Finite type conditions for real hypersurfaces in $\mathbb{C}^n$

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In Several Complex Variables and Complex Geometry, a fundamental problem is to solve the Cauchy-Riemann equations.

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Let D be a domian in $\mathbb{C}^n$. Study the existence and regularity of

$$\overline{\partial} u = f \quad \text{in} \ \ D.$$

Here $0 \leq p \leq n$, $1 \leq q \leq n$, f is a (p, q) form satisfying the solvable condition:

$$\overline{\partial} f = 0 \quad \text{in} \ \ D.$$

Theorem

If D *is a bounded pseudoconvex domain and* $f \in L_2^{(p,q)}(D)$. *Then there exits a* u *with* $\|u\|_{L^2} \leq c_q \|f\|_{L^2}$.

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Theorem (J. Kohn 1973)

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 2)$ with smooth boundary. For every $f \in C^\infty_{(p,q)}(\overline{D})$, there exists a $u \in C^\infty_{(p,q-1)}(\overline{D})$ such that $\overline{\partial} u = f$.

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Kohn-Nirenberg: The answer is POSITIVE if the domain has subelliptic estimates.

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For $f \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*)$. Then

$$\|f\|_\epsilon^2 \leq \|\partial f\|^2 + \|\overline{\partial}^* f\|^2 + \|f\|^2 \; with \; \epsilon = \frac{1}{2}.$$

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4. contact order by holomorphic curves $\Delta_1(M, p)$.

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T. Bloom (1981):

When $M \subset \mathbb{C}^n$. For each integer $1 \leq s \leq n-1$, we can define corresponding integer invaiants $a^{(s)}(M,p)$, $t^{(s)}(M,p)$ and $c^{(s)}(M,p)$ as follows.

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(i): The s-contact type $a^{(s)}(M,p)$:

$$a^{(s)}(M,p) = \sup_X \{r \mid \exists \text{ an } s\text{-dimensional complex submanifold } X \text{ whose order of vanishing of } \rho|_X \text{ at } p \text{ is } r\}.$$

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Let B be an s-dimensional subbundle of $T^{1,0}M$. We let $\mathcal{M}_1(B)$ be the $C^\infty(M)$-module spanned by the smooth tangential $(1,0)$ vector fields L with $L|_q \in B|_q$ for each $q \in M$, together with the conjugate of these vector fields.

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For $\mu \geq 1$, we let $\mathcal{M}_\mu(B)$ denote the $C^\infty(M)$-module spanned by commutators of length less than or equal to μ of vector fields from $\mathcal{M}_1(B)$. A commutator of length μ of vector fields in $\mathcal{M}_1(B)$ is a vector field of the following form: $[Y_\mu, [Y_{\mu-1}, \cdots, [Y_2, Y_1] \cdots]$. Here $Y_j \in \mathcal{M}_1(B)$.

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Define $t^{(s)}(B,p) = m$ if $\langle F, \partial\rho \rangle(p) = 0$ for any $F \in \mathcal{M}_{m-1}(B)$ but $\langle G, \partial\rho \rangle(p) \neq 0$ for a certain $G \in \mathcal{M}_m(B)$. Then

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$$t^{(s)}(M,p) = \sup_B \{t(B,p) |\ B \text{ is an } s\text{-dimensional subbundle of } T^{1,0}M\}.$$

$t^{(s)}(B,p)$ is the smallest length of the commutators by vector fields in $\mathcal{M}_1(B)$ to recover the complex contact direction in $\mathbb{C}T_pM$. $t^{(s)}(M,p)$ is the largest possible value among all $t^{(s)}(B,p)'s$. Namely, $t^{(s)}(M,p)$ describes the most degenerate s-subbundle of $T^{1,0}M$.

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Let B be as in (ii). Let $\mathcal{L}_{M,p}$ be a Levi form associated with a defining function ρ near p of M. For $V_B = \{L_1, \cdots, L_s\}$, a basis of smooth sections of B near p, we define the trace of $\mathcal{L}_{M,p}$ along V_B by

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$$\mathrm{tr}_{V_B}\mathcal{L}_{M,p} = \sum_{j=1}^{s} \langle [L_j, \overline{L_j}], \partial\rho \rangle (p).$$

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We define $c(V_B, p) = m$ if for any $m-3$ vector fields $F_1, \cdots, F_{m-3}$ of $\mathcal{M}_1(B)$, and any basis of sections of B, it holds that

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and for a certain choice of $m-2$ vector fields $G_1, \cdots, G_{m-2}$ of $\mathcal{M}_1(B)$, and a certain choice of sections of B, we have

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Then

$$c^{(s)}(M,p) = \sup_B\{c(V_B,p) : \ B \ \text{ is an } s\text{-dimensional subbundle of } T^{1,0}M\}.$$

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- The third invariant is defined by the degeneracy of the Levi form, it is always more easily to be applied.

- **Bloom-Graham (1977):** $a^{(n-1)}(M,p) = t^{(n-1)}(M,p)$.
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For these results, pseudo-convexity is not necessary.

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- **Conjecture:** When M is pseudo-convex, for $1 \leq s \leq n-1$, $a^{(s)}(M,p) = t^{(s)}(M,p) = c^{(s)}(M,p)$.

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- pseudo-convexity is necessary in this conjecture:

  Let $\rho = 2\text{Re}(w) + (z_2 + \overline{z_2} + |z_1|^2)^2$ and let $M = \{(z_1, z_2, w) \in \mathbb{C}^3 |\ \rho = 0\}$. Let $p = (0,0,0)$. Then $a^{(1)}(M,p) = 4$ but $c^{(1)}(M,p) = t^{(1)}(M,p) = \infty$.

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- When $M \subset \mathbb{C}^3$, $a^{(1)}(M,p) = c^{(1)}(M,p)$.

Huang-Y. (2021): When M is pseudo-convex,

$$a^{(n-2)}(M,p) = t^{(n-2)}(M,p) = c^{(n-2)}(M,p).$$

In particular, this gives a complete solution for $n = 3$.

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(In this case, $a^{(1)}(M,p) = c^{(1)}(M,p)$ is due to Abdallah TALHAOUI (1983))

A Conjecture of D'Angelo (1986)

Suppose M is pseudoconvex. Then for any fixed $(1, 0)$ tangent vector field L, we have $t^{(1)}(L, p) = c^{(1)}(L, p)$.

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The D'Angelo finite type

The first D'Angelo finite type:

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Here $\phi : (\mathbb{C}^{n-q+1},0) \to (\mathbb{C}^n,z_0)$ is a linear embedding.

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When z is required to be regular, this is exactly the regular finite type.

The Catlin multitype

Let Γ_n denote the set of all $n-$tuple of numbers $\Lambda = (\lambda_1, \cdots, \lambda_n)$ with $1 \leq \lambda_i \leq \infty$ such that $\lambda_1 \leq \cdots \leq \lambda_n$.

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Γ_n is called a weight if for each k, either $\lambda_k = +\infty$ or there is a set of nonnegative integers $a_1, \cdots, a_k$ with $a_k > 0$ such that $\sum_{j=1}^{k} \frac{a_j}{\lambda_j} = 1$

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Order of the weights: Let $\Lambda' = (\lambda'_1, \cdots, \lambda'_n)$ and $\Lambda'' = (\lambda''_1, \cdots, \lambda''_n)$.
$\Lambda' < \Lambda'$ if for some k, $\lambda'_j = \lambda''_j$ for $j < k$ and $\lambda'_k < \lambda''_k$.

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A weight $\Lambda \in \Gamma_n$ is said to be distinguished if there exist holomorphic coordinates $(z_1, \cdots, z_n)$ about z_0 with z_0 mapped to the origin such that

$$\sum_{j=1}^{n} D^{\alpha}\overline{D^{\beta}}\rho(z_0) = 0 \text{ for } \sum_{j=1}^{n} \frac{\alpha_j + \beta_j}{\lambda_j} < 1.$$

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The multitype $\mathcal{M}(z_0)$ is defined to be the smallest $(m_1, \cdots, m_n) \in \Gamma_n$ such that for every distinguished weight Λ, we have $\mathcal{M}(z_0) \geq \Lambda$.

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Notice that the Catlin multitype has a equivalent description by means of the degeneracy of the Levi form (in some sense) similar to the definition of $c^{(s)}(M,p)$, which is crucial to Catlin's solution of Kohn's subelliptic estimates problem.

Relation between these invariants

Example: Let $M \subset \mathbb{C}^4$ be a real hypersurface defined by

$$r = -2\text{Im}w + |z_1|^4 + |z_1|^2|z_2|^2 + |z_1|^2|z_3|^2 + |z_2^2 - z_3^3|^4.$$

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The Caltin multitypes at 0 are $4,\ 4,\ 4$,

The Bloom regular contact types are $4,\ 8,\ 12$,

The D'Angelo finite types are $4,\ 8,\ +\infty$.

Relation between these invariants

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It seems to me that the Bloom Conjecture for the boundary of a convex domain is also unknown.

Kohn's finite ideal type

As before, let D be a smooth pseudoconvex domain in $\mathbb{C}^n$. $x_0 \in M = bD$.

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Denote by $I^q(x_0)$ the set of germs of multipliers satisfying the following:

$\exists$ a neighborhood U of x_0, $f \in C_0^\infty(U \cap \overline{D})$ such that there are $C, \epsilon > 0$ for which

$$|||f\phi|||_\epsilon^2 \leq C(\|\overline{\partial}\phi\|^2 + \|\overline{\partial}^*\phi\|^2)$$

for all $\phi \in \mathcal{D}^{(p,q)}(U \cap D)$.

Kohn's finite ideal type

J. Kohn inductively defined the ideals $I_k^q(x_0)$ as follows:

$$I_1^q(x_0) = \sqrt[\mathbb{R}]{r, coeff.\{\partial r \wedge \overline{\partial} r \wedge (\partial\overline{\partial} r)^{n-q}\}}.$$

$$I_{k+1}^q(x_0) = \sqrt[\mathbb{R}]{I_k^q(x_0), coeff.\{\partial f_1 \wedge \cdots \wedge \partial f_j \wedge \partial r \wedge \overline{\partial} r \wedge (\partial\overline{\partial} r)^{n-q-j}\}}.$$

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Here $f_1, \cdots, f_j \in I_k^q(x_0)$.

We say x_0 is of finite ideal type with respect to (p, q) forms if there is a integer k such that $1 \in I_k^q(x_0)$.

Back to the subelliptic estimates

Theorem (J. Kohn 1979:)

Let D be a pseudoconvex domain in $\mathbb{C}^n$ with real analytic boundary. Then $1 \in I_k^q(x_0)$ if and only if $\Delta_q(M, x_0) < \infty$.

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Theorem (D. Catlin 1987:)

Let D be a pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then subelliptic estimates holds for (p, q) forms if and only if $\Delta_q(M, x_0) < \infty$.

Effectiveness

Let the domain is defined by $r = 2\mathrm{Re}(w) + |f_1(z)|^2 + \cdots + |f_m(z)|^2$, which is of D'Angelo finite type at the boundary point x_0.

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Siu(2010,2017): $1 \in I_\epsilon(x_0)$ with some ϵ bounded by constant depends on the finite type.

Kim-Zaitsev (2021): give a explicit effective bound.

- Find sone special $L \in T^{(1,0)}M'$, with M' another pseudoconvex hypersurface and L with weighted homogeneous coefficients.

Sketch of the proof for $n = 3$

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- The given condition means that the T direction is always transversal to N^0 at any point of N^0. Hence the dimension of N^0 must be 3 or 4.
- Comparing with Bloom's proof of $a^{(1)}(M,0) = c^{(1)}(M,0)$, we need to replace two deep theorems by K. Diederich and J. Fornaess (Annals, 1978).

Sketch of the proof

Theorem 1: Let S be a C^2-submanifold of a pseudoconvex C^4-hypersurface $M \subset \mathbb{C}^n$. Let X, Y be C^1-vector fields on S with values in $T^N S$. Then the vector field $[X, Y]$ also has values in $T^N S$ along S.

For all $p \in S$,

$$T_p^N S = \{X \in T_p S : \ X = \mathrm{Re} Y, \ Y \in T_p^{(1,0)} M, \ \partial\overline{\partial}\rho(Y, \overline{Y})(p) = 0\}.$$

Sketch of the proof

Theorem 2: Let $M \subset \mathbb{C}^n$ be a pseudoconvex C^∞ hypersurface with $0 \in M$ and $S \subset M$ a $C^\infty-$CR submanifold, $0 \in S$, with the following properties:

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Theorem 2: Let $M \subset \mathbb{C}^n$ be a pseudoconvex C^∞ hypersurface with $0 \in M$ and $S \subset M$ a C^∞−CR submanifold, $0 \in S$, with the following properties:

- $S \subset \mathbb{C}^{n-1} \times \{0\}$, rank $T^{(1,0)} = q,$ dim$_{\mathbb{R}} S = 2q + r$ with $q + r = n - 1$.
- $TS = T^N S$
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- By taking subsequent brackets of C^∞ vector fields with values in $T^h S$ one generates the whole tangent bundle TS.

Then in any neighborhood of 0, there is a relatively open set $\widehat{U}$ on M such that $\mathbb{C}^{n-1} \times \{0\}$ is tangent to bM of infinite order at all points $z \in \widehat{U}$.

Sketch of the proof

Theorem 1': Let N be a real analytic hypersurface in $\mathbb{C}^{n-1}$ with $0 \in N$ with $n \geq 3$. Let $\rho(z, \overline{z})$ be a real analytic plurisubharmonic function with $\rho = O(|z|^2)$ as $z \to 0$ defined over a neighborhood of $\mathbb{C}^{n-1}$. Assume that N is of finite type in the sense of Hömander–Bloom-Graham and $N \subset \{\rho = 0\}$. Then $\rho \equiv 0$.

Sketch of the proof

Theorem 2': Define the weight of z_1 and $\overline{z_1}$ to be 1, the weight of z_2 and $\overline{z_2}$ to be $k \in \mathbb{N}$ with $k > 1$. Let $A = A(z_1, \overline{z_1})$ be a homogenous polynomial of degree $k-1$ in $(z_1, \overline{z_1})$ without holomorphic terms. Suppose that f is a weighted homogeneous polynomial in $(z, \overline{z})$ of weighted degree $m > k$. Further assume that Re(f) is plurisubharmonic, contains no non-trivial holomorphic terms and assume that f satisfies the following equation:

$$f_{\overline{z_1}}(z, \overline{z}) + \overline{A(z_1, \overline{z_1})} f_{\overline{z_2}}(z, \overline{z}) = 0. \tag{0.1}$$

Then Re$(f) \equiv 0$.

The main difficulties

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- In Theorem 2', we need to solve a PDE with the real part plurisubharmonic.
- For higher dimensional case, we have to deal with the case:
  N CR-singular manifold and $r(z_1, \cdots, z_{n-1}, 0)|_N$ satisfies some PDE but is non-zero.

Thank you for your attention!