Weighted L^2 estimate for $\overline{\partial}$ and application to Corona problem

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1. Overview of $\overline{\partial}$ estimate
2. Our results
3. Corona problems in SCV
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Background

- Let Ω be a bounded domains (open and connected set) in $\mathbb{C}^n$
- Let $f = \sum_{j=1}^{n} f_j(z)d\bar{z}_j$ be a $(0, 1)$-form
- Cauchy-Riemann equation:

$$\overline{\partial} u = f \tag{1}$$

Remarks:

1) u is holomorphic in Ω if and only if $f = 0$ in Ω

2) Solution of (1) is not unique

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3) A necessary condition for $\overline{\partial} u = f$ having a solution is:
f must be $\overline{\partial}$-closed since $\overline{\partial}^2 = 0$.

Question. If f is $\overline{\partial}$-closed $(0, 1)$-form or $(0, q)$-form in Ω, does $\overline{\partial} u = f$ have a solution in Ω?

Answers: Yes when $n = 1$;
No, when $n > 1$ for general Ω;
Yes, if Ω is pseudoconvex.

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Basic Setting

- ϕ is a plurisubharmonic function on Ω
- $L^2(\Omega, \phi)$ denote the set of all measurable functions u with

$$\|u\|_\phi^2 = \int_\Omega |u(z)|^2 e^{-\phi(z)} dv(z) < \infty.$$

- $L^2_{(0,1)}(\Omega, \phi)$ denote the set of all $(0,1)$-forms $f = \sum_{j=1}^n f_j d\overline{z}_j$

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Hörmander's theorem on L^2 estimate

Assume that

- Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$,
- ϕ is a plurisubharmonic function on Ω,
- $A^2(\Omega, \phi)$ is the holomorohic subspace of $L^2(\Omega, \phi)$.

Then for any $\overline{\partial}$-closed $f \in L^2_{(0,1)}(\Omega, \phi)$, the $\overline{\partial}$-equation

$$\overline{\partial} u = f$$

has the unique solution $u_0 \perp A^2(\Omega, \phi)$ satisfying

$$\|u_0\|_\phi \le C_\Omega \|f\|_\phi$$

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Notation

- $\phi \in C^2(\Omega)$ is a strictly plurisubharmonic function on Ω;
- Complex Hessian matrix of ϕ and its inverse are defined by

$$H(\phi)(z) = \left[\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}\right]_{n\times n} \text{ and } \left[\phi^{i\bar{j}}(z)\right]^t = H(\phi)(z)^{-1}.$$

- For any $(0,1)$-form $f = \sum_{j=1}^n f_j d\bar{z}_j$, we let

$$\|f(z)\|^2_{i\partial\bar{\partial}\phi} := \sum_{i,j=1}^n \phi^{i\bar{j}}(z)\bar{f}_i(z) f_j(z)$$

Restatement of the Hörmander's theorem

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Integral solutions and uniform estimate

• Another very important method to solve $\overline{\partial}$-equation is to construct a kernel function $B(z, w)$ on $\Omega \times \Omega$ which is $(0, 1)$-form in z and $(n, n-1)$-form in w such that

$$u(z) = \int_{\Omega} B(z, w) \wedge f(w) \tag{2}$$

• In 1970, G. M. Henkin; Grauart and Lieb constructed an integral formula and obtained a uniform estimate:

Theorem

Let Ω be a smoothly bounded strictly pseudoconvex domain in C^n. There is a kernel $B(z, w)$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty_{(0,1)}(\Omega)}$$

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Improvement for uniform estimate

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Let Ω be a smoothly bounded strictly pseudoconvex domain in C^n. There is a solution u to $\overline{\partial}u = f$ such that

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Gong's theorem

• Xianghong Gong (Math Ann. 2019) was able to improve the above theorem for domaind Ω with $\partial\Omega \in C^2$.

Theorem

Let Ω be a bounded strictly pseudoconvex domain in C^n with C^2 boundary. There is a solution u to $\overline{\partial} u = f$ such that

$$\|u\|_{C^{1/2+\gamma}(\overline{\Omega})} \leq C\|f\|_{C^{\gamma}(\Omega)},$$

where $\frac{1}{2} + \gamma$ is not integer.

• When $\partial\Omega$ is not smooth, problems was studied by Range and Siu (1972 and 1973), Krantz (1976), Shaw, etc.

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Weakly pseudoconvex domains

• When Ω is a weakly pseudoconvex domain, $\bar\partial$-estimate becomes much more complicated.

• Range (1976) proved: *Uniform estimate holds for $\bar\partial$ when Ω is a convex domain with real analytic boundary.*

• B. Berndtsson (1993), J. Fornaess (1986), and N. Sibony (1980) constructed weakly pseudoconvex domains in $\mathbb{C}^2$, $\mathbb{C}^3$ respectively for which the uniform estimate fail.

• J. Fornaess and N. Sibony (1990s) constructed weakly pseudoconvex domains in $\mathbb{C}^2$ with one weakly pseudoconvex point for which $L^\infty - L^p$ estimate fails for any $p > 2$.

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Solution for $\bar\partial$ on Bidisc

- Bidisc: $\Omega = D(0,1)^2$

A special weakly pseudoconvex domain without C^1 boundary.

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Let $\Omega = D(0,1)^2$. For any $\bar\partial$-closed $(0,1)$-form f with $f \in C^1_{(0,1)}(\overline{\Omega})$. Then there is a solution u of $\bar\partial u = f$ on Ω satisfies

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$\overline{\partial}$-estimate on Polydis in $\mathbb{C}^n$

- Recently, the Henkin's result has been generalized by Chen and McNeal (2018) and Fassina and Pan (2019)

- However, problem the uniform estimate for $\overline{\partial}$:

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Does the uniform estimate hold for $\overline{\partial}$ on the classical bounded symmetric domains ?

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Donnelly-Fefferman weighted L^2 estimate (1997)

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Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$.
ϕ and ψ are plurisubharmonic in Ω satisfying

$$|\overline{\partial}\phi|^2_{i\partial\overline{\partial}\phi} \leq 1/4.$$

If u is the solution of $\overline{\partial}u = f$ and $u \perp A^2(\Omega, \phi/2 + \psi)$ then

$$\|u\|_{L^2(\Omega,\psi)} \leq 4\, \||f(\cdot)|_{i\partial\overline{\partial}\phi}\|_{L^2(\Omega,\psi)}.$$

Berndtsson's approach

• Use the Donnelly-Fefferman's weighted L^2 estimate with the plurisubharmonic weight function $\phi(z) = -\log(1 - |z|^2)$. Which is the potential function for the Bergman metric g. Moreover,

$$\phi^{i\bar{j}}(z) = (1 - |z|^2)(\delta_{ij} - z_i\overline{z}_j)$$

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Berndtsson's estemate

• Berbdtsson's Theorem: Let g be the Bergman metric over the unit ball B_n. If f is $\overline{\partial}$-closed $(0,1)$-form and if $u \perp A^2(\Omega)$ is a solution of $\overline{\partial} u = f$, then

$$|u(z)| \leq C \sup_{w \in B_n} \left\{ |f(w)|_g \right\} \log \frac{4}{1-|z|^2} \tag{3}$$

and

$$\|u\|_\infty \leq C_\epsilon \sup_{z \in B_n} \left\{ (1-|z|^2)^{-\epsilon} |f(z)|_g \right\}$$

for any $\epsilon > 0$.

• Estimate (3) is sharp
since $u(z) = \log(1-|z|^2) - c$ is the sharp solution.

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A joint work with X. Dong and J. Treuer

Theorem (D-L-T, APDE, 2021) Let g be the Bergman metric over a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. If f is $\bar\partial$-closed $(0,1)$-form then there is a solution u for $\bar\partial u = f$ satisfying

$$|u(z)| \leq C \sup_{w \in B_n} \left\{ |f(w)|_g \right\} \log K(z) \tag{4}$$

where K is the Bergman kernel on Ω. Moreover,

$$\|u\|_\infty^2 \leq C_\epsilon \sup_{z \in B_n} \left\{ |f(z)|_g^2 (\log(1 + K(z)))^p \right\} \tag{5}$$

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Definitions of BSD

- $M^{m,n}(\mathbb{C})$ denotes $m \times n$ matrices with entries in $\mathbb{C}$.
- Bounded symmetric domain of type I is defined by

$$\mathbf{I}(m,n) = \{\mathbf{z} \in M^{m,n}(\mathbb{C}) : I_m - \mathbf{z}\mathbf{z}^* > 0\}$$

- BSD of type II is defined by

$$\mathbf{II}(n) = \{\mathbf{z} \in \mathbf{I}(n,n) : \mathbf{z} \text{ is symmetric}\}$$

- BSD of type III is defined by

$$\mathbf{III}(n) = \{\mathbf{z} \in \mathbf{I}(n,n) : \mathbf{z}^t = -\mathbf{z}\}$$

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$$\mathbf{II}(n) = \{\mathbf{z} \in \mathbf{I}(n,n) : \mathbf{z} \text{ is symmetric}\}$$

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Definitions of BSD

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BSD of type 4

- BSD of type IV is defined by

$$\mathbf{IV}(n) = \{z \in \mathbb{C}^n : r(z) > 0, \quad |s(z)| < 1\}$$

where $r(z) = 1 + |s(z)|^2 - 2|z|^2, \quad s(z) = \sum_{j=1}^n z_j^2$.

- When $n = 2$, IV(2) is biholomorphic to bidisk $D(0,1)^2$.

Our second results

Theorem (D-L-T, 2019). Let Ω be a classical bounded symmetric domain or polydisc in $\mathbb{C}^n$, K is the Bergman kernel function of Ω and g is the Bergman metric. Let f is a closed $(0,1)$-form satisfying

$$\|f\|_{g,\infty}^2 = \sup\{|f(z)|_g^2 : z \in \Omega\} < \infty.$$

Let u be the canonical solution to $\overline{\partial} u = f$. Then

$$|u(z)| \leq C\|f\|_{g,\infty} \int_\Omega |K(z,w)| dv(w), \quad z \in \Omega.$$

Our third result

Theorem (D-L-T, 2019). Let g be the Bergman metric over the polydisc in $\mathbb{C}^n$. If f is $\overline{\partial}$-closed $(0,1)$-form then the canonical solution u to $\overline{\partial}u = f$ satisfies

$$\|u\|_\infty^2 \le C_p \sup_{z\in B_n} \Big\{ |f(z)|_g^2 \prod_{j=1}^n (\log \frac{2}{1-|z_j|}))^p \Big\}, \quad z \in D(0,1)^n$$

for any $p > 1$.

Our Example

Example. (D-L-T). Let g be the Bergman metric over a bounded symmetric domain Ω in $\mathbb{C}^N$. Let

$$u(z) = \log\det(I - zz^*), \quad z \in \Omega.$$

Then (i) $P[u] = c_\Omega$

(ii) $|\overline{\partial} u|_g^2 = (zz^*)$ is bounded by 1 on Ω

(iii) $\overline{\partial} u$ is bounded in the Bergman metric on Ω, but $u - P[u]$ is not bounded on Ω.

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Idea of the proof

Let u be the canonical solution of $\overline{\partial} u = f$. We try to prove

$$\oint_{B(a)} |u(z) dv(z) \le C \int_{\Omega} |U(z)| dv(z)$$

where U is the canonical solution in $L^2(\Omega, \psi + \frac{\phi}{2})$.
Here ϕ, ψ are two plurisubharmonic functions.
Main technique is how to choose the weighted functions ϕ and ψ to get the estimate you want.

Open problems

Even if there are partial result for uniform or pointwise estimate, the following problem is still open.

Problem. Let $f \in L^{\infty}_{(0,1)}(\Omega)$ be a $\overline{\partial}$-closed $(0,1)$-form on a bounded symmetric domain Ω. Does there is $u \in L^{\infty}(\Omega)$ such that $\overline{\partial} u = f$?

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Corona Problem

- Let Ω be a bounded domains in $\mathbb{C}^n$
- $H^\infty(\Omega)$ is the set of bounded holomorphic functions on Ω with sup-norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \Omega\}$$

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Question.

Given m elements $f_1, \cdots, f_m \in H^\infty(\Omega)$.
Under what conditions, one has

$$\langle f_1, f_2, \cdots, f_m \rangle = H^\infty(\Omega)?$$

Here, $\langle f_1, f_2, \cdots, f_m \rangle$ is the ideal generated by $f_1, f_2, \cdots, f_m$.

- A necessary condition:

If there are $g_1, \cdots, g_m \in H^\infty(\Omega)$ such that

$$1 = \sum_{j=1}^{m} f_j(z) g_j(z)$$

holds.

Necessary Condition

Write

$$|f(z)|^2 = \sum_{j=1}^{m} |f_j(z)|^2, \quad |g(z)|^2 = \sum_{j=1}^{m} |g_j(z)|^2 \le C^2.$$

Then $1 = \sum_{j=1}^{m} f_j(z) g_j(z)$ implies

$$1 \le |f(z)||g(z)| \le C|f(z)|.$$

Therefore,

$$\frac{1}{C} \le |f(z)|, \quad z \in \Omega$$

We may normalized it as:

$$0 < \delta^2 \le |f(z)|^2 = \sum_{j=1}^{m} |f_j(z)|^2 \le 1, \quad z \in \Omega. \tag{1}$$

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Statement of Corona Problem

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Let Ω be a bounded domain in $\mathbb{C}^n$.

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Are there $g_1, g_2, \cdots, g_m \in H^\infty(\Omega)$ such that

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History

- The Corona problem was first formulated by Kakutani in 1941 from the point of view in the function algebra.

Are the point evaluation functionals dense in the maximal ideal space $H^\infty(D)$ on the unit disc $D \subset \mathbf{C}$?

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Carleson's idea of the solutions

Carleson's idea to solve the Corona problem is as follows:

1) Construct bounded functions $\phi_1(z), \cdots, \phi_m(z)$ such that

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2) Try to modify ϕ_j to get $g_j \in H^\infty(D)$ such that

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Q: How to modify it?

He defined

$$g_j(z) = \phi_j - \sum_{k=1}^{m} f_k A_{k,j} \quad \text{with} \quad A_{k,j} = -A_{j,k}$$

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- g_j is holomorphic if and only if

$$0 = \overline{\partial} g_j(z) = \overline{\partial}\phi_j(z) - \sum_{k=1}^{n} f_k(z)\overline{\partial}A_{jk}(z)$$

One can see that if

$$\overline{\partial}A_{kj} = \phi_k\overline{\partial}\phi_j - \phi_j\overline{\partial}\phi_k = \psi_{k,j}$$

then g_j is holomorphic.

Then the Corona problem is solved if $\overline{\partial}A_{k,j} = \psi_{k,j}$ has a solution $A_{jk} \in L^\infty(D)$.

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Tom Wolff's idea

Carleson's solution of Corona problem is very complicated based on his construction of $\phi_j \in L^\infty(D)$ with $\overline{\partial}\phi_j$ is a Carleson measure and

$$\sum_{j=1}^{m} f_j(z)\phi_j(z) = 1, \quad z \in D.$$

In 1980, T. Wolff came up a new idea to choose

$$\phi_j(z) = \frac{\overline{f}_j(z)}{|f(z)|^2}, \quad |f(z)|^2 = \sum_{k=1}^{m} |f_k(z)|^2.$$

Change to the difficulty of the problem to solve $\overline{\partial}$-equation:

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Tom Wolff's proof

Let $S : L^2(\partial D) \to H^2(D)$ be the Szegö projection. He try to solve

$$\overline{\partial} B_{k,j} = \phi_k \overline{\partial} \phi_j, \quad A_{k,j} = B_{k,j} - B_{j,k}$$

with $B_{j,k} \in MBO(\partial D) \cap H^2(D)^{\perp}$. Then

$$B_{k,j} = (I - S)B_{k,j} = \overline{S}_0 B_{k,j} \quad \text{on } \partial D.$$

Apply Stein-Weiss (Fefferman's duality) theorem

$$BMOA(D) = S(L^\infty(\partial D)), \quad BMOA(D) = H^1(D)^*.$$

There is a $b_{k,j} \in L^\infty(\partial D)$ such that

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$$g_j(z) = \phi_j(z) - \sum_{k=1}^{m} f_k(B_{k,j} - B_{j,k}) \quad \Big(B_{k,j} = b_{k,j} - S(b_{k,j})\Big)$$

are holomorphic in D and $g_j \in H^p(D)$ for any $p < \infty$.

We extend $b_{k,j}$ from ∂D to D by letting:

$$b_{k,j} = B_{k,j} + S(b_{k,j}).$$

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$\overline{\partial}$-solution

$$\overline{\partial} B_{k,j} = \phi_k \overline{\partial} \phi_j, \quad B_{k,j} \perp H^2(D)$$

For $h \in H^1(D)$ with $h(0) = 0$. Write $h(z) = h_1(z) h_2(z)$ with

$$h_j \in H^2(D) \quad \text{and} \quad \|h\|_{H^1} = \|h_1\|_{H^2} \|h_2\|_{H^2}.$$

It suffices to prove

$$\Big| \int_{\partial D} B_{k,j}(z) h(z) d\sigma(z) \Big| \le C \|h\|_{H^1}.$$

$$
\begin{aligned}
&\Big|\int_{\partial D} B_{k,j} h_1(z) h_2(z) d\sigma(z)\Big| \\
&= \Big|\int_D \Delta\Big(B_{k,j} h_1(z) h_2(z)\Big) \log\frac{1}{|z|} dA(z)\Big| \\
&= \Big|\int_D [\Delta B_{k,j} h_1 h_2 + 4\overline{\partial}\Big(B_{k,j})\partial(h_1(z)h_2(z)\Big) \log\frac{1}{|z|}\Big| \\
&= 4\Big|\int_D [-\partial(\phi_k \overline{\partial}\phi_j) h_1 h_2 + \phi_k \overline{\partial}\phi_j (h_2 \partial h_1 + h_2 \partial h_1)] \log\frac{1}{|z|} \\
&\leq \frac{C}{\delta^3} \|h_2\|_{H^2} \|h_1\|_{H^2}. \quad \text{(Green's theorem).}
\end{aligned}
$$

Remark.

• Carleson's theorem is true for any bounded domain with C^1 boundary.
However, the following problem is still open.

Problem. Let $f_1, \cdots, f_m, h \in H^\infty(D)$ such that

$$\sum_{j=1} |f_j(z)|^2 \geq |h(z)|^2, \quad z \in D.$$

Then there are $g_1, \cdots, g_m \in H^\infty(D)$ such that

$$\sum_{j=1}^{m} f_j(z) g_j(z) = h(z)^2.$$

Remark continued

From the Tom Wolff's proof, one can prove:

There are $g_1, \cdots, g_m \in H^\infty(D)$ such that

$$\sum_{j=1}^{m} f_j(z)g_j(z) = h(z)^3$$

This problem is equivalent to estimate for the best constant upper bound in original Corona Problem (J. Garnett and P. Jones) .

$$\|g_j\|_\infty \le \frac{C}{\delta^2 \log \frac{1}{\delta}}$$

Question: Can one improve the upper bound to $\frac{C}{\delta^2}$?

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Corona problem in higher dimensions

- When $n > 1$.

Counter example was constructed by Fornaess and Sibony in 1993.

There is a bounded pseudoconvex domain in $\mathbf{C}^2$ with smooth boundary. Where the Corona problem is not solvable.

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Varopoules in 1977, generalized the Carleson's idea to the higher dimension on the unit ball $B_n \subset \mathbf{C}^n$.

He constructed L^∞ solutions ϕ_1, ϕ_2 (as Carleson did) such that

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has a solution $A \in BMO(\partial B_n)$.

Wen $n > 1$, $A \in BMO(\partial B_n)$ can have

$$g_1(z) = \phi_j(z) - f_2(z)A, \quad g_2(z) = \phi_2(z) + f_1(z)A$$

are holomorphic in B_n and

$$f_1(z)g_1(z) + f_2(z)g_2(z) = 1,$$

But, in general, A **can not** be written as

$$A = B - H$$

with $B \in L^\infty(B_n)$ and $H \in H^1(B_n)$.

Multiplication operator

From the operator theory point of view, one can define a multiplication operator S_f associated to the Corona dada $f = (f_1, \cdots, f_m)$ which satisfies

$$0 < \delta^2 \leq \sum_{j=1}^{m} |f_j(z)|^2 \leq 1.$$

If X is function space over Ω, one define

$$S_f : X^m = X \oplus X \oplus \cdots \oplus X \to X, \quad S_f(g) = \sum_{j=1}^{m} f_j g_j$$

Carleson Theorem.

$S_f : H^\infty(D)^m \to H^\infty(D)$ is bounded and onto.

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Arveson, using Foias lifting theorem, prove

Theorem. $S_f : H^\infty(D)^m \to H^\infty(D)$ is onto if and only if $S_f : H^p(D)^m \to H^p(D)$ is onto.
(True, for a bounded domain $\Omega \subset$ with $C^{1,\epsilon}$ boundary by L)

When $n > 1$, whether the above theorem is true or not, it is not known for the most standard domains Ω, like B_n, $D(0,1)^n$. We know

Theorem

$S_f : X^m \to X$ is bounded and onto when $X = H^p(B_n)$ or $X = H^p(D(0,1)^n)$.

Either m is finite or infinity, the above theorem proved by several people, including: Varopoules, K. C. Lin, S-Y Li, E. Amar, M. Anderson, etc.

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For more general function space X, $S_f : X^m \to X$ may not be bounded. Under the boundedness assumption, there are more results are obtained.

Theorem

Let $S_f : X^m \to X$ be bounded. Then $S_f : X^m \to X$ is onto when $X = B^p_{m,\sigma}(B_n)$ with $p > 1$ and some restriction on m and σ.

Where $h \in B^p_\sigma(B_n)$ if h is holomorphic and

$$\|f\|^p_{p,\sigma} =: \int_{B_n} \Big(|D^m f(z)| r(z)^\sigma\Big)^p K(z,z) dv(z) < \infty$$

When $\sigma = m = 1$ and $p > n$, the above space is the standard Besov space. When $n = 1$ and $p = 2$, it is Dirichlet space.

Contributors

The above theorem was proved by several mathematicians with various conditions, I listed part of them as follows.
M. Anderson (1994, 2011)
J. Xiao (1998, $n = 1$ Dirichlet space)
Arcozzi, Rochberg and Sawyer (2006)
E. T. Sawyer (2009)
Costea, Sawyer and Wick (2010)
Costea, Sawyer and Wick (2011)

Several papers written by Krantz, Li, Treil, Trent, Wick and Zhang, etc.

Tool to study the Corona problem

Recently, I used the Hörmander's weighted L^2 estimate to study the Corona problem and obtained the following results.

Result 1

Theorem

Let $f_1, \cdots, f_m \in H^2(B_n) \cap C^\gamma(\overline{B}_n)$ *with* $0 < \alpha < 1$ *satisfy*

(1) $$0 < \delta^2 \leq \sum_{j=1}^{m} |f_j(z)|^2 \leq 1, \quad z \in B_n.$$

Then there are $g_j \in H^2(B_n) \cap C^\gamma(\overline{B}_n)$ *such that*

(2) $$\sum_{j=1}^{m} f_j(z) g_j(z) = 1.$$

Remark. when $D = D(0,1)^n$, the above was proved by Krantz and Li (1995)

Result 2

We give a simpler proof of the following theorem:

Theorem

Let $f_1, f_2 \in H^\infty(B_n)$ satisfy Corona condition (1) and

$$\phi_j = \overline{f_j(z)}/|f(z)|^2, \quad |f(z)|^2 = \sum_{j=1}^{2} |f_j(z)|^2.$$

Then the canonical solution of $\overline{\partial}$-equation:

$$\overline{\partial} A = \phi_1 \overline{\partial} \phi_2 - \phi_2 \overline{\partial} \phi_1$$

satisfies $A \in BMO(\partial B_n)$ and $|\nabla A|^2(1-|z|^2)dv$ is a Carleson measure on B_n.

Two Function Spaces

Let

$$r(z) = 1 - |z|^2, \quad \phi_0(z) = -\log r(z).$$

• Let $\mathcal{L}B_p(B_n)$ be the space of all holomorphic functions h on B_n with

$$\|h\|_{\mathcal{L}B_p} = \sup\{|\log r(z)|^p |\overline{\partial h}(z)|_{i\partial\overline{\partial}\phi_0} : z \in B_n\} < \infty.$$

• Let $\mathcal{S}(B_n)$ denote the space of all holomorphic functions h on B_n with

$$\|h\|_{\mathcal{S}(B_n)} = \sup_{a \in B_n} \int_{B_n} |\overline{\partial h}(z)|_{i\partial\overline{\partial}\phi_0} |K(z,a)| dv(z) < \infty.$$

Result 3

Theorem

Let $f_1, f_2 \in H^\infty(B_n)$ satisfy Corona condition (1) and $f_j \in \mathcal{S}(B_n) \cap \mathcal{L}B_p(B_n)$. Then there are $g_1, g_2 \in H^\infty(B_n)$ such that

$$f_1(z)g_1(z) + f_2(z)g_2(z) = 1.$$

Moreover, $g_j \in \mathcal{S}(B_n) \cap \mathcal{L}B_p(B_n)$. Here, $p > 1/2$ be any real number.

- The Corona Problem for the unit ball B_n or for polydisc $D(0,1)^n$ with $n > 1$ is still open.

Thank you very much for your attention!