Transitory canard cycles

P. De Maesschalck, F. Dumortier, R. Roussarie

February 3, 2014

- The talk is about planar vector fields

$$\begin{cases} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

- More particularly about limit cycles
- Llibre: any configuration of limit cycles can be realized by polynomial vector fields
- Inversely: Hilbert 16th problem: find *uniform* upper bound for limit cycles depending on degree
- Extensive studies, several breakthroughs (Ilyashenko, Ecalle, Mourtada, Kaloshin, ...)
- Despite efforts, still no answer on H16. Degree 2: ≥ 4 cycles, expected ≤ 4 cycles
  Coefficient space = already 12 dimensional

- The talk is about planar vector fields

$$\begin{cases} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

- More particularly about limit cycles
- Llibre: any configuration of limit cycles can be realized by polynomial vector fields
- Inversely: Hilbert 16th problem: find *uniform* upper bound for limit cycles depending on degree
- Extensive studies, several breakthroughs (Ilyashenko, Ecalle, Mourtada, Kaloshin, ...)
- Despite efforts, still no answer on H16. Degree 2: ≥ 4 cycles, expected ≤ 4 cycles
  Coefficient space $=$ already 12 dimensional

- The talk is about planar vector fields

$$\begin{cases} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

- More particularly about limit cycles
- Llibre: any configuration of limit cycles can be realized by polynomial vector fields
- Inversely: Hilbert 16th problem: find *uniform* upper bound for limit cycles depending on degree
- Extensive studies, several breakthroughs (Ilyashenko, Ecalle, Mourtada, Kaloshin, ...)
- Despite efforts, still no answer on H16. Degree 2: ≥ 4 cycles, expected ≤ 4 cycles
  Coefficient space = already 12 dimensional

- The talk is about planar vector fields

$$\begin{cases} \dot{x} & = & f(x,y) \\ \dot{y} & = & g(x,y) \end{cases}$$

- More particularly about limit cycles
- Llibre: any configuration of limit cycles can be realized by polynomial vector fields
- Inversely: Hilbert 16th problem: find *uniform* upper bound for limit cycles depending on degree
- Extensive studies, several breakthroughs (Ilyashenko, Ecalle, Mourtada, Kaloshin, ...)
- Despite efforts, still no answer on H16. Degree 2: ≥ 4 cycles, expected ≤ 4 cycles
  Coefficient space = already 12 dimensional

- The talk is about planar vector fields

$$\begin{cases} \dot{x} & = & f(x,y) \\ \dot{y} & = & g(x,y) \end{cases}$$

- More particularly about limit cycles
- Llibre: any configuration of limit cycles can be realized by polynomial vector fields
- Inversely: Hilbert 16th problem: find *uniform* upper bound for limit cycles depending on degree
- Extensive studies, several breakthroughs (Ilyashenko, Ecalle, Mourtada, Kaloshin, ...)
- Despite efforts, still no answer on H16. Degree 2: ≥ 4 cycles, expected ≤ 4 cycles

  Coefficient space = already 12 dimensional

- The talk is about planar vector fields

$$\begin{cases} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

- More particularly about limit cycles
- Llibre: any configuration of limit cycles can be realized by polynomial vector fields
- Inversely: Hilbert 16th problem: find *uniform* upper bound for limit cycles depending on degree
- Extensive studies, several breakthroughs (Ilyashenko, Ecalle, Mourtada, Kaloshin, ...)
- Despite efforts, still no answer on H16. Degree 2: ≥ 4 cycles, expected ≤ 4 cycles
  Coefficient space = already 12 dimensional

Now it becomes a question of studying a complete coefficient space.

$$\begin{cases} \dot{x} &= \sum_{i+j\leq N} a_{ij}x^i y^j \\ \dot{y} &= \sum_{i+j\leq N} b_{ij}x^i y^j \end{cases}$$

Idea of Dumortier-Roussarie-Rousseau: compactify parameter space + phase space

Reduces the global study to local study around limit periodic sets.

Limit periodic sets can be

- Single points
- Annuli of periodic orbits (Hamiltonian systems)
- Double saddle connections
- More general graphics
- Degenerate graphics (containing nonisolated singular points)

For degree $N = 2$: reduction to 121 cases to be treated. Leftover are difficult ones.

Now it becomes a question of studying a complete coefficient space.

$$\begin{cases} \dot{x} &= \sum_{i+j\leq N} a_{ij}x^i y^j \\ \dot{y} &= \sum_{i+j\leq N} b_{ij}x^i y^j \end{cases}$$

Idea of Dumortier-Roussarie-Rousseau: compactify parameter space + phase space

Reduces the global study to local study around limit periodic sets.

Limit periodic sets can be

- Single points
- Annuli of periodic orbits (Hamiltonian systems)
- Double saddle connections
- More general graphics
- Degenerate graphics (containing nonisolated singular points)

For degree $N = 2$: reduction to 121 cases to be treated. Leftover are difficult ones.

Now it becomes a question of studying a complete coefficient space.

$$\begin{cases} \dot{x} &= \sum_{i+j\leq N} a_{ij}x^i y^j \\ \dot{y} &= \sum_{i+j\leq N} b_{ij}x^i y^j \end{cases}$$

Idea of Dumortier-Roussarie-Rousseau: compactify parameter space + phase space

Reduces the global study to local study around limit periodic sets.

Limit periodic sets can be

- Single points
- Annuli of periodic orbits (Hamiltonian systems)
- Double saddle connections
- More general graphics
- Degenerate graphics (containing nonisolated singular points)

For degree $N = 2$: reduction to 121 cases to be treated. Leftover are difficult ones.

In 2000, Smale repeated Hilbert's 16th problem as Smale's 13th problem. Now specifically for Liénard systems

$$\begin{cases} \dot{x} &= y - F(x) \\ \dot{y} &= -x \end{cases}$$

This contains only 1 singular point, and configurations of limit cycles are easy: they all surround each other.

Recent results up to degree 4 cyclicity is one (Llibre, Li). In degree 5 not yet known whether cycl is either 2 or 3.

Conjecture Lins de Melo Pugh [1976]: $\leq (n-1)/2$ cycles for degree n. It reveals not to be true.

Best lower bound known until now: paper submitted by Huzak-PDM (5 days ago): $\geq n-2$. Still linear where Smale predicts at most n^d. (My guess at most quadratic)

In 2000, Smale repeated Hilbert's 16th problem as Smale's 13th problem. Now specifically for Liénard systems

$$\begin{cases} \dot{x} &= y - F(x) \\ \dot{y} &= -x \end{cases}$$

This contains only 1 singular point, and configurations of limit cycles are easy: they all surround each other.
Recent results up to degree 4 cyclicity is one (Llibre, Li). In degree 5 not yet known whether cycl is either 2 or 3.
Conjecture Lins de Melo Pugh [1976]: $\leq (n-1)/2$ cycles for degree n. It reveals not to be true.
Best lower bound known until now: paper submitted by Huzak-PDM (5 days ago): $\geq n-2$. Still linear where Smale predicts at most n^d. (My guess at most quadratic)

In 2000, Smale repeated Hilbert's 16th problem as Smale's 13th problem. Now specifically for Liénard systems

$$\begin{cases} \dot{x} & = & y - F(x) \\ \dot{y} & = & -x \end{cases}$$

This contains only 1 singular point, and configurations of limit cycles are easy: they all surround each other.
Recent results up to degree 4 cyclicity is one (Llibre, Li). In degree 5 not yet known whether cycl is either 2 or 3.
Conjecture Lins de Melo Pugh [1976]: $\leq (n-1)/2$ cycles for degree n. It reveals not to be true.
Best lower bound known until now: paper submitted by Huzak-PDM (5 days ago): $\geq n-2$. Still linear where Smale predicts at most n^d. (My guess at most quadratic)

In 2000, Smale repeated Hilbert's 16th problem as Smale's 13th problem. Now specifically for Liénard systems

$$\begin{cases} \dot{x} &= y - F(x) \\ \dot{y} &= -x \end{cases}$$

This contains only 1 singular point, and configurations of limit cycles are easy: they all surround each other.
Recent results up to degree 4 cyclicity is one (Llibre, Li). In degree 5 not yet known whether cycl is either 2 or 3.
Conjecture Lins de Melo Pugh [1976]: $\leq (n-1)/2$ cycles for degree n. It reveals not to be true.
Best lower bound known until now: paper submitted by Huzak-PDM (5 days ago): $\geq n-2$. Still linear where Smale predicts at most n^d. (My guess at most quadratic)

In 2000, Smale repeated Hilbert's 16th problem as Smale's 13th problem. Now specifically for Liénard systems

$$\begin{cases} \dot{x} &= y - F(x) \\ \dot{y} &= -x \end{cases}$$

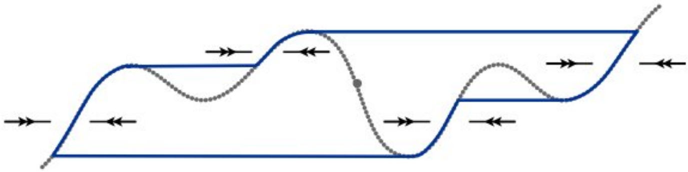
This contains only 1 singular point, and configurations of limit cycles are easy: they all surround each other.
Recent results up to degree 4 cyclicity is one (Llibre, Li). In degree 5 not yet known whether cycl is either 2 or 3.
Conjecture Lins de Melo Pugh [1976]: $\leq (n-1)/2$ cycles for degree n. It reveals not to be true.
Best lower bound known until now: paper submitted by Huzak-PDM (5 days ago): $\geq n-2$. Still linear where Smale predicts at most n^d. (My guess at most quadratic)

Compactification argument applied to Liénard vector fields $\Longrightarrow$ only degenerate graphics need to be studied.

$$\begin{cases} \dot{x} &= y - F(x) \\ \dot{y} &= -\epsilon x \end{cases}$$

This is a *slow-fast vector field*. We have relaxation oscillations and annuli of periodic orbits.

- Common cycles
- Canard cycles and breaking mechanisms
- Single layer canard cycles

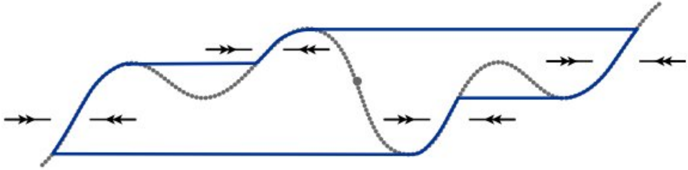


Compactification argument applied to Liénard vector fields $\Longrightarrow$ only degenerate graphics need to be studied.

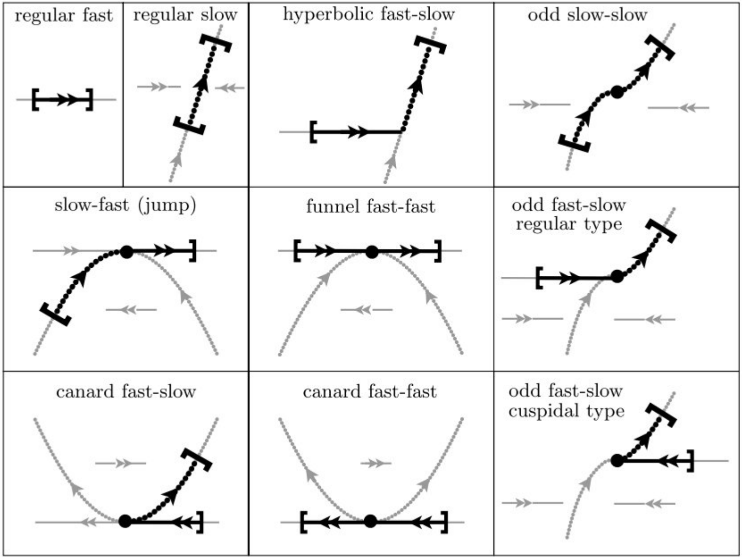
$$\begin{cases} \dot{x} &= y - F(x) \\ \dot{y} &= -\epsilon G(x) \end{cases}$$

This is a *slow-fast vector field*. We have relaxation oscillations and annuli of periodic orbits.

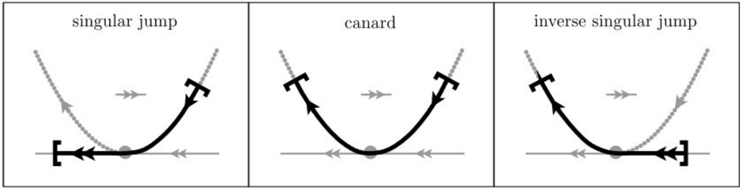
- Common cycles
- Canard cycles and breaking mechanisms
- Single layer canard cycles



Common cycles:



Canard cycles:

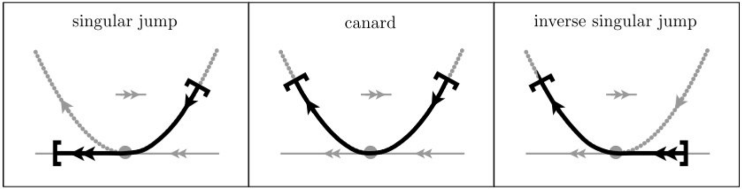


Canard cycles where first studied in Van der Pol:

$$\begin{cases} \dot{x} &= y - \frac{1}{2}x^2 - \frac{1}{3}x^3 \\ \dot{y} &= \epsilon(\lambda - x) \end{cases}$$

Canard cycles are due to a *Hopf mechanism*.
Canard cycles can also be created in a *jump breaking mechanism*.

Canard cycles:



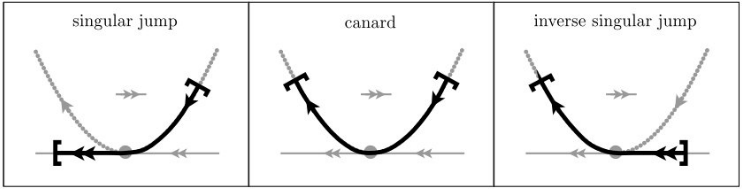
Canard cycles where first studied in Van der Pol:

$$\begin{cases} \dot{x} &= y - \frac{1}{2}x^2 - \frac{1}{3}x^3 \\ \dot{y} &= \epsilon(\lambda - x) \end{cases}$$

Canard cycles are due to a *Hopf mechanism*.

Canard cycles can also be created in a *jump breaking mechanism*.

Canard cycles:



Canard cycles where first studied in Van der Pol:

$$\begin{cases} \dot{x} &= y - \frac{1}{2}x^2 - \frac{1}{3}x^3 \\ \dot{y} &= \epsilon(\lambda - x) \end{cases}$$

Canard cycles are due to a *Hopf mechanism*.
Canard cycles can also be created in a *jump breaking mechanism*.

Study of normally hyperbolic arcs: Slow divergency integral
Study of a simple slow-fast system

$$\begin{cases} \dot{x} & = & \lambda(y)x \\ \dot{y} & = & \epsilon \end{cases}$$

Clearly,

$$x(y) = x_0 \exp \frac{1}{\epsilon} \int_{y_0}^{y} \lambda(s) ds$$

Since $x = 0$ is invariant we interpret this as: orbits approach the invariant manifold exponentially fast, and the exponential speed can be measured by looking at the divergence integral.

For the system

$$\begin{cases} \dot{x} & = & y - F(x) \\ \dot{y} & = & \epsilon G(x) \end{cases}$$

We write the slow divergence integral from $p_1 = (x_1, y_1)$ to $p2 = (x_2, y_2)$ as

$$I = \int_{y_1}^{y_2} F'(x) \frac{dy}{G(x)} = \int_{x_1}^{x_2} \frac{F'(x)^2}{G(x)} dx.$$

$\implies$ exponential attraction along invariant manifold can be expressed using I.

Points of nonhyperbolicity: studied via blow-up
We consider generic points

$$\begin{cases} \dot{x} &= y - x^n + h.o.t. + unfolding \\ \dot{y} &= \epsilon(-1 + h.o.t.) \end{cases}$$

Hopf points

$$\begin{cases} \dot{x} &= y - x^n + h.o.t. + unfolding \\ \dot{y} &= \epsilon(\lambda - x + h.o.t.) \end{cases}$$

BT points

$$\begin{cases} \dot{x} &= y - x^n + h.o.t. + unfolding \\ \dot{y} &= \epsilon(\lambda + \mu x \pm x^2 + h.o.t.) \end{cases}$$

codim-3 points . . .

Points of nonhyperbolicity: studied via blow-up
We consider generic points

$$\left\{\begin{array}{rcl}\dot{x} & = & y - x^n + h.o.t. + unfolding \\ \dot{y} & = & \epsilon(-1 + h.o.t.)\end{array}\right.$$

Hopf points

$$\left\{\begin{array}{rcl}\dot{x} & = & y - x^n + h.o.t. + unfolding \\ \dot{y} & = & \epsilon(\lambda - x + h.o.t.)\end{array}\right.$$

BT points

$$\left\{\begin{array}{rcl}\dot{x} & = & y - x^n + h.o.t. + unfolding \\ \dot{y} & = & \epsilon(\lambda + \mu x \pm x^2 + h.o.t.)\end{array}\right.$$

codim-3 points ...

Theorem

Any common cycle or canard cycle containing only generic or hopf points of nonhyperbolicity, and with a slow divergence integral that is nonzero perturbs to at most 1 cycle.

Study of a limit periodic set of canard type with $I = 0$:

- Consider the poincaré-map P from a transverse section to itself and find fixed points
- Decompose $P = R \circ L^{-1}$ and deduce $P(x) = x$ if and only if $L(x) = R(x)$ (study zeros of difference maps)
- Write

  $$L(x) = \phi_L(\epsilon) + \exp \frac{I_L(x, \epsilon)}{\epsilon}$$

  and

  $$R(x) = \phi_R(\epsilon) + \exp \frac{I_R(x, \epsilon)}{\epsilon}$$

  and deduce using Rolle it suffices to discuss $I_L = I_R$.

The equation $I = I_L - I_R = 0$ is not smooth, but only continuous in ϵ, and smooth in s and bifurcation parameters.
The equation for $\epsilon = 0$ is the slow divergence integral and can be computed exactly.
$\implies$ the bifurcation diagram of $I = 0$ is strongly related to the bifurcation diagram of fixed points of the Poincaré map.
$\implies$ this leads to numerous results in various papers.
Example: When I has a simple zero, there perturb at most 2 cycles.
Can be extended to any codimension of I.

The equation $I = I_L - I_R = 0$ is not smooth, but only continuous in ϵ, and smooth in s and bifurcation parameters.
The equation for $\epsilon = 0$ is the slow divergence integral and can be computed exactly.
$\Longrightarrow$ the bifurcation diagram of $I = 0$ is strongly related to the bifurcation diagram of fixed points of the Poincaré map.
$\Longrightarrow$ this leads to numerous results in various papers.
Example: When I has a simple zero, there perturb at most 2 cycles.
Can be extended to any codimension of I.

The equation $I = I_L - I_R = 0$ is not smooth, but only continuous in ϵ, and smooth in s and bifurcation parameters.
The equation for $\epsilon = 0$ is the slow divergence integral and can be computed exactly.
$\implies$ the bifurcation diagram of $I = 0$ is strongly related to the bifurcation diagram of fixed points of the Poincaré map.
$\implies$ this leads to numerous results in various papers.
Example: When I has a simple zero, there perturb at most 2 cycles.
Can be extended to any codimension of I.

Above technique only works for 1-layer cycles.
For generalized Liénard systems

$$\left\{ \begin{array}{rcl} \dot{x} & = & y - F(x) \\ \dot{y} & = & -\epsilon g(x) \end{array} \right.$$

we can have multi-layer canard cycles.
$\Longrightarrow$ fixed points of the Poincaré map are now related to fixed points of iterations of power functions

$$x \mapsto a + x^r.$$

Open problem: found lower/upper bounds for the number of fixed points.

Above technique only works for 1-layer cycles.
For generalized Liénard systems

$$\begin{cases} \dot{x} &= y - F(x) \\ \dot{y} &= -\epsilon g(x) \end{cases}$$

we can have multi-layer canard cycles.
$\Longrightarrow$ fixed points of the Poincaré map are now related to fixed points of iterations of power functions

$$x \mapsto a + x^r.$$

Open problem: found lower/upper bounds for the number of fixed points.

Above technique only works on the inside of a layer of limit periodic sets, not at the boundary!

Boundary from below: birth of canard cycles.

We blow-up and find a nonsingular Hopf bifurcation (but possibly degenerate).

$$\left\{\begin{array}{rcl}\dot{x} & = & y - x^2F(x) \\ \dot{y} & = & \epsilon(\lambda - x)\end{array}\right.$$

Write

$$x = uX, y = u^2Y, \epsilon = u^2, \lambda = uL$$

$$\left\{\begin{array}{rcl}\dot{X} & = & Y - X^2F(0) \\ \dot{Y} & = & L - X\end{array}\right.$$

Limit cycles are bounded by computing Abelian integrals. Small canard cycles are dealt with using relations between Abelian integrals and slow divergence integrals.

Boundary from above: transitions.

Above technique only works on the inside of a layer of limit periodic sets, not at the boundary!
Boundary from below: birth of canard cycles.
We blow-up and find a nonsingular Hopf bifurcation (but possibly degenerate).

$$\begin{cases} \dot{x} &= y - x^2 F(x) \\ \dot{y} &= \epsilon(\lambda - x) \end{cases}$$

Write

$$x = uX, y = u^2 Y, \epsilon = u^2, \lambda = uL$$

$$\begin{cases} \dot{X} &= Y - X^2 F(0) \\ \dot{Y} &= L - X \end{cases}$$

Limit cycles are bounded by computing Abelian integrals. Small canard cycles are dealt with using relations between Abelian integrals and slow divergence integrals.
Boundary from above: transitions.

Above technique only works on the inside of a layer of limit periodic sets, not at the boundary!
Boundary from below: birth of canard cycles.
We blow-up and find a nonsingular Hopf bifurcation (but possibly degenerate).

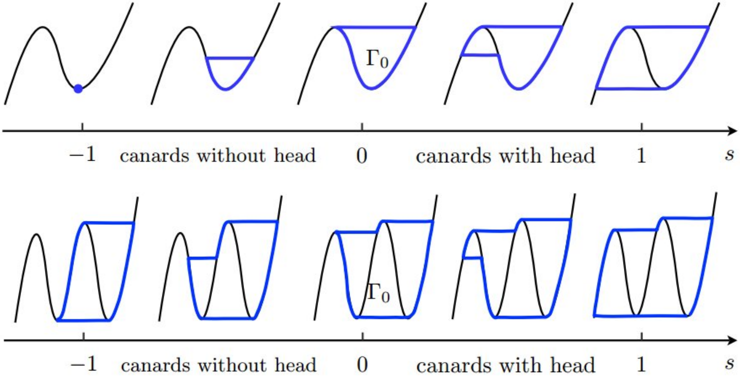
$$\left\{ \begin{array}{rcl} \dot{x} & = & y - x^2 F(x) \\ \dot{y} & = & \epsilon(\lambda - x) \end{array} \right.$$

Write

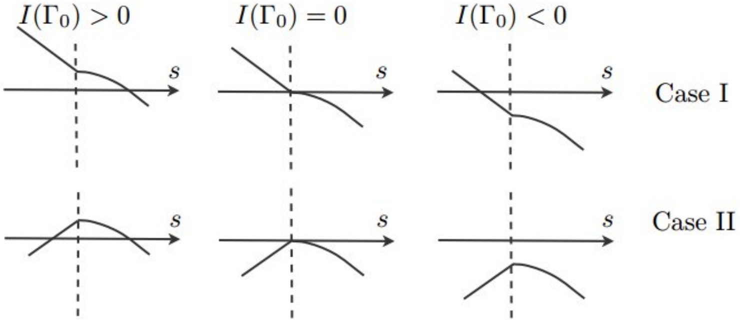
$$x = uX, y = u^2 Y, \epsilon = u^2, \lambda = uL$$

$$\left\{ \begin{array}{rcl} \dot{X} & = & Y - X^2 F(0) \\ \dot{Y} & = & L - X \end{array} \right.$$

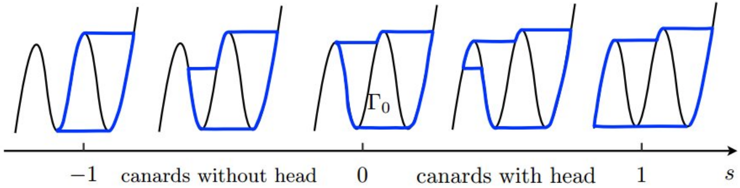
Limit cycles are bounded by computing Abelian integrals. Small canard cycles are dealt with using relations between Abelian integrals and slow divergence integrals.
Boundary from above: transitions.



Γ_0
-1
canards without head
0
canards with head
1
s
Γ_0
-1
canards without head
0
canards with head
1
s


$I(\Gamma_0) > 0$
$I(\Gamma_0) = 0$
$I(\Gamma_0) < 0$
s
s
s
Case I
s
s
s
Case II

1. We have seen that the **slow divergence integral** of the canard cycle $\Gamma_\mu(y)$, is the tool to study the generic canard cycles with one breaking parameter (in the interior of the layer) . This integral is also defined for the transitory canard cycles.
2. If the **slow divergence integral of a canard cycle (transitory or not) is not zero,** then the canard cycle bifurcates in a single hyperbolic limit cycle. It is the easy case.
3. **For this reason, we will assume that the slow divergence integral of the transitory canard cycle $\Gamma_{\mu_0}(0)$ is zero, for some parameter value μ_0 and we will consider $\mu \sim \mu_0$.**

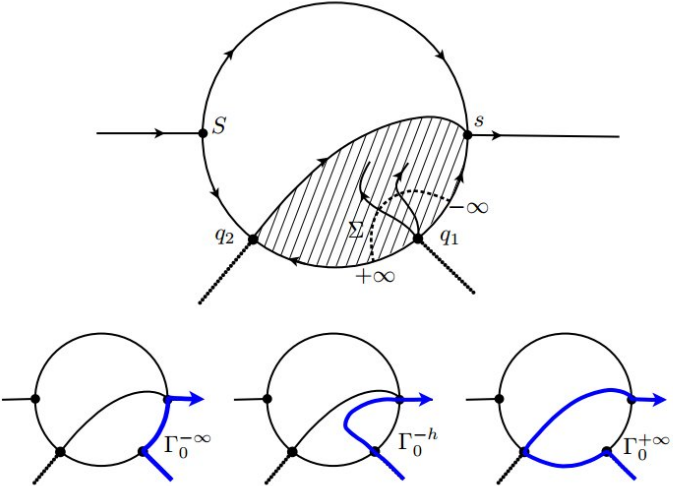


Theorem

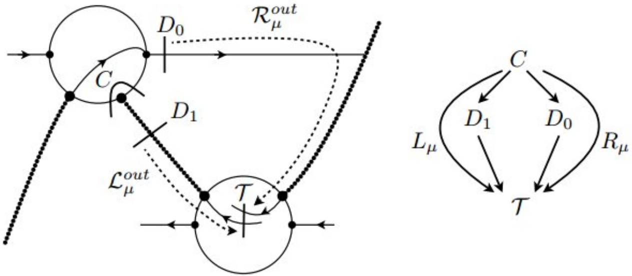
At most two cycles bifurcate from the transitory cycle in case 1 and at most three cycles in case 2 (counted with multiplicity).

In a complete unfolding of a transitory cycle, there will be a cycle of multiplicity 3, but the location is fixed.

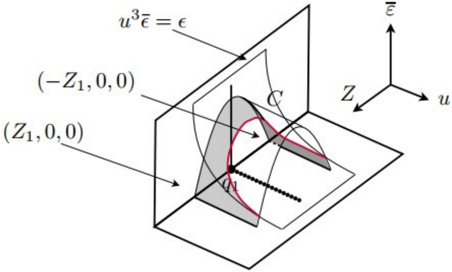
The transitory canard blows up to a full family of limit periodic sets, family with boundary.

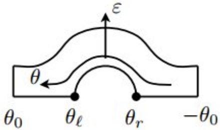


Overview of the forward and backward transition maps. The regularity of those maps is limited, but just enough information can be deduced.

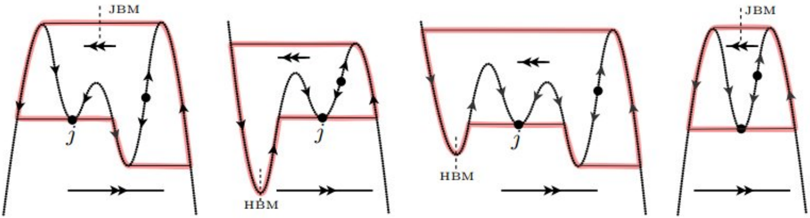


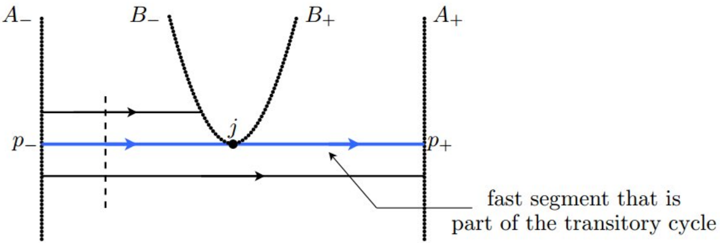
The difference map is from a section C to another section T, where C is not a simple rectangle parameterized by (y, ϵ). Instead it is parameterized by blow-up variables $(u, \bar{\epsilon})$, in a foliated region.

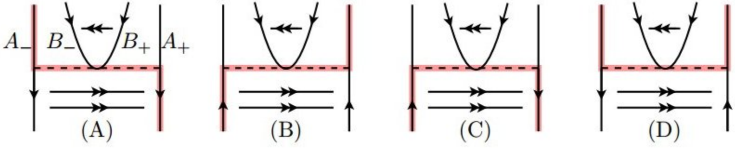




A second transition case




A_-
B_-
B_+
A_+
j
p_-
p_+
fast segment that is part of the transitory cycle



A_-
B_-
B_+
A_+
(A)
(B)
(C)
(D)

Exactly as for a regular value of the layer variable we can introduce a notion of codimension for a canard cycle at the transitory level $y = 0$. To this end we use smooth function I_- defined for $y \leq 0$:

Definition

We we say that the fast-fast transitory canard cycle $\Gamma_{\mu_0}(0)$ has a codimension equal to $k \geq 1$ if and only if

$$I_-(0, \mu_0, 0) = \cdots = \frac{\partial^{k-1} I_-}{\partial y^{k-1}}(0, \mu_0, 0) = 0 \text{ and } \frac{\partial^k I_-}{\partial y^k}(0, \mu_0, 0) \neq 0.$$

We can of course extend the definition for $k = 0$ by $I(0, \mu_0, 0) \neq 0$, but here we will just consider a codimension ≥ 1. The codimension is equal to one in the cases (A) and (B). The study in these two cases is rather trivial in comparison to the other cases. Then a codimension > 1 may just occur in the cases (C) and (D). We will denote by $(C)_k$ and $(D)_k$ the corresponding cases of codimension $k \geq 1$.

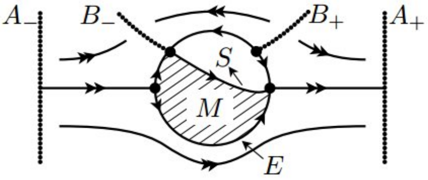
Result for the fast-fast canard cycles

Theorem *Let us consider a fast-fast transitory canard cycle* $\Gamma_{\mu_0}(0)$ *of codimension equal to* $k \geq 1$. *Then :*

If the $k = 1$ *(i.e. if* $\frac{\partial I_l}{\partial y}(0, \mu_0, 0)/\frac{\partial I_r}{\partial y}(0, \mu_0, 0) \neq 1$*), the cyclicity of* $\Gamma_{\mu_0}(0)$ *is less than* 4 *when* $\frac{\partial I_l}{\partial y}(0, \mu_0, 0)/\frac{\partial I_r}{\partial y}(0, \mu_0, 0) \notin]0, 1[$ *and is less than* 5 *when* $\frac{\partial I_l}{\partial y}(0, \mu_0, 0)/\frac{\partial I_r}{\partial y}(0, \mu_0, 0) \in]0, 1[$,

If the $k \geq 2$ *(i.e. if* $\frac{\partial I_l}{\partial y}(0, \mu_0, 0)/\frac{\partial I_r}{\partial y}(0, \mu_0, 0) = 1$*), the cyclicity of* Γ *is less than* $k + 4$.

Remark The bounds are probably not optimal .



Conclusions:

- Smale's 13th problem reduces to slow-fast equations
- All limit periodic sets are slow-fast cycles.
- Those cycles that have slow divergence integral nonzero are treated completely.
- Multilayer cycles are still problematic.
- Left to study: transitory canard cycles, of which are already studied:
  - The birth of canards (for generic Hopf)
  - slow-fast transition
  - fast-fast transition

Thank you for your attention!