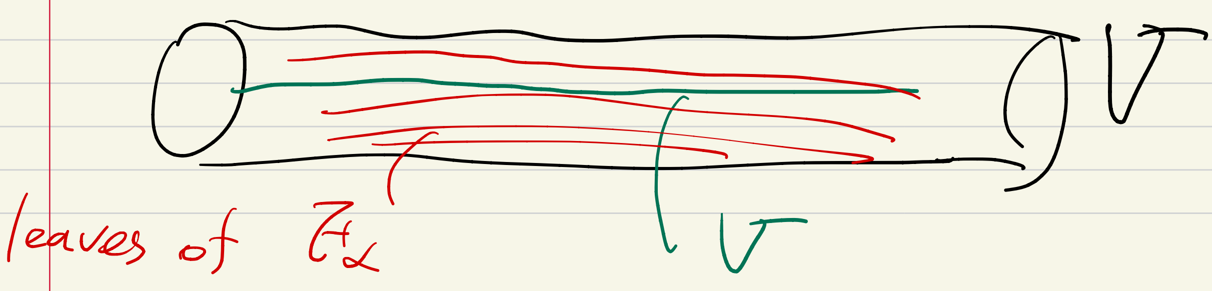
$\rightsquigarrow$ Case I or Case III

$\Downarrow$ $N_{Y/X}^{\otimes m}$: hol'ly triv for $\exists m > 0$

$\Downarrow$ $N_{Y/X}^{\otimes m}$: not triv for $\forall m > 0$

In Case III,

$\mathcal{F}_\alpha$ can be regarded as a (non-sing) hol. foliatn on V (by adding Y as a leaf)



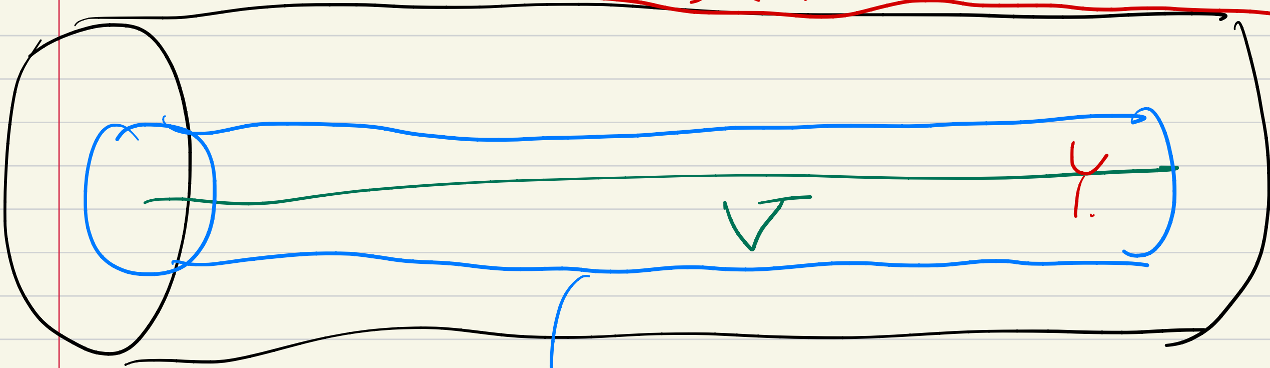
Apply Pérez-Marco's "Hedgehog theory"

to show the $U(1)$-linearizability of the holonomy!

Fix $\gamma \in \pi_1(Y, *)$ (suitably)

$f := \mathrm{Hol}_{\mathcal{F},Y}(\gamma) \in \mathrm{Diff}(\mathbb{C},0)$

$f'(0)$: non-tor $\in U(1)$



$\{\Psi = \text{constant}\}$

: cpt Levi-flat in V.

($^\forall$leaf is also a leaf of $\mathcal{F}_\alpha$)

c.f. [K-Ogawa] ArXiv:1808.10219

Not $U(1)$-linearizable $\xrightarrow{\text{[Pérez-Marco]}}$ $^\exists$leaf of $\mathcal{F}_\alpha$ in V which approaches to Y