Some applications of Ru-Vojta theorem

Min Ru

Univeresity of Houston

Congratulations to Xiaojun for his Montel Chair award

Congratulations to Xiaojun for his Montel Chair award

Geometry vs Arithmetic

Geometry vs Arithmetic

In 1983, Faltings (Fields Medal 1986) proved that *For the Fermat's equation $x^n + y^n = z^n$, when $n \geq 4$, it has only finitely many solutions in k where k is any number field* (the finite extension of $\mathbb{Q}$).

Let $V := \{[x : y : z] \mid x^n + y^n = z^n\} \subset \mathbb{P}^2(\mathbb{C})$.

Let $V := \{[x : y : z] \mid x^n + y^n = z^n\} \subset \mathbb{P}^2(\mathbb{C})$. It is known that $n \geq 4$ if and only if the genus of the Riemann surface V is ≥ 2.

Let $V := \{[x : y : z] \mid x^n + y^n = z^n\} \subset \mathbb{P}^2(\mathbb{C})$. It is known that $n \geq 4$ if and only if the genus of the Riemann surface V is ≥ 2.

Faltings actually proved the following stronger version (known as Mordell's conjecture):

Faltings actually proved the following stronger version (known as Mordell's conjecture): *if V is a Riemann surface defined over k whose geneus is ≥ 2, then there are only finitely many k-points on $V(k)$ for any number field k*.

Faltings actually proved the following stronger version (known as Mordell's conjecture): *if V is a Riemann surface defined over k whose geneus is ≥ 2, then there are only finitely many k-points on $V(k)$ for any number field k*. On geometric side, if a Riemann surface M is of geneus ≥ 2, then it carries a metric of curvature -1.

Faltings actually proved the following stronger version (known as Mordell's conjecture): *if V is a Riemann surface defined over k whose geneus is ≥ 2, then there are only finitely many k-points on $V(k)$ for any number field k*. On geometric side, if a Riemann surface M is of geneus ≥ 2, then it carries a metric of curvature -1. On analytic side, M is Brody hyperbolic, i.e. every holomorphic map $f : \mathbb{C} \to M$ must be constant (Nevanlinna theory).

Faltings actually proved the following stronger version (known as Mordell's conjecture): *if V is a Riemann surface defined over k whose geneus is ≥ 2, then there are only finitely many k-points on $V(k)$ for any number field k*. On geometric side, if a Riemann surface M is of geneus ≥ 2, then it carries a metric of curvature -1. On analytic side, M is Brody hyperbolic, i.e. every holomorphic map $f : \mathbb{C} \to M$ must be constant (Nevanlinna theory). It is believed that geometry determines arithmetic (as well as the analytic properties).

Faltings actually proved the following stronger version (known as Mordell's conjecture): *if V is a Riemann surface defined over k whose geneus is ≥ 2, then there are only finitely many k-points on $V(k)$ for any number field k*. On geometric side, if a Riemann surface M is of geneus ≥ 2, then it carries a metric of curvature -1. On analytic side, M is Brody hyperbolic, i.e. every holomorphic map $f : \mathbb{C} \to M$ must be constant (Nevanlinna theory). It is believed that geometry determines arithmetic (as well as the analytic properties).

Lang's conjecture:

Faltings actually proved the following stronger version (known as Mordell's conjecture): *if V is a Riemann surface defined over k whose geneus is ≥ 2, then there are only finitely many k-points on $V(k)$ for any number field k.* On geometric side, if a Riemann surface M is of geneus ≥ 2, then it carries a metric of curvature -1. On analytic side, M is Brody hyperbolic, i.e. every holomorphic map $f : \mathbb{C} \to M$ must be constant (Nevanlinna theory). It is believed that geometry determines arithmetic (as well as the analytic properties).

Lang's conjecture: Let V be a projective variety defined over a number field, then $\#V(k) < \infty$ if and only if $V(\mathbb{C})$ is hyperbolic.

Faltings actually proved the following stronger version (known as Mordell's conjecture): *if V is a Riemann surface defined over k whose geneus is ≥ 2, then there are only finitely many k-points on $V(k)$ for any number field k*. On geometric side, if a Riemann surface M is of geneus ≥ 2, then it carries a metric of curvature -1. On analytic side, M is Brody hyperbolic, i.e. every holomorphic map $f : \mathbb{C} \to M$ must be constant (Nevanlinna theory). It is believed that geometry determines arithmetic (as well as the analytic properties).

Lang's conjecture: Let V be a projective variety defined over a number field, then $\#V(k) < \infty$ if and only if $V(\mathbb{C})$ is hyperbolic. More specically, an infinite set of k-rational points in $V(k)$ corresponds to a non-consant holomorphic curve $f : \mathbb{C} \to V(\mathbb{C})$.

Faltings actually proved the following stronger version (known as Mordell's conjecture): *if V is a Riemann surface defined over k whose geneus is ≥ 2, then there are only finitely many k-points on $V(k)$ for any number field k.* On geometric side, if a Riemann surface M is of geneus ≥ 2, then it carries a metric of curvature -1. On analytic side, M is Brody hyperbolic, i.e. every holomorphic map $f : \mathbb{C} \to M$ must be constant (Nevanlinna theory). It is believed that geometry determines arithmetic (as well as the analytic properties).

Lang's conjecture: Let V be a projective variety defined over a number field, then $\# V(k) < \infty$ if and only if $V(\mathbb{C})$ is hyperbolic. More specically, an infinite set of k-rational points in $V(k)$ corresponds to a non-consant holomorphic curve $f : \mathbb{C} \to V(\mathbb{C})$. It is the so-called Lang's program.

The Weil function and the First Main Theorem

The Weil function and the First Main Theorem

Complex case:

The Weil function and the First Main Theorem

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor.

The Weil function and the First Main Theorem

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$.

The Weil function and the First Main Theorem

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c[\log \|s_D\|^2] = -D + c_1([D])$.

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c[\log \|s_D\|^2] = -D + c_1([D])$. Let $f : \mathbb{C} \to X$ be a holomorphic.

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c[\log \|s_D\|^2] = -D + c_1([D])$. Let $f : \mathbb{C} \to X$ be a holomorphic. By pulling back through f and applying $\int_1^t \frac{dt}{t} \int_{|z|<t}$, we get $m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1)$ (First Main Theorem),

The Weil function and the First Main Theorem

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c[\log \|s_D\|^2] = -D + c_1([D])$. Let $f: \mathbb{C} \to X$ be a holomorphic. By pulling back through f and applying $\int_1^t \frac{dt}{t} \int_{|z|<t}$, we get $m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1)$ (First Main Theorem), where $\lambda_D(x) = -\log \|s_D(x)\| =-$ log distance from x to D (Weil function for D).

The Weil function and the First Main Theorem

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c[\log \|s_D\|^2] = -D + c_1([D])$. Let $f\colon \mathbb{C} \to X$ be a holomorphic. By pulling back through f and applying $\int_1^t \frac{dt}{t} \int_{|z|<t}$, we get $m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1)$ (First Main Theorem), where $\lambda_D(x) = -\log \|s_D(x)\| =-\log$ distance from x to D (Weil function for D).

The Weil function and the First Main Theorem

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c[\log \|s_D\|^2] = -D + c_1([D])$. Let $f: \mathbb{C} \to X$ be a holomorphic. By pulling back through f and applying $\int_1^t \frac{dt}{t} \int_{|z|<t}$, we get $m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1)$ (First Main Theorem), where $\lambda_D(x) = -\log \|s_D(x)\| = -\log$ distance from x to D (Weil function for D). $m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi}$ (Approximation function).

The Weil function and the First Main Theorem

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c[\log \|s_D\|^2] = -D + c_1([D])$. Let $f: \mathbb{C} \to X$ be a holomorphic. By pulling back through f and applying $\int_1^t \frac{dt}{t} \int_{|z|<t}$, we get $m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1)$ (First Main Theorem), where $\lambda_D(x) = -\log \|s_D(x)\| = -\log$ distance from x to D (Weil function for D). $m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi}$ (Approximation function). $T_{f,L}(r) := \int_1^r \frac{dt}{t} \int_{|z|<t} f^*c_1(L)$ (Height function).

The Weil function and the First Main Theorem

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c[\log \|s_D\|^2] = -D + c_1([D])$. Let $f: \mathbb{C} \to X$ be a holomorphic. By pulling back through f and applying $\int_1^t \frac{dt}{t} \int_{|z|<t}$, we get $m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1)$ (First Main Theorem), where $\lambda_D(x) = -\log \|s_D(x)\| = -\log$ distance from x to D (Weil function for D). $m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi}$ (Approximation function). $T_{f,L}(r) := \int_1^r \frac{dt}{t} \int_{|z|<t} f^*c_1(L)$ (Height function).

The Weil function and the First Main Theorem

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c[\log \|s_D\|^2] = -D + c_1([D])$. Let $f: \mathbb{C} \to X$ be a holomorphic. By pulling back through f and applying $\int_1^t \frac{dt}{t} \int_{|z|<t}$, we get $m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1)$ (First Main Theorem), where $\lambda_D(x) = -\log \|s_D(x)\| = -\log$ distance from x to D (Weil function for D). $m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi}$ (Approximation function). $T_{f,L}(r) := \int_1^r \frac{dt}{t} \int_{|z|<t} f^*c_1(L)$ (Height function). The Second Main Theorem seeks the control the boundary term $m_f(r, D)$ in terms of the height function.

Complex case: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\| \ \|$ be an hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c[\log \|s_D\|^2] = -D + c_1([D])$. Let $f : \mathbb{C} \to X$ be a holomorphic. By pulling back through f and applying $\int_1^t \frac{dt}{t} \int_{|z|<t}$, we get $m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1)$ (First Main Theorem), where $\lambda_D(x) = -\log \|s_D(x)\| =-$ log distance from x to D (Weil function for D). $m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi}$ (Approximation function). $T_{f,L}(r) := \int_1^r \frac{dt}{t} \int_{|z|<t} f^*c_1(L)$ (Height function). The Second Main Theorem seeks the control the boundary term $m_f(r, D)$ in terms of the height function.

Weil function: Let k be a number field, and M_k the set of places.

Weil function: Let k be a number field, and M_k the set of places. Assume X and D are both defined over k.

Weil function: Let k be a number field, and M_k the set of places. Assume X and D are both defined over k. The Weil-function, for $v \in M_k$, $\lambda_{D,v}(x)$ can be defined using the notion of v-adically metrized line sheaf due to S. Zhang [Zhang (1992)]

Weil function: Let k be a number field, and M_k the set of places. Assume X and D are both defined over k. The Weil-function, for $v \in M_k$, $\lambda_{D,v}(x)$ can be defined using the notion of v-adically metrized line sheaf due to S. Zhang [Zhang (1992)] The main properties of Weil functions are additivity, functoriality, Weil function: Let k be a number field, and M_k the set of places. Assume X and D are both defined over k. The Weil-function, for $v \in M_k$, $\lambda_{D,v}(x)$ can be defined using the notion of v-adically metrized line sheaf due to S. Zhang [Zhang (1992)] The main properties of Weil functions are additivity, functoriality, and if D is effective, then, for all $v \in M_k$, $\lambda_{D,v}$ is bounded from below.

Weil function: Let k be a number field, and M_k the set of places. Assume X and D are both defined over k. The Weil-function, for $v \in M_k$, $\lambda_{D,v}(x)$ can be defined using the notion of v-adically metrized line sheaf due to S. Zhang [Zhang (1992)] The main properties of Weil functions are additivity, functoriality, and if D is effective, then, for all $v \in M_k$, $\lambda_{D,v}$ is bounded from below. For a finite set $S \subset M_k$, $m_S(x, D) = \sum_{v \in S} \lambda_{D,v}(x)$ and the height function is

$$h_D(x) = \sum_{v \in M_k} \lambda_{D,v}(x).$$

The Griffiths' and Vojta's conjecture

The Griffiths' and Vojta's conjecture

Griffiths' Conjecture.

Griffiths' Conjecture. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image.

The Griffiths' and Vojta's conjecture

Griffiths' Conjecture. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Let D be a divisor on X with a simple normal crossing and A be an ample divisor on X.

The Griffiths' and Vojta's conjecture

Griffiths' Conjecture. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Let D be a divisor on X with a simple normal crossing and A be an ample divisor on X. Then for any $\epsilon > 0$,

$$m_f(r, D) + T_{f,K_X}(r) \leq_{exc} \epsilon T_{f,A}(r).$$

The Griffiths' and Vojta's conjecture

Griffiths' Conjecture. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Let D be a divisor on X with a simple normal crossing and A be an ample divisor on X. Then for any $\epsilon > 0$,

$$m_f(r, D) + T_{f,K_X}(r) \leq_{exc} \epsilon T_{f,A}(r).$$

Vojta's Conjecture

The Griffiths' and Vojta's conjecture

Griffiths' Conjecture. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Let D be a divisor on X with a simple normal crossing and A be an ample divisor on X. Then for any $\epsilon > 0$,

$$m_f(r, D) + T_{f,K_X}(r) \leq_{exc} \epsilon T_{f,A}(r).$$

Vojta's Conjecture Let k be a number field and S be a finite set of places of k. Let X be a projective variety and D be a simple normal crossing divisor, both defined over k.

The Griffiths' and Vojta's conjecture

Griffiths' Conjecture. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Let D be a divisor on X with a simple normal crossing and A be an ample divisor on X. Then for any $\epsilon > 0$,

$$m_f(r, D) + T_{f,K_X}(r) \leq_{exc} \epsilon T_{f,A}(r).$$

Vojta's Conjecture Let k be a number field and S be a finite set of places of k. Let X be a projective variety and D be a simple normal crossing divisor, both defined over k. Let A be an ample divisor.

The Griffiths' and Vojta's conjecture

Griffiths' Conjecture. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Let D be a divisor on X with a simple normal crossing and A be an ample divisor on X. Then for any $\epsilon > 0$,

$$m_f(r, D) + T_{f,K_X}(r) \leq_{exc} \epsilon T_{f,A}(r).$$

Vojta's Conjecture Let k be a number field and S be a finite set of places of k. Let X be a projective variety and D be a simple normal crossing divisor, both defined over k. Let A be an ample divisor. Then, for $\epsilon > 0$, there exits a Zariski closed variety Z of X such that for all $P \in X(k)$ with $P \notin Z$ we have

$$m_S(P, D) + h_{K_X}(P) < \epsilon h_A(P) + O(1),$$

where $m_S(P, D) = \sum_{v \in S} \lambda_{D,v}(P)$.

Note, from the First Main Theorem, $m_f(r, D) = T_{f,D}(r) + O(1)$ if $f(\mathbb{C})$ omits D.

Note, from the First Main Theorem, $m_f(r, D) = T_{f,D}(r) + O(1)$ if $f(\mathbb{C})$ omits D. Hence, when $\dim X = 1$, Griffiths' conjecture implies that $\mathbb{P}^1$ minus three points is hyperbolic, Note, from the First Main Theorem, $m_f(r, D) = T_{f,D}(r) + O(1)$ if $f(\mathbb{C})$ omits D. Hence, when $\dim X = 1$, Griffiths' conjecture implies that $\mathbb{P}^1$ minus three points is hyperbolic, X minus one points is hyperbolic when the genus g of X is 1, Note, from the First Main Theorem, $m_f(r, D) = T_{f,D}(r) + O(1)$ if $f(\mathbb{C})$ omits D. Hence, when $\dim X = 1$, Griffiths' conjecture implies that $\mathbb{P}^1$ minus three points is hyperbolic, X minus one points is hyperbolic when the genus g of X is 1, and X is hyperbolic if $g \geq 2$.

Note, from the First Main Theorem, $m_f(r, D) = T_{f,D}(r) + O(1)$ if $f(\mathbb{C})$ omits D. Hence, when $\dim X = 1$, Griffiths' conjecture implies that $\mathbb{P}^1$ minus three points is hyperbolic, X minus one points is hyperbolic when the genus g of X is 1, and X is hyperbolic if $g \geq 2$. In the arithmetic case, Vojta's conjecture implies Roth's theorem, Siegel's theorem and Faltings' theorem about the Mordell conjecture.

Note, from the First Main Theorem, $m_f(r, D) = T_{f,D}(r) + O(1)$ if $f(\mathbb{C})$ omits D. Hence, when $\dim X = 1$, Griffiths' conjecture implies that $\mathbb{P}^1$ minus three points is hyperbolic, X minus one points is hyperbolic when the genus g of X is 1, and X is hyperbolic if $g \geq 2$. In the arithmetic case, Vojta's conjecture implies Roth's theorem, Siegel's theorem and Faltings' theorem about the Mordell conjecture.

When $\dim X = 2$, Griffiths' conjecture gives the so-called Green-Griffiths conjecture: If X is a complex surface of general type, then every holomorphic map $f : \mathbb{C} \to X$ must be degenerate.

Note, from the First Main Theorem, $m_f(r, D) = T_{f,D}(r) + O(1)$ if $f(\mathbb{C})$ omits D. Hence, when $\dim X = 1$, Griffiths' conjecture implies that $\mathbb{P}^1$ minus three points is hyperbolic, X minus one points is hyperbolic when the genus g of X is 1, and X is hyperbolic if $g \geq 2$. In the arithmetic case, Vojta's conjecture implies Roth's theorem, Siegel's theorem and Faltings' theorem about the Mordell conjecture.

When $\dim X = 2$, Griffiths' conjecture gives the so-called Green-Griffiths conjecture: If X is a complex surface of general type, then every holomorphic map $f : \mathbb{C} \to X$ must be degenerate. Vojta's conjecture gives Bomberie's conjecture: If X is a projective surface defined over a $\overline{\mathbb{Q}}$, then the set of k-rational points of X is degenerate, for any number field k.

The result of Ru-Vojta

The result of Ru-Vojta

Define $\beta(L, D) = \limsup_{N\to+\infty} \frac{\sum_{m\geq 1} h^0(L^N(-mD))}{Nh^0(L^N)}$.

The result of Ru-Vojta

Define $\beta(L, D) = \limsup_{N\to+\infty} \frac{\sum_{m\geq 1} h^0(L^N(-mD))}{Nh^0(L^N)}$.

Theorem (Ru-Vojta, 2020, Amer. J. Math.)

The result of Ru-Vojta

Define $\beta(L, D) = \limsup_{N \to +\infty} \frac{\sum_{m \geq 1} h^0(L^N(-mD))}{Nh^0(L^N)}$.

Theorem (Ru-Vojta, 2020, Amer. J. Math.) Let X be a complex projective variety, and $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X.

The result of Ru-Vojta

Define $\beta(L, D) = \limsup_{N\to+\infty} \frac{\sum_{m\geq 1} h^0(L^N(-mD))}{Nh^0(L^N)}$.

Theorem (Ru-Vojta, 2020, Amer. J. Math.) Let X be a complex projective variety, and $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let L be a line bundle over X with $\dim H^0(X, L^N) \geq 1$ for N big enough.

The result of Ru-Vojta

Define $\beta(L, D) = \limsup_{N\to+\infty} \frac{\sum_{m\geq 1} h^0(L^N(-mD))}{Nh^0(L^N)}$.

Theorem (Ru-Vojta, 2020, Amer. J. Math.) Let X be a complex projective variety, and $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let L be a line bundle over X with $\dim H^0(X, L^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image.

The result of Ru-Vojta

Define $\beta(L, D) = \limsup_{N\to+\infty} \frac{\sum_{m\geq 1} h^0(L^N(-mD))}{Nh^0(L^N)}$.

Theorem (Ru-Vojta, 2020, Amer. J. Math.) Let X be a complex projective variety, and $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let L be a line bundle over X with $\dim H^0(X, L^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Then, for every $\epsilon > 0$, $\sum_{i=1}^{q} \beta(L, D_j) m_f(r, D_i) \leq_{exc} (1+\epsilon)\, T_{f,L}(r)$.

The result of Ru-Vojta

Define $\beta(L, D) = \limsup_{N\to+\infty} \frac{\sum_{m\geq 1} h^0(L^N(-mD))}{Nh^0(L^N)}$.

Theorem (Ru-Vojta, 2020, Amer. J. Math.) Let X be a complex projective variety, and $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let L be a line bundle over X with $\dim H^0(X, L^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Then, for every $\epsilon > 0$, $\sum_{i=1}^q \beta(L, D_j) m_f(r, D_i) \leq_{exc} (1+\epsilon)\, T_{f,L}(r)$.

Theorem (Ru-Vojta) (arithmetic).

The result of Ru-Vojta

Define $\beta(L, D) = \limsup_{N \to +\infty} \frac{\sum_{m \geq 1} h^0(L^N(-mD))}{N h^0(L^N)}$.

Theorem (Ru-Vojta, 2020, Amer. J. Math.) Let X be a complex projective variety, and $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let L be a line bundle over X with $\dim H^0(X, L^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Then, for every $\epsilon > 0$, $\sum_{i=1}^{q} \beta(L, D_j) m_f(r, D_i) \leq_{exc} (1 + \epsilon)\, T_{f,L}(r)$.

Theorem (Ru-Vojta) (arithmetic). Same assumption, assume that X and $D_1, \ldots, D_q$ are defined over a number filed k. Then, for every $\epsilon > 0$,

$$\sum_{j=1}^{q} \beta(L, D_j) m_S(x, D_j) \leq_{exc} (1 + \epsilon) h_L(x).$$

The result of Ru-Vojta

Define $\beta(L, D) = \limsup_{N\to+\infty} \frac{\sum_{m\geq 1} h^0(L^N(-mD))}{Nh^0(L^N)}$.

Theorem (Ru-Vojta, 2020, Amer. J. Math.) Let X be a complex projective variety, and $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let L be a line bundle over X with $\dim H^0(X, L^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to X$ be a holomorphic mapping with Zariski-dense image. Then, for every $\epsilon > 0$, $\sum_{i=1}^q \beta(L, D_j) m_f(r, D_i) \leq_{exc} (1+\epsilon)\, T_{f,L}(r)$.

Theorem (Ru-Vojta) (arithmetic). Same assumption, assume that X and $D_1, \ldots, D_q$ are defined over a number filed k. Then, for every $\epsilon > 0$,

$$\sum_{j=1}^{q} \beta(L, D_j) m_S(x, D_j) \leq_{exc} (1+\epsilon) h_L(x).$$

Note: by taking $L = -K_X$, if $\beta(-K_X, D) \geq 1$, the Grrffiths' conjecture and Vojta's conjecture hold.

Silverman's "blow-up method" in the study of divisibility and GCD problem

Silverman's "blow-up method" in the study of divisibility and GCD problem

As a special case of Pisot's conjecure,

Silverman's "blow-up method" in the study of divisibility and GCD problem

As a special case of Pisot's conjecure, Given integers $a, b > 1$, if $(a^n - 1)|(b^n - 1)$ for $\forall n >> 0$,

Silverman's "blow-up method" in the study of divisibility and GCD problem

As a special case of Pisot's conjecure, Given integers $a, b > 1$, if $(a^n - 1)|(b^n - 1)$ for $\forall n >> 0$, then a is a power of b (about divisibility)

Silverman's "blow-up method" in the study of divisibility and GCD problem

As a special case of Pisot's conjecure, Given integers $a, b > 1$, if $(a^n - 1)|(b^n - 1)$ for $\forall n >> 0$, then a is a power of b (about divisibility) It was solved by Van den Porteen in 1988.

Silverman's "blow-up method" in the study of divisibility and GCD problem

As a special case of Pisot's conjecure, Given integers $a, b > 1$, if $(a^n - 1)|(b^n - 1)$ for $\forall n >> 0$, then a is a power of b (about divisibility) It was solved by Van den Porteen in 1988. Instead of considering the condition $(a^n - 1)|(b^n - 1)$ (divisibility), one considers $\gcd(a^n - 1, b^n - 1)$ (GCD problem).

Silverman's "blow-up method" in the study of divisibility and GCD problem

As a special case of Pisot's conjecure, Given integers $a, b > 1$, if $(a^n - 1)|(b^n - 1)$ for $\forall n >> 0$, then a is a power of b (about divisibility) It was solved by Van den Porteen in 1988. Instead of considering the condition $(a^n - 1)|(b^n - 1)$ (divisibility), one considers $\gcd(a^n - 1, b^n - 1)$ (GCD problem).

Theorem (Bugeaud, Corvaja, Zannier, 2003).

Silverman's "blow-up method" in the study of divisibility and GCD problem

As a special case of Pisot's conjecure, Given integers $a, b > 1$, if $(a^n - 1)|(b^n - 1)$ for $\forall n >> 0$, then a is a power of b (about divisibility) It was solved by Van den Porteen in 1988. Instead of considering the condition $(a^n - 1)|(b^n - 1)$ (divisibility), one considers $\gcd(a^n - 1, b^n - 1)$ (GCD problem).

Theorem (Bugeaud, Corvaja, Zannier, 2003). Let a, b be multiplicatively independent integers ≥ 2.

Silverman's "blow-up method" in the study of divisibility and GCD problem

As a special case of Pisot's conjecure, Given integers $a, b > 1$, if $(a^n - 1)|(b^n - 1)$ for $\forall n >> 0$, then a is a power of b (about divisibility) It was solved by Van den Porteen in 1988. Instead of considering the condition $(a^n - 1)|(b^n - 1)$ (divisibility), one considers $\gcd(a^n - 1, b^n - 1)$ (GCD problem).

Theorem (Bugeaud, Corvaja, Zannier, 2003). Let a, b be multiplicatively independent integers ≥ 2. Then, for $\epsilon > 0$, there is $N(a, b, \epsilon)$ such that for $n > N$,

$$gcd(a^n - 1, b^n - 1) < 2^{\epsilon n}.$$

Silverman's "blow-up method" in the study of divisibility and GCD problem

As a special case of Pisot's conjecure, Given integers $a, b > 1$, if $(a^n - 1)|(b^n - 1)$ for $\forall n >> 0$, then a is a power of b (about divisibility) It was solved by Van den Porteen in 1988. Instead of considering the condition $(a^n - 1)|(b^n - 1)$ (divisibility), one considers $\gcd(a^n - 1, b^n - 1)$ (GCD problem).

Theorem (Bugeaud, Corvaja, Zannier, 2003). Let a, b be multiplicatively independent integers ≥ 2. Then, for $\epsilon > 0$, there is $N(a, b, \epsilon)$ such that for $n > N$,

$$gcd(a^n - 1, b^n - 1) < 2^{\epsilon n}.$$

Corvaja and Zannier generalized the above result by replacing a^n and b^n with arbitrary elements from a fixed finitely generated subgroup of $\overline{\mathbb{Q}}^*$

Theorem (Corvaja-Zannier, 2004).

Theorem (Corvaja-Zannier, 2004). Let $S = \{\infty, p_1, \ldots, p_t\}$ be a finite set of primes. If $\alpha, \beta \in \mathbb{Z}$ are S-units, then, for $\epsilon > 0$.

Theorem (Corvaja-Zannier, 2004). Let $S = \{\infty, p_1, \ldots, p_t\}$ be a finite set of primes. If $\alpha, \beta \in \mathbb{Z}$ are S-units, then, for $\epsilon > 0$. Then

$$\gcd(\alpha - 1, \beta - 1) \leq \max(|\alpha|, |\beta|)^{\epsilon},$$

except for some obvious families of exceptions together with a finite number of additional exceptions.

Theorem (Corvaja-Zannier, 2004). Let $S = \{\infty, p_1, \ldots, p_t\}$ be a finite set of primes. If $\alpha, \beta \in \mathbb{Z}$ are S-units, then, for $\epsilon > 0$. Then

$$\gcd(\alpha - 1, \beta - 1) \leq \max(|\alpha|, |\beta|)^{\epsilon},$$

except for some obvious families of exceptions together with a finite number of additional exceptions. Here a is a S-unit means that $a = \pm p_1^{a_1} \cdots p_t^{a_t}$.

The analogy result of Corvaja and Zannier in Nevanlinna theory is as follows:

The analogy result of Corvaja and Zannier in Nevanlinna theory is as follows:

Theorem (Noguchi, Winkelmann and Yamanoi, 2002).

The analogy result of Corvaja and Zannier in Nevanlinna theory is as follows:

Theorem (Noguchi, Winkelmann and Yamanoi, 2002). Let f, g be entire functions without zeros (i.e., units of entire functions), and suppose that f, g are multiplicatively independent(i.e., for $\forall (m, n) \in \mathbb{Z} \times \mathbb{Z} \backslash (0, 0)$, we have $f^m \cdot g^n \notin \mathbb{C}$).

The analogy result of Corvaja and Zannier in Nevanlinna theory is as follows:

Theorem (Noguchi, Winkelmann and Yamanoi, 2002). Let f, g be entire functions without zeros (i.e., units of entire functions), and suppose that f, g are multiplicatively independent(i.e., for $\forall (m, n) \in \mathbb{Z} \times \mathbb{Z} \backslash (0, 0)$, we have $f^m \cdot g^n \notin \mathbb{C}$). Then, for every $\epsilon > 0$,

$$N(f-1, g-1, r) \leq_{exc} \epsilon max\{T_f(r), T_g(r)\},$$

The analogy result of Corvaja and Zannier in Nevanlinna theory is as follows:

Theorem (Noguchi, Winkelmann and Yamanoi, 2002). Let f, g be entire functions without zeros (i.e., units of entire functions), and suppose that f, g are multiplicatively independent(i.e., for $\forall (m, n) \in \mathbb{Z} \times \mathbb{Z} \backslash (0, 0)$, we have $f^m \cdot g^n \notin \mathbb{C}$). Then, for every $\epsilon > 0$,

$$N(f-1, g-1, r) \leq_{exc} \epsilon max\{T_f(r), T_g(r)\},$$

where $n(f, g, r) := \sum_{|z| \leq r} \min\{ord_z^+(f), ord_z^+(g)\}$.

The analogy result of Corvaja and Zannier in Nevanlinna theory is as follows:

Theorem (Noguchi, Winkelmann and Yamanoi, 2002). Let f, g be entire functions without zeros (i.e., units of entire functions), and suppose that f, g are multiplicatively independent(i.e., for $\forall (m, n) \in \mathbb{Z} \times \mathbb{Z} \backslash (0, 0)$, we have $f^m \cdot g^n \notin \mathbb{C}$). Then, for every $\epsilon > 0$,

$$N(f-1, g-1, r) \leq_{exc} \epsilon max\{T_f(r), T_g(r)\},$$

where $n(f, g, r) := \sum_{|z| \leq r} \min\{ord_z^+(f), ord_z^+(g)\}$.

Their full -statement is as follows.

The analogy result of Corvaja and Zannier in Nevanlinna theory is as follows:

Theorem (Noguchi, Winkelmann and Yamanoi, 2002). Let f, g be entire functions without zeros (i.e., units of entire functions), and suppose that f, g are multiplicatively independent(i.e., for $\forall (m, n) \in \mathbb{Z} \times \mathbb{Z} \backslash (0, 0)$, we have $f^m \cdot g^n \notin \mathbb{C}$). Then, for every $\epsilon > 0$,

$$N(f-1, g-1, r) \leq_{exc} \epsilon max\{T_f(r), T_g(r)\},$$

where $n(f, g, r) := \sum_{|z| \leq r} \min\{ord_z^+(f), ord_z^+(g)\}$.

Their full -statement is as follows.

Let $f : \mathbb{C} \to A$ be a holomorphic map to a semi-abelian variety A with Zariski-dense image.

The analogy result of Corvaja and Zannier in Nevanlinna theory is as follows:

Theorem (Noguchi, Winkelmann and Yamanoi, 2002). Let f, g be entire functions without zeros (i.e., units of entire functions), and suppose that f, g are multiplicatively independent(i.e., for $\forall (m, n) \in \mathbb{Z} \times \mathbb{Z} \backslash (0, 0)$, we have $f^m \cdot g^n \notin \mathbb{C}$). Then, for every $\epsilon > 0$,

$$N(f-1, g-1, r) \leq_{exc} \epsilon max\{T_f(r), T_g(r)\},$$

where $n(f, g, r) := \sum_{|z|\leq r} \min\{ord_z^+(f), ord_z^+(g)\}$.

Their full -statement is as follows.

Let $f : \mathbb{C} \to A$ be a holomorphic map to a semi-abelian variety A with Zariski-dense image. Let Y be a closed subscheme of A with $\operatorname{codim} Y \geq 2$.

The analogy result of Corvaja and Zannier in Nevanlinna theory is as follows:

Theorem (Noguchi, Winkelmann and Yamanoi, 2002). Let f, g be entire functions without zeros (i.e., units of entire functions), and suppose that f, g are multiplicatively independent(i.e., for $\forall (m, n) \in \mathbb{Z} \times \mathbb{Z} \backslash (0, 0)$, we have $f^m \cdot g^n \notin \mathbb{C}$). Then, for every $\epsilon > 0$,

$$N(f-1, g-1, r) \leq_{exc} \epsilon max\{T_f(r), T_g(r)\},$$

where $n(f, g, r) := \sum_{|z| \leq r} \min\{ord_z^+(f), ord_z^+(g)\}$.

Their full -statement is as follows.

Let $f : \mathbb{C} \to A$ be a holomorphic map to a semi-abelian variety A with Zariski-dense image. Let Y be a closed subscheme of A with codim$Y \geq 2$. Then, for any $\epsilon > 0$, we have

$$N_f(Y, r) \leq_{exc} \epsilon T_f(r).$$

The analogy result of Corvaja and Zannier in Nevanlinna theory is as follows:

Theorem (Noguchi, Winkelmann and Yamanoi, 2002). Let f, g be entire functions without zeros (i.e., units of entire functions), and suppose that f, g are multiplicatively independent(i.e., for $\forall(m, n) \in \mathbb{Z} \times \mathbb{Z}\backslash(0, 0)$, we have $f^m \cdot g^n \notin \mathbb{C}$). Then, for every $\epsilon > 0$,

$$N(f-1, g-1, r) \leq_{exc} \epsilon max\{T_f(r), T_g(r)\},$$

where $n(f, g, r) := \sum_{|z|\leq r} \min\{ord_z^+(f), ord_z^+(g)\}$.

Their full -statement is as follows.

Let $f : \mathbb{C} \to A$ be a holomorphic map to a semi-abelian variety A with Zariski-dense image. Let Y be a closed subscheme of A with codim$Y \geq 2$. Then, for any $\epsilon > 0$, we have

$$N_f(Y, r) \leq_{exc} \epsilon T_f(r).$$

Note: The GCD problem eventually gets to to estimate $N_f(Y, r)$ (or $T_{f,Y}(r)$ or $h_Y(x)$ in the arithmetic case) for closed subscheme Y with codim$Y \geq 2$.

J. Silverman in 2005 proposed a method of applying Vojta's Conjecture (Griffiths' conjecture) to varieties blown up along smooth subschemes which leads to a very general statement about greatest common divisors that encompasses many known results and previous conjectures.

J. Silverman in 2005 proposed a method of applying Vojta's Conjecture (Griffiths' conjecture) to varieties blown up along smooth subschemes which leads to a very general statement about greatest common divisors that encompasses many known results and previous conjectures. Silverman's blow-up method: assuming Griffiths conjecture holds (in arithmetic case, assuming Vojta's conjecture).

J. Silverman in 2005 proposed a method of applying Vojta's Conjecture (Griffiths' conjecture) to varieties blown up along smooth subschemes which leads to a very general statement about greatest common divisors that encompasses many known results and previous conjectures. Silverman's blow-up method: assuming Griffiths conjecture holds (in arithmetic case, assuming Vojta's conjecture). Let f_1, f_2 be two mero. functions, alg. indep.

J. Silverman in 2005 proposed a method of applying Vojta's Conjecture (Griffiths' conjecture) to varieties blown up along smooth subschemes which leads to a very general statement about greatest common divisors that encompasses many known results and previous conjectures. Silverman's blow-up method: assuming Griffiths conjecture holds (in arithmetic case, assuming Vojta's conjecture). Let f_1, f_2 be two mero. functions, alg. indep. Let $f := (f_1, f_2) : \mathbb{C} \to X := \mathbb{P}^1 \times \mathbb{P}^1$.

J. Silverman in 2005 proposed a method of applying Vojta's Conjecture (Griffiths' conjecture) to varieties blown up along smooth subschemes which leads to a very general statement about greatest common divisors that encompasses many known results and previous conjectures. Silverman's blow-up method: assuming Griffiths conjecture holds (in arithmetic case, assuming Vojta's conjecture). Let f_1, f_2 be two mero. functions, alg. indep. Let $f := (f_1, f_2) : \mathbb{C} \to X := \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : \tilde{X} \to X$ be the blowup at $(1, 1) \in X$ and E be the exceptional divisor.

J. Silverman in 2005 proposed a method of applying Vojta's Conjecture (Griffiths' conjecture) to varieties blown up along smooth subschemes which leads to a very general statement about greatest common divisors that encompasses many known results and previous conjectures. Silverman's blow-up method: assuming Griffiths conjecture holds (in arithmetic case, assuming Vojta's conjecture). Let f_1, f_2 be two mero. functions, alg. indep. Let $f := (f_1, f_2) : \mathbb{C} \to X := \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : \tilde{X} \to X$ be the blowup at $(1, 1) \in X$ and E be the exceptional divisor. Applying Griffiths' conjecture with $D := -\pi^* K_X$ on $\tilde{X}$,

J. Silverman in 2005 proposed a method of applying Vojta's Conjecture (Griffiths' conjecture) to varieties blown up along smooth subschemes which leads to a very general statement about greatest common divisors that encompasses many known results and previous conjectures. Silverman's blow-up method: assuming Griffiths conjecture holds (in arithmetic case, assuming Vojta's conjecture). Let f_1, f_2 be two mero. functions, alg. indep. Let $f := (f_1, f_2) : \mathbb{C} \to X := \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : \tilde{X} \to X$ be the blowup at $(1, 1) \in X$ and E be the exceptional divisor. Applying Griffiths' conjecture with $D := -\pi^* K_X$ on $\tilde{X}$, we get $m_{\tilde{f}}(r, -\pi^* K_X) + T_{\tilde{f}, K_{\tilde{X}}}(r) \leq_{exc} \epsilon T_{\tilde{f}, -K_{\tilde{X}}}(r)$.

J. Silverman in 2005 proposed a method of applying Vojta's Conjecture (Griffiths' conjecture) to varieties blown up along smooth subschemes which leads to a very general statement about greatest common divisors that encompasses many known results and previous conjectures. Silverman's blow-up method: assuming Griffiths conjecture holds (in arithmetic case, assuming Vojta's conjecture). Let f_1, f_2 be two mero. functions, alg. indep. Let $f := (f_1, f_2) : \mathbb{C} \to X := \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : \tilde{X} \to X$ be the blowup at $(1,1) \in X$ and E be the exceptional divisor. Applying Griffiths' conjecture with $D := -\pi^* K_X$ on $\tilde{X}$, we get $m_{\tilde{f}}(r, -\pi^* K_X) + T_{\tilde{f}, K_{\tilde{X}}}(r) \leq_{exc} \epsilon T_{\tilde{f}, -K_{\tilde{X}}}(r)$. This implies that, noticing $K_{\tilde{X}} = \pi^* K_X + E$,

$$-N_{\tilde{f}}(r, -\pi^* K_X) + (1+\epsilon) T_{\tilde{f}, E}(r) \leq_{exc} \epsilon T_{\tilde{f}, -\pi^* K_X}(r).$$

J. Silverman in 2005 proposed a method of applying Vojta's Conjecture (Griffiths' conjecture) to varieties blown up along smooth subschemes which leads to a very general statement about greatest common divisors that encompasses many known results and previous conjectures. Silverman's blow-up method: assuming Griffiths conjecture holds (in arithmetic case, assuming Vojta's conjecture). Let f_1, f_2 be two mero. functions, alg. indep. Let $f := (f_1, f_2) : \mathbb{C} \to X := \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : \tilde{X} \to X$ be the blowup at $(1, 1) \in X$ and E be the exceptional divisor. Applying Griffiths' conjecture with $D := -\pi^* K_X$ on $\tilde{X}$, we get $m_{\tilde{f}}(r, -\pi^* K_X) + T_{\tilde{f}, K_{\tilde{X}}}(r) \leq_{exc} \epsilon T_{\tilde{f}, -K_{\tilde{X}}}(r)$. This implies that, noticing $K_{\tilde{X}} = \pi^* K_X + E$,

$$-N_{\tilde{f}}(r, -\pi^* K_X) + (1 + \epsilon) T_{\tilde{f}, E}(r) \leq_{exc} \epsilon T_{\tilde{f}, -\pi^* K_X}(r).$$

Since $K_X = -\{0\} \times \mathbb{P}^1 - \{\infty\} \times \mathbb{P}^1 - \mathbb{P}^1 \times \{0\} - \mathbb{P}^1 \times \{\infty\}$,

J. Silverman in 2005 proposed a method of applying Vojta's Conjecture (Griffiths' conjecture) to varieties blown up along smooth subschemes which leads to a very general statement about greatest common divisors that encompasses many known results and previous conjectures. Silverman's blow-up method: assuming Griffiths conjecture holds (in arithmetic case, assuming Vojta's conjecture). Let f_1, f_2 be two mero. functions, alg. indep. Let $f := (f_1, f_2) : \mathbb{C} \to X := \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : \tilde{X} \to X$ be the blowup at $(1,1) \in X$ and E be the exceptional divisor. Applying Griffiths' conjecture with $D := -\pi^* K_X$ on $\tilde{X}$, we get $m_{\tilde{f}}(r, -\pi^* K_X) + T_{\tilde{f}, K_{\tilde{X}}}(r) \leq_{exc} \epsilon T_{\tilde{f}, -K_{\tilde{X}}}(r)$. This implies that, noticing $K_{\tilde{X}} = \pi^* K_X + E$,

$$-N_{\tilde{f}}(r, -\pi^* K_X) + (1+\epsilon) T_{\tilde{f}, E}(r) \leq_{exc} \epsilon T_{\tilde{f}, -\pi^* K_X}(r).$$

Since $K_X = -\{0\} \times \mathbb{P}^1 - \{\infty\} \times \mathbb{P}^1 - \mathbb{P}^1 \times \{0\} - \mathbb{P}^1 \times \{\infty\}$, we get

$$(1+\epsilon) N(f_1 - 1, f_2 - 1, r) \leq_{exc} \sum_{i=1}^{2} (N_{f_i}(r, 0) + N_{f_i}(r, \infty))$$

Following Silverma's approach, Ji Guo and J. Wang used the result of Ru-Vojta to replace Girffiths' conjecture to obtain:
Theorem (Ji Guo and J. Wang, Trans. AMS, 2019).

Following Silverma's approach, Ji Guo and J. Wang used the result of Ru-Vojta to replace Girffiths' conjecture to obtain:

Theorem (Ji Guo and J. Wang, Trans. AMS, 2019). Let f_1 and f_2 be algebraically independent meromorphic functions. For any $\epsilon > 0$, there exists n_0 such that for all $n \geq n_0$,

$$N(f_1^n - 1, f_2^n - 1, r) \leq_{exc} \left(\frac{1}{2} + \epsilon\right) \max\{T_{f_1^n}(r), T_{f_2^n}(r)\}.$$

Following Silverma's approach, Ji Guo and J. Wang used the result of Ru-Vojta to replace Girffiths' conjecture to obtain:

Theorem (Ji Guo and J. Wang, Trans. AMS, 2019). Let f_1 and f_2 be algebraically independent meromorphic functions. For any $\epsilon > 0$, there exists n_0 such that for all $n \geq n_0$,

$$N(f_1^n - 1, f_2^n - 1, r) \leq_{exc} \left(\frac{1}{2} + \epsilon\right) \max\{T_{f_1^n}(r), T_{f_2^n}(r)\}.$$

Corollary.

Following Silverma's approach, Ji Guo and J. Wang used the result of Ru-Vojta to replace Girffiths' conjecture to obtain:

Theorem (Ji Guo and J. Wang, Trans. AMS, 2019). Let f_1 and f_2 be algebraically independent meromorphic functions. For any $\epsilon > 0$, there exists n_0 such that for all $n \geq n_0$,

$$N(f_1^n - 1, f_2^n - 1, r) \leq_{exc} \left(\frac{1}{2} + \epsilon\right) \max\{T_{f_1^n}(r), T_{f_2^n}(r)\}.$$

Corollary. Let f, g be transcendental entire functions with $T_f(r) \sim T_g(r)$.

Following Silverma's approach, Ji Guo and J. Wang used the result of Ru-Vojta to replace Girffiths' conjecture to obtain:

Theorem (Ji Guo and J. Wang, Trans. AMS, 2019). Let f_1 and f_2 be algebraically independent meromorphic functions. For any $\epsilon > 0$, there exists n_0 such that for all $n \geq n_0$,

$$N(f_1^n - 1, f_2^n - 1, r) \leq_{exc} \left(\frac{1}{2} + \epsilon\right) \max\{T_{f_1^n}(r), T_{f_2^n}(r)\}.$$

Corollary. Let f, g be transcendental entire functions with $T_f(r) \sim T_g(r)$. Suppose that $(f^n - 1)|(g^n - 1)$ for infinitely many integers n. Then f, g are multiplicatively dependent.

In the case of Guo-Wang, they proved the following: Let p be a point in $\mathbb{P}^1$ and let $D_1 = \{p\} \times \mathbb{P}^1$ and $D_2 = \mathbb{P}^1 \times \{p\}$. Then $\beta(-K_{\tilde{X}}, \pi^* D_i) \geq \frac{7}{8}$ for $i = 1, 2$.

In the case of Guo-Wang, they proved the following: Let p be a point in $\mathbb{P}^1$ and let $D_1 = \{p\} \times \mathbb{P}^1$ and $D_2 = \mathbb{P}^1 \times \{p\}$. Then $\beta(-K_{\tilde{X}}, \pi^* D_i) \geq \frac{7}{8}$ for $i = 1, 2$. Applying Ru-Vojta, by taking $L := -K_{\tilde{X}}$ and $D := -\pi^* K_X$, we get

$$T_{\tilde{f}}(r, D) - \left(\frac{8}{7} + \epsilon\right) T_{\tilde{f}, -K_{\tilde{X}}} \leq_{exc} N_{\tilde{f}}^{(M)}(r, D).$$

In the case of Guo-Wang, they proved the following: Let p be a point in $\mathbb{P}^1$ and let $D_1 = \{p\} \times \mathbb{P}^1$ and $D_2 = \mathbb{P}^1 \times \{p\}$. Then $\beta(-K_{\tilde{X}}, \pi^* D_i) \geq \frac{7}{8}$ for $i = 1, 2$. Applying Ru-Vojta, by taking $L := -K_{\tilde{X}}$ and $D := -\pi^* K_X$, we get

$$T_{\tilde{f}}(r, D) - \left(\frac{8}{7} + \epsilon\right) T_{\tilde{f}, -K_{\tilde{X}}} \leq_{exc} N_{\tilde{f}}^{(M)}(r, D).$$

Comparing with Griffiths' conjecture:

$$T_{\tilde{f}}(r, D) - (1 + \epsilon) T_{\tilde{f}, -K_{\tilde{X}}}(r) \leq_{exc} N_{\tilde{f}}(r, D).$$

In the case of Guo-Wang, they proved the following: Let p be a point in $\mathbb{P}^1$ and let $D_1 = \{p\} \times \mathbb{P}^1$ and $D_2 = \mathbb{P}^1 \times \{p\}$. Then $\beta(-K_{\tilde{X}}, \pi^* D_i) \geq \frac{7}{8}$ for $i = 1, 2$. Applying Ru-Vojta, by taking $L := -K_{\tilde{X}}$ and $D := -\pi^* K_X$, we get

$$T_{\tilde{f}}(r, D) - \left(\frac{8}{7} + \epsilon\right) T_{\tilde{f}, -K_{\tilde{X}}} \leq_{exc} N_{\tilde{f}}^{(M)}(r, D).$$

Comparing with Griffiths' conjecture:

$$T_{\tilde{f}}(r, D) - (1 + \epsilon) T_{\tilde{f}, -K_{\tilde{X}}}(r) \leq_{exc} N_{\tilde{f}}(r, D).$$

In the case of Guo-Wang, they proved the following: Let p be a point in $\mathbb{P}^1$ and let $D_1 = \{p\} \times \mathbb{P}^1$ and $D_2 = \mathbb{P}^1 \times \{p\}$. Then $\beta(-K_{\tilde{X}}, \pi^* D_i) \geq \frac{7}{8}$ for $i = 1, 2$. Applying Ru-Vojta, by taking $L := -K_{\tilde{X}}$ and $D := -\pi^* K_X$, we get

$$T_{\tilde{f}}(r, D) - \left(\frac{8}{7} + \epsilon\right) T_{\tilde{f}, -K_{\tilde{X}}} \leq_{exc} N_{\tilde{f}}^{(M)}(r, D).$$

Comparing with Griffiths' conjecture:

$$T_{\tilde{f}}(r, D) - (1 + \epsilon) T_{\tilde{f}, -K_{\tilde{X}}}(r) \leq_{exc} N_{\tilde{f}}(r, D).$$

Work of Wang and Yasufuku

Work of Wang and Yasufuku Very recently, Julie Wang and Yu Yasufuku again used Ru-Vojta theorem to replace Griffiths conjecture.

Work of Wang and Yasufuku Very recently, Julie Wang and Yu Yasufuku again used Ru-Vojta theorem to replace Griffiths conjecture. More importantly, the approach is very simple and recovers most previous important results in GCD problem, including Levin's 2019 Invention Math paper result.

Work of Wang and Yasufuku Very recently, Julie Wang and Yu Yasufuku again used Ru-Vojta theorem to replace Griffiths conjecture. More importantly, the approach is very simple and recovers most previous important results in GCD problem, including Levin's 2019 Invention Math paper result.
Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$.

Work of Wang and Yasufuku   Very recently, Julie Wang and Yu Yasufuku again used Ru-Vojta theorem to replace Griffiths conjecture.   More importantly, the approach is very simple and recovers most previous important results in GCD problem, including Levin's 2019 Invention Math paper result.

Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$.   Let Y be a closed subscheme of X.

Work of Wang and Yasufuku Very recently, Julie Wang and Yu Yasufuku again used Ru-Vojta theorem to replace Griffiths conjecture. More importantly, the approach is very simple and recovers most previous important results in GCD problem, including Levin's 2019 Invention Math paper result.

Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$. Let Y be a closed subscheme of X. Let $\pi : \tilde{X} \to X$ be the blowup along Y and E be the exceptional divisor.

Work of Wang and Yasufuku Very recently, Julie Wang and Yu Yasufuku again used Ru-Vojta theorem to replace Griffiths conjecture. More importantly, the approach is very simple and recovers most previous important results in GCD problem, including Levin's 2019 Invention Math paper result.

Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$. Let Y be a closed subscheme of X. Let $\pi : \tilde{X} \to X$ be the blowup along Y and E be the exceptional divisor. Wang-Yasufuku used, instead of taking $L = -K_{\tilde{X}}$, $L := m\pi^* A - E$,

Work of Wang and Yasufuku   Very recently, Julie Wang and Yu Yasufuku again used Ru-Vojta theorem to replace Griffiths conjecture.   More importantly, the approach is very simple and recovers most previous important results in GCD problem, including Levin's 2019 Invention Math paper result.

Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$.   Let Y be a closed subscheme of X. Let $\pi : \tilde{X} \to X$ be the blowup along Y and E be the exceptional divisor.   Wang-Yasufuku used, instead of taking $L = -K_{\tilde{X}}$, $L := m\pi^* A - E$,   and showed that, for m big enough,

$$\beta^{-1}(L, \pi^* D_i) \leq \frac{1}{m}\left(1 + \frac{1}{m\sqrt{m}}\right).$$

Work of Wang and Yasufuku Very recently, Julie Wang and Yu Yasufuku again used Ru-Vojta theorem to replace Griffiths conjecture. More importantly, the approach is very simple and recovers most previous important results in GCD problem, including Levin's 2019 Invention Math paper result.

Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$. Let Y be a closed subscheme of X. Let $\pi : \tilde{X} \to X$ be the blowup along Y and E be the exceptional divisor. Wang-Yasufuku used, instead of taking $L = -K_{\tilde{X}}$, $L := m\pi^* A - E$, and showed that, for m big enough,

$$\beta^{-1}(L, \pi^* D_i) \leq \frac{1}{m}\left(1 + \frac{1}{m\sqrt{m}}\right).$$

Theorem (Wang-Yasufuku), Algebra & Number Theory, 2021.

Theorem (Wang-Yasufuku), Algebra & Number Theory, 2021. Let X be a smooth algebraic variety. Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$.

Theorem (Wang-Yasufuku), Algebra & Number Theory, 2021. Let X be a smooth algebraic variety. Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$. Let Y be a closed subscheme of X which does not contain any point of the set $\cup_{i=1}^{n+1}(\cap_{1\leq j\neq i\leq n+1} D_j)$.

Theorem (Wang-Yasufuku), Algebra & Number Theory, 2021. Let X be a smooth algebraic variety. Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$. Let Y be a closed subscheme of X which does not contain any point of the set $\cup_{i=1}^{n+1}(\cap_{1\leq j\neq i\leq n+1} D_j)$. Let $f : \mathbb{C} \to X \backslash (\cup_{j=1}^{n+1} D_j)$ be a holomorphic mapping with Zariski-dense image.

Theorem (Wang-Yasufuku), Algebra & Number Theory, 2021. Let X be a smooth algebraic variety. Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$. Let Y be a closed subscheme of X which does not contain any point of the set $\cup_{i=1}^{n+1}(\cap_{1\leq j\neq i\leq n+1} D_j)$. Let $f : \mathbb{C} \to X \backslash (\cup_{j=1}^{n+1} D_j)$ be a holomorphic mapping with Zariski-dense image. Then, for a given $\epsilon > 0$. We have

$$T_{f,Y}(r) \leq_{exc} \epsilon T_{f,A}(r).$$

Theorem (Wang-Yasufuku), Algebra & Number Theory, 2021. Let X be a smooth algebraic variety. Let $D_1, \dots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$. Let Y be a closed subscheme of X which does not contain any point of the set $\cup_{i=1}^{n+1}(\cap_{1\leq j\neq i\leq n+1} D_j)$. Let $f : \mathbb{C} \to X \backslash (\cup_{j=1}^{n+1} D_j)$ be a holomorphic mapping with Zariski-dense image. Then, for a given $\epsilon > 0$. We have

$$T_{f,Y}(r) \leq_{exc} \epsilon T_{f,A}(r).$$

Outline of proof. Let $\pi : \tilde{X} \to X$ be the blowup along Y and E be the exceptional divisor.

Theorem (Wang-Yasufuku), Algebra & Number Theory, 2021. Let X be a smooth algebraic variety. Let $D_1, \ldots, D_{n+1}$ be effective divisors on X in general position, and assume that $D_i \equiv d_i A$. Let Y be a closed subscheme of X which does not contain any point of the set $\cup_{i=1}^{n+1}(\cap_{1\le j\neq i\le n+1} D_j)$. Let $f : \mathbb{C} \to X \backslash (\cup_{j=1}^{n+1} D_j)$ be a holomorphic mapping with Zariski-dense image. Then, for a given $\epsilon > 0$. We have

$$T_{f,Y}(r) \le_{exc} \epsilon T_{f,A}(r).$$

Outline of proof. Let $\pi : \tilde{X} \to X$ be the blowup along Y and E be the exceptional divisor. Applying Ru-Vojta's theorem with $\epsilon = m^{-\frac{5}{2}}$, we have

$$\sum_{j=1}^{n+1} m_{\tilde{f}}(r, \tilde{D}_i) \le_{exc} \frac{1}{m}\left(1 + \frac{1}{m\sqrt{m}} + \epsilon\right) T_{\tilde{f}, m\pi^*A-E}(r)$$

$$= \left(1 + \frac{2}{m\sqrt{m}}\right) T_{\tilde{f},A}(r) - \left(\frac{1}{m} + \frac{2}{m^2\sqrt{m}}\right) T_{\tilde{f},E}(r).$$

Divisibilities and integral points (and hyperbolicity

Work of E. Rousseau, A. Turchet and Julie Tzu-Yueh Wang

Work of E. Rousseau, A. Turchet and Julie Tzu-Yueh Wang For their abstract:

Work of E. Rousseau, A. Turchet and Julie Tzu-Yueh Wang For their abstract: We prove several statements about arithmetic hyperbolicity of certain blow-up varieties. As a corollary we obtain multiple examples of simply connected quasi-projective varieties that are pseudo arithmetically hyperbolic. This generalizes results of Corvaja and Zannier obtained in dimension 2 to arbitrary dimension. The key input is an application of the Ru-Vojta's strategy.

Theorem. Let $n \geq 2$, $F_1, ..., F_r, G \in \mathbb{C}[X_1, ..., X_n]$ be polynomials in general position (i.e. the associated hypersurfaces are in general position) with $\deg(F_i) \geq \deg(G)$ for $i = 1, \ldots, r$.

Theorem. Let $n \geq 2$, $F_1, ..., F_r, G \in \mathbb{C}[X_1, ..., X_n]$ be polynomials in general position (i.e. the associated hypersurfaces are in general position) with $\deg(F_i) \geq \deg(G)$ for $i = 1, \ldots, r$. Let $h_1, ..., h_n$ be holomorphic functions on $\mathbb{C}$ such that one of the following holds

(i) $r \geq 2n$ and $\frac{G(h_1,...,h_n)}{F_i(h_1,...,h_n)}$ is holomorphic, for $i = 1, ..., r$; or

(ii) $r \geq n+1$ and $\frac{G(h_1,...,h_n)}{\prod_{i=1}^r F_i(h_1,...,h_n)}$ is holomorphic.

Theorem. Let $n \geq 2$, $F_1, ..., F_r, G \in \mathbb{C}[X_1, ..., X_n]$ be polynomials in general position (i.e. the associated hypersurfaces are in general position) with $\deg(F_i) \geq \deg(G)$ for $i = 1, \ldots, r$. Let $h_1, ..., h_n$ be holomorphic functions on $\mathbb{C}$ such that one of the following holds

(i) $r \geq 2n$ and $\frac{G(h_1,...,h_n)}{F_i(h_1,...,h_n)}$ is holomorphic, for $i = 1, ..., r$; or

(ii) $r \geq n+1$ and $\frac{G(h_1,...,h_n)}{\prod_{i=1}^r F_i(h_1,...,h_n)}$ is holomorphic.
Then $h_1, \ldots, h_n$ are algebraically dependent.

Theorem. Let $n \geq 2$, $F_1, ..., F_r, G \in \mathbb{C}[X_1, ..., X_n]$ be polynomials in general position (i.e. the associated hypersurfaces are in general position) with $\deg(F_i) \geq \deg(G)$ for $i = 1, \ldots, r$. Let $h_1, ..., h_n$ be holomorphic functions on $\mathbb{C}$ such that one of the following holds

(i) $r \geq 2n$ and $\frac{G(h_1,...,h_n)}{F_i(h_1,...,h_n)}$ is holomorphic, for $i = 1, ..., r$; or

(ii) $r \geq n+1$ and $\frac{G(h_1,...,h_n)}{\prod_{i=1}^r F_i(h_1,...,h_n)}$ is holomorphic.

Then $h_1, \ldots, h_n$ are algebraically dependent.

This can be seen as a generalization of Borel Lemma stating that nowhere vanishing entire functions $h_1, ..., h_{n+1}$ satisfying the identity $h_1 + \cdots + h_{n+1} = 1$ are linearly dependent.

Theorem. Let $n \geq 2$, $F_1, ..., F_r, G \in \mathbb{C}[X_1, ..., X_n]$ be polynomials in general position (i.e. the associated hypersurfaces are in general position) with $\deg(F_i) \geq \deg(G)$ for $i = 1, \ldots, r$. Let $h_1, ..., h_n$ be holomorphic functions on $\mathbb{C}$ such that one of the following holds

(i) $r \geq 2n$ and $\frac{G(h_1,...,h_n)}{F_i(h_1,...,h_n)}$ is holomorphic, for $i = 1, ..., r$; or

(ii) $r \geq n+1$ and $\frac{G(h_1,...,h_n)}{\prod_{i=1}^r F_i(h_1,...,h_n)}$ is holomorphic.

Then $h_1, \ldots, h_n$ are algebraically dependent.

This can be seen as a generalization of Borel Lemma stating that nowhere vanishing entire functions $h_1, ..., h_{n+1}$ satisfying the identity $h_1 + \cdots + h_{n+1} = 1$ are linearly dependent. Indeed, we have the following corollary.

Corollary. Let $h_1, ..., h_{n+1}$ be holomorphic functions on $\mathbb{C}$ such that $\frac{1}{(h_1 \cdots h_n)(1 - \sum_{i=1}^n h_i)}$ is holomorphic. Then $h_1, \ldots, h_n$ are linearly dependent.

Theorem. Let $n \geq 2$. Let k be a number field, let S be a finite set of places including the Archimedean ones, and let $\mathcal{O}_S$ be the ring of S-integers.

Theorem. Let $n \geq 2$. Let k be a number field, let S be a finite set of places including the Archimedean ones, and let $\mathcal{O}_S$ be the ring of S-integers. Let $F_1, ..., F_r, G \in \mathcal{O}_S[X_1, ..., X_n]$ be absolutely irreducible homogeneous polynomials of the same degree. Suppose that the hypersurfaces defined by $F_1, ..., F_r$ and G are in general position, and $\deg(F_i) \geq \deg(G)$ for $i = 1, ..., r$.

Theorem. Let $n \geq 2$. Let k be a number field, let S be a finite set of places including the Archimedean ones, and let $\mathcal{O}_S$ be the ring of S-integers. Let $F_1, ..., F_r, G \in \mathcal{O}_S[X_1, ..., X_n]$ be absolutely irreducible homogeneous polynomials of the same degree. Suppose that the hypersurfaces defined by $F_1, ..., F_r$ and G are in general position, and $\deg(F_i) \geq \deg(G)$ for $i = 1, ..., r$. Then there exists a closed subset $Z \subset \mathbb{P}^n$, independent of k and S, such that there are only finitely many points $(x_0, ..., x_n) \in \mathbb{P}^n(\mathcal{O}_S) \backslash Z$ such that one of the following holds:

(i) $r \geq 2n+1$ and $F_i(x_0, \ldots, x_n) | G(x_0, ..., x_n)$ in the ring $\mathcal{O}_S$, for $i = 0, ..., r$; or

(ii) $r \geq n+2$ and an $\prod_{i=1}^r F_i(x_0, ..., x_n) | G(x_0, ..., x_n)$ in the ring $\mathcal{O}_S$.

Theorem. Let V be a Cohen–Macaulay complex projective variety of dimension n.

Theorem. Let V be a Cohen–Macaulay complex projective variety of dimension n. Let $D_0, D_1, ..., D_r$, $r \geq n+1$, be effective Cartier divisors of V in general position.

Theorem. Let V be a Cohen–Macaulay complex projective variety of dimension n. Let $D_0, D_1, ..., D_r$, $r \geq n+1$, be effective Cartier divisors of V in general position. Suppose that there exist an ample Cartier divisor A on V and positive integers d_i such that $D_i \equiv d_i A$ and $d_i \geq d_0$ for all $0 \leq i \leq r$. Let $f : \mathbb{C} \rightarrow X$ be a holomorphic map.

Theorem. Let V be a Cohen–Macaulay complex projective variety of dimension n. Let $D_0, D_1, ..., D_r$, $r \geq n+1$, be effective Cartier divisors of V in general position. Suppose that there exist an ample Cartier divisor A on V and positive integers d_i such that $D_i \equiv d_i A$ and $d_i \geq d_0$ for all $0 \leq i \leq r$. Let $f : \mathbb{C} \to X$ be a holomorphic map. Assume that the following

(i) $r \geq 2n+1$ and $\frac{1}{d_i} f^* D_i \leq \frac{1}{d_0} f^* D_0 + O(1)$ for all $i = 0, \ldots, r$; or

(ii) $r \geq n+2$ and $\sum_{i=1}^{r} \frac{1}{d_i} f^* D_i \leq \frac{1}{d_0} f^* D_0 + O(1)$.

Theorem. Let V be a Cohen–Macaulay complex projective variety of dimension n. Let $D_0, D_1, ..., D_r$, $r \geq n+1$, be effective Cartier divisors of V in general position. Suppose that there exist an ample Cartier divisor A on V and positive integers d_i such that $D_i \equiv d_i A$ and $d_i \geq d_0$ for all $0 \leq i \leq r$. Let $f : \mathbb{C} \to X$ be a holomorphic map. Assume that the following

(i) $r \geq 2n+1$ and $\frac{1}{d_i} f^* D_i \leq \frac{1}{d_0} f^* D_0 + O(1)$ for all $i = 0, \ldots, r$; or

(ii) $r \geq n+2$ and $\sum_{i=1}^{r} \frac{1}{d_i} f^* D_i \leq \frac{1}{d_0} f^* D_0 + O(1)$. Then f is algebraic degenerate.

Proof: WLOG, we assume that A is very ample, and $D_i \equiv A$ for $i = 0, 1, \ldots, r$.

Proof: WLOG, we assume that A is very ample, and $D_i \equiv A$ for $i = 0, 1, \ldots, r$. Let $Y_i = D_i \cap D_0$ and $Y = \cup_{i=1}^r Y_i$.

Proof: WLOG, we assume that A is very ample, and $D_i \equiv A$ for $i = 0, 1, \ldots, r$. Let $Y_i = D_i \cap D_0$ and $Y = \cup_{i=1}^r Y_i$. Let $\pi : \tilde{V} \to V$ be the blowup along Y, and $E = E_1 + \cdots + E_r$, $E_i = \pi^{-1}(Y_i)$ be the exceptional divisors.

Proof: WLOG, we assume that A is very ample, and $D_i \equiv A$ for $i = 0, 1, \ldots, r$. Let $Y_i = D_i \cap D_0$ and $Y = \cup_{i=1}^r Y_i$. Let $\pi : \tilde{V} \to V$ be the blowup along Y, and $E = E_1 + \cdots + E_r$, $E_i = \pi^{-1}(Y_i)$ be the exceptional divisors. Consider the line sheaf $\mathcal{L} = \mathcal{O}(\ell(n+1)\pi^* A - E)$ which is ample and, by a computation, a fixed sufficiently large integer ℓ,

$$\beta_{\mathcal{L},\pi^* D_i}^{-1} \leq \frac{1}{\ell}\left(1 + \frac{1}{\ell\sqrt{\ell}}\right).$$

Proof: WLOG, we assume that A is very ample, and $D_i \equiv A$ for $i = 0, 1, \ldots, r$. Let $Y_i = D_i \cap D_0$ and $Y = \cup_{i=1}^r Y_i$. Let $\pi : \tilde{V} \to V$ be the blowup along Y, and $E = E_1 + \cdots + E_r$, $E_i = \pi^{-1}(Y_i)$ be the exceptional divisors. Consider the line sheaf $\mathcal{L} = \mathcal{O}(\ell(n+1)\pi^* A - E)$ which is ample and, by a computation, a fixed sufficiently large integer ℓ,

$$\beta_{\mathcal{L},\pi^* D_i}^{-1} \leq \frac{1}{\ell}\left(1 + \frac{1}{\ell\sqrt{\ell}}\right).$$

If f is not algebraically degenerate, then Ru-Vojta implies that, with $\epsilon' = \ell^{-5/2}$,

$$\sum_{i=1}^r m_{\tilde{f}}(r, \pi^* D_i) \leq_{exc} \left(\frac{1}{\ell}\left(1 + \frac{1}{\ell\sqrt{\ell}}\right) + \epsilon'\right) T_{\tilde{f},\mathcal{O}(\ell(n+1)\pi^* A - E)}(r)$$

$$\leq \left(1 + \frac{2}{\ell\sqrt{\ell}}\right)(n+1) T_{\tilde{f},\mathcal{O}(\pi^* A)}(r) - \frac{1}{\ell} T_{\tilde{f},E}(r).$$

Proof: WLOG, we assume that A is very ample, and $D_i \equiv A$ for $i = 0, 1, \ldots, r$. Let $Y_i = D_i \cap D_0$ and $Y = \cup_{i=1}^{r} Y_i$. Let $\pi : \tilde{V} \to V$ be the blowup along Y, and $E = E_1 + \cdots + E_r$, $E_i = \pi^{-1}(Y_i)$ be the exceptional divisors. Consider the line sheaf $\mathcal{L} = \mathcal{O}(\ell(n+1)\pi^* A - E)$ which is ample and, by a computation, a fixed sufficiently large integer ℓ,

$$\beta^{-1}_{\mathcal{L}, \pi^* D_i} \leq \frac{1}{\ell}\left(1 + \frac{1}{\ell\sqrt{\ell}}\right).$$

If f is not algebraically degenerate, then Ru-Vojta implies that, with $\epsilon' = \ell^{-5/2}$,

$$\sum_{i=1}^{r} m_{\tilde{f}}(r, \pi^* D_i) \leq_{exc} \left(\frac{1}{\ell}\left(1 + \frac{1}{\ell\sqrt{\ell}}\right) + \epsilon'\right) T_{\tilde{f}, \mathcal{O}(\ell(n+1)\pi^* A - E)}(r)$$

$$\leq \left(1 + \frac{2}{\ell\sqrt{\ell}}\right)(n+1) T_{\tilde{f}, \mathcal{O}(\pi^* A)}(r) - \frac{1}{\ell} T_{\tilde{f}, E}(r).$$

Hence,

$$\left(r-(n-1)-\frac{2(n+1)}{\ell\sqrt{\ell}}\right)T_{f,A}(r)+\frac{1}{\ell}T_{Y,f}(r)\leq\sum_{i=1}^{r}N_f(r,D_i).$$

Hence,

$$\left(r-(n-1)-\frac{2(n+1)}{\ell\sqrt{\ell}}\right)T_{f,A}(r)+\frac{1}{\ell}T_{Y,f}(r)\leq\sum_{i=1}^{r}N_f(r,D_i).$$

Since $D_1,\ldots,D_r$ are in general position,

$$\sum_{i=1}^{r}N_f(r,D_i)\leq nN_f(r,D_0)+O(1)\leq\left(n+\frac{1}{\ell^2}\right)T_{f,A}(r).$$

Hence,

$$\left(r-(n-1)-\frac{2(n+1)}{\ell\sqrt{\ell}}\right)T_{f,A}(r)+\frac{1}{\ell}T_{Y,f}(r)\leq\sum_{i=1}^{r}N_f(r,D_i).$$

Since $D_1,\ldots,D_r$ are in general position,

$$\sum_{i=1}^{r}N_f(r,D_i)\leq nN_f(r,D_0)+O(1)\leq\left(n+\frac{1}{\ell^2}\right)T_{f,A}(r).$$

Thus

$$\left(r-2n-1+\frac{1}{\ell}-\frac{2(n+1)}{\ell\sqrt{\ell}}-\frac{2}{\ell^2}\right)T_{f,A}(r)\leq O(1)$$

which gives a contradiction.

Examples of weakly special, but not special.

Examples of weakly special, but not special.

- Let $p > 0$. A rank one saturated coherent sheaf $\mathcal{L} \subset \Omega_X^p$ is called a Bogomolov sheaf if $\kappa(X, \mathcal{L}) = p$, i.e. if $\mathcal{L}$ has the largest possible Iitaka dimension.

Examples of weakly special, but not special.

- Let $p > 0$. A rank one saturated coherent sheaf $\mathcal{L} \subset \Omega_X^p$ is called a Bogomolov sheaf if $\kappa(X, \mathcal{L}) = p$, i.e. if $\mathcal{L}$ has the largest possible Iitaka dimension.
- A nonsingular variety X is said to be special (or of special type) if there is no Bogomolov sheaf on X. A projective variety is said to be special if some (or any) of its resolutions are special.

Examples of weakly special, but not special.

- Let $p > 0$. A rank one saturated coherent sheaf $\mathcal{L} \subset \Omega_X^p$ is called a Bogomolov sheaf if $\kappa(X, \mathcal{L}) = p$, i.e. if $\mathcal{L}$ has the largest possible Iitaka dimension.
- A nonsingular variety X is said to be special (or of special type) if there is no Bogomolov sheaf on X. A projective variety is said to be special if some (or any) of its resolutions are special.
- A variety X is special if and only if it has no fibrations of general type. (Campana)

Examples of weakly special, but not special.

- Let $p > 0$. A rank one saturated coherent sheaf $\mathcal{L} \subset \Omega_X^p$ is called a Bogomolov sheaf if $\kappa(X, \mathcal{L}) = p$, i.e. if $\mathcal{L}$ has the largest possible Iitaka dimension.
- A nonsingular variety X is said to be special (or of special type) if there is no Bogomolov sheaf on X. A projective variety is said to be special if some (or any) of its resolutions are special.
- A variety X is special if and only if it has no fibrations of general type. (Campana)
- A smooth projective variety X over a field k is weakly special if for every finite étale morphism $u : X \to X_k$ the variety X' does not admit a dominant rational map $f : X \to Z$ to a positive dimensional variety Z' of general type.

Examples of weakly special, but not special.

- Let $p > 0$. A rank one saturated coherent sheaf $\mathcal{L} \subset \Omega_X^p$ is called a Bogomolov sheaf if $\kappa(X, \mathcal{L}) = p$, i.e. if $\mathcal{L}$ has the largest possible Iitaka dimension.
- A nonsingular variety X is said to be special (or of special type) if there is no Bogomolov sheaf on X. A projective variety is said to be special if some (or any) of its resolutions are special.
- A variety X is special if and only if it has no fibrations of general type. (Campana)
- A smooth projective variety X over a field k is weakly special if for every finite étale morphism $u : X \to X_k$ the variety X' does not admit a dominant rational map $f : X \to Z$ to a positive dimensional variety Z' of general type.

- Conjecture The set of rational points on X is potentially dense if and only if X is weakly special.

- Conjecture The set of rational points on X is potentially dense if and only if X is weakly special.
- Conjecture Let X be a complex proj. variety. X is weakly special if and only if: (1) there exists an entire curve $\mathbb{C} \to X$ with Zariski dense image; (2) X is pseudo algebraic hyperbolic.

- Conjecture The set of rational points on X is potentially dense if and only if X is weakly special.
- Conjecture Let X be a complex proj. variety. X is weakly special if and only if: (1) there exists an entire curve $\mathbb{C} \to X$ with Zariski dense image; (2) X is pseudo algebraic hyperbolic.
- Rousseau, A. Turchet and Julie Tzu-Yueh Wang (Forum of Mathematics, 2021) constructed examples of 3-dimensional projective varieties which are weakly special, but not special.

- Conjecture The set of rational points on X is potentially dense if and only if X is weakly special.
- Conjecture Let X be a complex proj. variety. X is weakly special if and only if: (1) there exists an entire curve $\mathbb{C} \to X$ with Zariski dense image; (2) X is pseudo algebraic hyperbolic.
- Rousseau, A. Turchet and Julie Tzu-Yueh Wang (Forum of Mathematics, 2021) constructed examples of 3-dimensional projective varieties which are weakly special, but not special. They showed that the examples contradict the conjecture above.

- Conjecture The set of rational points on X is potentially dense if and only if X is weakly special.
- Conjecture Let X be a complex proj. variety. X is weakly special if and only if: (1) there exists an entire curve $\mathbb{C} \to X$ with Zariski dense image; (2) X is pseudo algebraic hyperbolic.
- Rousseau, A. Turchet and Julie Tzu-Yueh Wang (Forum of Mathematics, 2021) constructed examples of 3-dimensional projective varieties which are weakly special, but not special. They showed that the examples contradict the conjecture above.

Theorem(RTW). Let $X \subset \mathbb{P}^m$ be a smooth projective surface and $D = D_1 + \cdots + D_q$ be a divisor with $q \geq 2$, such that

Theorem(RTW). Let $X \subset \mathbb{P}^m$ be a smooth projective surface and $D = D_1 + \cdots + D_q$ be a divisor with $q \geq 2$, such that
(1) No three of the components D_i meet at a point;

Theorem(RTW). Let $X \subset \mathbb{P}^m$ be a smooth projective surface and $D = D_1 + \cdots + D_q$ be a divisor with $q \geq 2$, such that
(1) No three of the components D_i meet at a point;
(2) There exists a choice of positive integers p_i such that the divisor $D_p := p_1 D_1 + p_2 D_2 + \cdots + p_q D_q$ is ample and the the following inequality holds:

$$2D_p^2 \xi_i > (D_p \cdot D_i)\xi_i^2 + 3D_p^2 p_i,$$

for every $i = 1, \ldots, q$ where ξ_i is the minimal positive solution of the equation $D_i^2 x^2 - 2(D_p \cdot D_i)x + D_p^2 = 0$.

Theorem(RTW). Let $X \subset \mathbb{P}^m$ be a smooth projective surface and $D = D_1 + \cdots + D_q$ be a divisor with $q \geq 2$, such that
(1) No three of the components D_i meet at a point;
(2) There exists a choice of positive integers p_i such that the divisor $D_p := p_1 D_1 + p_2 D_2 + \cdots + p_q D_q$ is ample and the the following inequality holds:

$$2D_p^2 \xi_i > (D_p \cdot D_i)\xi_i^2 + 3D_p^2 p_i,$$

for every $i = 1, \ldots, q$ where ξ_i is the minimal positive solution of the equation $D_i^2 x^2 - 2(D_p \cdot D_i)x + D_p^2 = 0$. Let $\triangle$ be the $\mathbb{Q}$-divisor defined as

$$\triangle = \sum_{j=1}^{q} \left(1 - \frac{1}{m_i}\right) D_i.$$

Theorem(RTW). Let $X \subset \mathbb{P}^m$ be a smooth projective surface and $D = D_1 + \cdots + D_q$ be a divisor with $q \geq 2$, such that
(1) No three of the components D_i meet at a point;
(2) There exists a choice of positive integers p_i such that the divisor $D_p := p_1 D_1 + p_2 D_2 + \cdots + p_q D_q$ is ample and the the following inequality holds:

$$2D_p^2 \xi_i > (D_p \cdot D_i)\xi_i^2 + 3D_p^2 p_i,$$

for every $i = 1, \ldots, q$ where ξ_i is the minimal positive solution of the equation $D_i^2 x^2 - 2(D_p \cdot D_i)x + D_p^2 = 0$. Let $\triangle$ be the $\mathbb{Q}$-divisor defined as

$$\triangle = \sum_{j=1}^{q} \left(1 - \frac{1}{m_i}\right) D_i.$$

Then, there exists a positive integer m such that, if $m_i \geq m$ for every i, every orbifold entire curve $f : \mathbb{C} \to (X, \triangle)$ is algebraically degenerate.

Proof.

Proof. By the Riemann-Roch Theorem, for N large enough we have that $2h^0(ND_p) = D_p^2 N^2 + O(N)$.

Proof. By the Riemann-Roch Theorem, for N large enough we have that $2h^0(ND_p) = D_p^2 N^2 + O(N)$. Also use Riemann-Roch,

$$\sum_{k=0}^{\xi_i N} h^0(ND_p - kD_i) \geq N^3 \left(\frac{\xi_i^2 (D_p \cdot D_i)}{2} - \frac{\xi_i^3 D_i^2}{3} \right) + O(N^2).$$

Proof. By the Riemann-Roch Theorem, for N large enough we have that $2h^0(ND_p) = D_p^2 N^2 + O(N)$. Also use Riemann-Roch,

$$\sum_{k=0}^{\xi_i N} h^0(ND_p - kD_i) \geq N^3 \left(\frac{\xi_i^2 (D_p \cdot D_i)}{2} - \frac{\xi_i^3 D_i^2}{3} \right) + O(N^2).$$

By definition of ξ_i, we have $D_i^2 x^2 - 2(D_p \cdot D_i)x + D_p^2 = 0$.

Proof. By the Riemann-Roch Theorem, for N large enough we have that $2h^0(ND_p) = D_p^2 N^2 + O(N)$. Also use Riemann-Roch,

$$\sum_{k=0}^{\xi_i N} h^0(ND_p - kD_i) \geq N^3 \left(\frac{\xi_i^2 (D_p \cdot D_i)}{2} - \frac{\xi_i^3 D_i^2}{3} \right) + O(N^2).$$

By definition of ξ_i, we have $D_i^2 x^2 - 2(D_p \cdot D_i)x + D_p^2 = 0$. This implies that

$$\beta(ND_p, D_i) \geq \frac{\frac{2}{3}\xi_i D_p^2 - \frac{1}{3}(D_p \cdot D_i)\xi_i^2}{D_p^2} > p_i.$$

Proof. By the Riemann-Roch Theorem, for N large enough we have that $2h^0(ND_p) = D_p^2 N^2 + O(N)$. Also use Riemann-Roch,

$$\sum_{k=0}^{\xi_i N} h^0(ND_p - kD_i) \geq N^3 \left(\frac{\xi_i^2 (D_p \cdot D_i)}{2} - \frac{\xi_i^3 D_i^2}{3} \right) + O(N^2).$$

By definition of ξ_i, we have $D_i^2 x^2 - 2(D_p \cdot D_i)x + D_p^2 = 0$. This implies that

$$\beta(ND_p, D_i) \geq \frac{\frac{2}{3}\xi_i D_p^2 - \frac{1}{3}(D_p \cdot D_i)\xi_i^2}{D_p^2} > p_i.$$

We conclude the theorem by the result of Ru-Vojta.