Winter School on PDEs
St Etienne de Tinée
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Hamiltonian partial differential equations and Painlevé transcendents

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- Cauchy problem for evolutionary PDEs with slow varying initial data

- Introduction into Hamiltonian PDEs

- On perturbative approach to integrability

- Phase transitions from regular to oscillatory behavior. Universality conjecture and Painlevé transcendents

System

$$\mathbf{u}_t = \mathbf{F}\left(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \dots\right) \qquad (1)$$

R.h.s. analytic in jets at $(\mathbf{u}; 0, 0, \dots)$

and $\mathbf{F}\left(\mathbf{u}; 0, 0, \dots\right) \equiv 0$

$\Rightarrow$ n-dimensional family of constant solutions (n=dim $\mathbf{u}$)

Slow-varying solutions $\mathbf{u}(x) = \mathbf{f}(\epsilon\, x), \quad \mathbf{u}_x = \mathcal{O}(\epsilon), \quad \mathbf{u}_{xx} = \mathcal{O}\left(\epsilon^2\right)$

Rescale $x \mapsto \epsilon\, x, \quad t \mapsto \epsilon\, t$ and expand

$$\begin{aligned} \mathbf{u}_t &= \tfrac{1}{\epsilon}\mathbf{F}\left(\mathbf{u}; \epsilon\, \mathbf{u}_x, \epsilon^2 \mathbf{u}_{xx}, \dots\right) \\ &= A(\mathbf{u})\mathbf{u}_x + \epsilon\left[B_1(\mathbf{u})\mathbf{u}_{xx} + B_2(\mathbf{u})\mathbf{u}_x^2\right] + \epsilon^2\left[C_1(\mathbf{u})\mathbf{u}_{xxx} + C_2(\mathbf{u})\mathbf{u}_{xx}\mathbf{u}_x + C_3(\mathbf{u})\mathbf{u}_x^3\right] + \dots \end{aligned}$$

ϵ -independent initial data $\mathbf{u}(x,t) = \mathbf{u}_0(x)$

Question 1: existence of solutions to (1) with slow varying initial data

Idea: for small times solutions to (1)

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + \epsilon\left[B_1(\mathbf{u})\mathbf{u}_{xx} + B_2(\mathbf{u})\mathbf{u}_x^2\right] + \epsilon^2\left[C_1(\mathbf{u})\mathbf{u}_{xxx} + C_2(\mathbf{u})\mathbf{u}_{xx}\mathbf{u}_x + C_3(\mathbf{u})\mathbf{u}_x^3\right]\ldots \quad (1)$$

and

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x \quad (2)$$

with the same initial date are close

before the time of gradient catastrophe of (2)

Question 2: comparison of solutions to (1) ("perturbed") and (2) ("unperturbed") near the point of gradient catastrophe of (2) = point of phase transition of (1)

Two main classes: dissipative perturbations,
e.g., Burgers equation

$$u_t + u\, u_x = \epsilon\, u_{xx}$$

shock waves;

or Hamiltonian (i.e., conservative) perturbations,
e.g., Korteweg - de Vries equation

$$u_t + u\, u_x + \epsilon^2 u_{xxx} = 0$$

oscillatory behaviour

Examples of Hamiltonian PDEs

1) KdV $\quad u_t + u\,u_x + \frac{\epsilon^2}{12} u_{xxx} = 0$

$$\Leftrightarrow \quad u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0 \qquad H = \int \left[\frac{u^3}{6} - \frac{\epsilon^2}{24} u_x^2 \right] dx$$

In the zero dispersion limit $\epsilon = 0 \quad \Rightarrow$ Hopf equation

$$u_t + u\,u_x = 0$$

2) Toda lattice

$$\left.\begin{aligned} \epsilon\, u_t &= v(x) - v(x-\epsilon) \\ \epsilon\, v_t &= e^{u(x+\epsilon)} - e^{u(x)} \end{aligned}\right\} \qquad (n=2)$$

Long wave limit

$$\left.\begin{aligned} u_t &= v_x \\ v_t &= e^u u_x \end{aligned}\right\}$$

More general class of systems of the Fermi-Pasta-Ulam type

$$H = \sum_{n=1}^{N} \frac{p_n^2}{2} + V(q_n - q_{n-1})$$

For large N the equations of motion can be replaced by

$$\left.\begin{aligned} u_t &= \tfrac{1}{\epsilon}[v(x) - v(x-\epsilon)] \\ v_t &= \tfrac{1}{\epsilon}\left[V'(u(x+\epsilon)) - V'(u(x)\right] \end{aligned}\right\}$$

$$p_n = v(n\epsilon), \quad q_n - q_{n-1} = u(n\epsilon), \quad \epsilon = \frac{1}{N}$$

In the leading term one obtains an integrable PDE

$$\left.\begin{aligned} u_t &= v_x \\ v_t &= V''(u)u_x \end{aligned}\right\}$$

3) Nonlinear Schrödinger equation

$$i\epsilon\,\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + |\psi|^2\psi = 0$$

In the real-valued variables

$$u = |\psi|^2, \quad v = \frac{\epsilon}{2i}\left(\frac{\psi_x}{\psi} - \frac{\bar{\psi}_x}{\bar{\psi}}\right)$$

can be recast into the form

$$u_t + (u\,v)_x = 0$$

$$v_t + v\,v_x - u_x + \frac{\epsilon^2}{4}\left(\frac{1}{2}\frac{u_x^2}{u^2} - \frac{u_{xx}}{u}\right)_x = 0$$

The Hamiltonian formulation

$$u_t + \frac{\partial}{\partial x}\frac{\delta H}{\delta v(x)} = 0$$

$$v_t + \frac{\partial}{\partial x}\frac{\delta H}{\delta u(x)} = 0$$

$$H = \int \left[\frac{1}{2}\left(u\,v^2 - u^2\right) + \frac{\epsilon^2}{8u}u_x^2\right] dx$$

Hamiltonian operator $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}$

General setting

Class of systems of PDEs depending on a small parameter ϵ

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + \epsilon\, A_2(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}) + \epsilon^2 A_3(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}) + \dots$$

$$\mathbf{u} = (u^1, \dots, u^n)$$

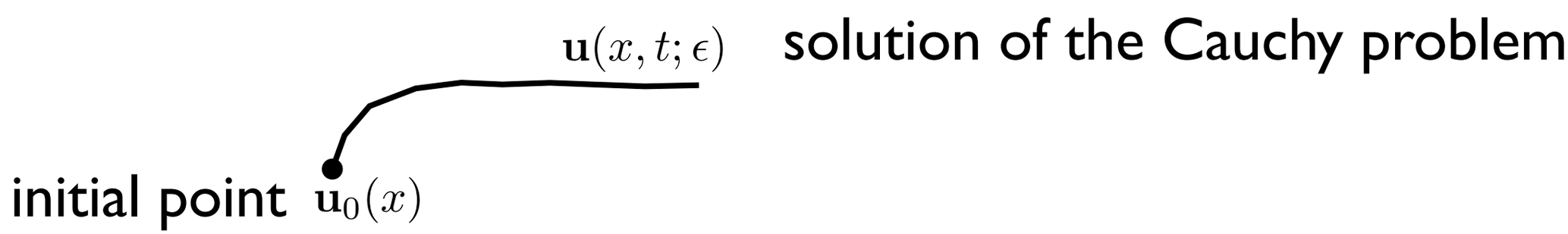
Terms of order ϵ^k are differential polynomals

of degree $k+1$

$$\deg u^{(m)} = m, \quad m = 1, 2, \dots$$

ε -dependent dynamical systems

on the space of vector-functions $\mathbf{u}(x)$



Hamiltonian formulation

$$\mathbf{u}_t = \mathbf{F}\left(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \dots\right) = P\frac{\delta H}{\delta \mathbf{u}(x)}$$

Hamiltonian *local* functional

$$H = \frac{1}{2\pi}\int_0^{2\pi} h\left(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \dots\right)\, dx$$

(the case of 2π-periodic function)

$$\frac{\delta H}{\delta \mathbf{u}(x)} =$$ Euler - Lagrange operator of h

$$\frac{\delta H}{\delta u^i(x)} = \frac{\partial h}{\partial u^i} - \partial_x \frac{\partial h}{\partial u^i_x} + \partial_x^2 \frac{\partial h}{\partial u^i_{xx}} - \dots$$

Finally, in the representation

$$\mathbf{u}_t = \mathbf{F}\left(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \dots\right) = P \frac{\delta H}{\delta \mathbf{u}(x)}$$

$P = \left(P^{ij}\right)$ is the matrix-valued operator of Poisson bracket

$$\{F, G\} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta F}{\delta u^i(x)} P^{ij} \frac{\delta G}{\delta u^j(x)} \, dx$$

- bilinearity
- skew symmetry $\{G, F\} = -\{F, G\}$
- Jacobi identity

$$\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0$$

For example, any linear skew-symmetric matrix operator with **constant** coefficients

$$P^{ij} = \sum_k P_k^{ij} \partial_x^k$$

$$P_k^{ji} = (-1)^{k+1} P_k^{ij}$$

defines a Poisson bracket

For KdV $P = \partial_x,$ for NLS $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x$

More general class

$$P^{ij} = \sum_k P_k^{ij}(\mathbf{u}; \mathbf{u}_x, \dots)\partial_x^k$$

After rescaling $x \mapsto \epsilon\, x$ obtain

$$P^{ij} = \sum_{k \geq 0} \epsilon^k \sum_{s=0}^{k+1} P^{ij}_{k,s}(\mathbf{u}; \mathbf{u}_x, \ldots, \mathbf{u}^{(s)}) \partial_x^{k-s+1}$$

$$\deg P^{ij}_{k,s}(\mathbf{u}; \mathbf{u}_x, \ldots, \mathbf{u}^{(s)}) = s$$

This class is invariant with respect of the group of *Miura-type transformations* of the form

$$\mathbf{u} \mapsto \tilde{\mathbf{u}} = F_0(\mathbf{u}) + \epsilon\, F_1(\mathbf{u}, \mathbf{u}_x) + \epsilon^2 F_2(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}) + \dots$$

$$\deg F_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)}) = k$$

$$\det\left(\frac{DF_0(\mathbf{u})}{D\mathbf{u}}\right) \neq 0$$

Thm. Assuming $\det\left(P^{ij}_{0,0}(\mathbf{u})\right) \neq 0$, any Hamiltonian operator

is equivalent to $P = \text{constant symmetric matrix} \cdot \partial_x$

Proof uses the theory of Poisson brackets of hydrodynamic type (B.D., S.Novikov, 1983) and triviality of Poisson cohomology (E. Getzler; F.Magri et al., 2001)

So, any Hamiltonian PDEs can be written in the form

$$u^i_t = \eta^{ij} \frac{\partial}{\partial x} \frac{\delta H}{\delta u^j(x)}, \quad i = 1, \ldots, n, \quad \det(\eta^{ij}) \neq 0, \quad \eta^{ji} = \eta^{ij}$$

Local Hamiltonians

$$H = H_0 + \epsilon\, H_1 + \epsilon^2 H_2 + \cdots = \int [h_0(\mathbf{u}) + \epsilon\, h_1(\mathbf{u}; \mathbf{u}_x) + \epsilon^2 h_2(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}) + \ldots] dx$$

Perturbative approach to integrability of PDEs

(B.D., Youjin Zhang)

Integrable Hamiltonian system $\dot{\mathbf{u}} = \{\mathbf{u}, H_0\}$

Complete family of commuting first integrals

$$\{H_0, F_0^{[i]}\} = 0, \quad i = 1, 2, \ldots$$

$$\{F_0^{[i]}, F_0^{[j]}\} = 0$$

Definition. Perturbed Hamiltonian $H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \ldots$

is integrable if **every** $F_0^{[i]}$ can be deformed

$$F^{[i]} = F_0^{[i]} + \epsilon F_1^{[i]} + \epsilon^2 F_2^{[i]} + \ldots, \quad i = 1, 2, \ldots$$

to a first integral of H, $\{H, F^{[i]}\} = 0$, and $\left\{F^{[i]}, F^{[j]}\right\} = 0$

Homological equation $\{H_0, F_1\} = \{F_0, H_1\}$ etc.

Remark. Regularity of H $\Rightarrow$commutativity of the centralizer

Example. Hopf equation

$$u_t = u\,u_x = \partial_x \frac{\delta H_0}{\delta u(x)}, \quad H_0 = \int \frac{1}{6} u^3(x)\,dx$$

is integrable: commuting first integrals are

$$F_0 = \int f\left(u(x)\right)\,dx$$

for an arbitrary function $f(u)$

Proof.

$$\{F_0, H_0\} = \int \frac{\delta F_0}{\delta u(x)} \partial_x \frac{\delta H_0}{\delta u(x)}\, dx = \int f'(u) u\, u_x\, dx = \int dF(u) = 0$$

where

$$\frac{dF}{du} = f'(u) u$$

An example of integrable deformation: KdV

$$H = H_0 + \epsilon^2 H_2 = \int \left[\frac{1}{6} u^3 - \frac{\epsilon^2}{24} u_x^2 \right] dx$$

Claim. For any function *f(u)* there exists a first integral

$$H_f = \int \left[f(u) - \frac{\epsilon^2}{24} f'''(u) u_x^2 + \epsilon^4 \left(\frac{1}{480} f^{(4)} u_{xx}^2 - \frac{1}{3456} f^{(6)} u_x^4 \right) + \dots \right] dx$$

of KdV equation $\{H, H_f\} = 0 \quad \forall\ f$

An explicit formula in terms of Lax operator

$$L = \frac{\epsilon^2}{2}\frac{d^2}{dx^2} + u(x)$$

Then

$$H_f = \int h_f \, dx$$

where

$$h_f = \operatorname{res} f^{(1/2)}(L)$$

More general second order perturbations

$$H = H_0 + \epsilon^2 H_2 = \int \left[\frac{1}{6} u^3 + \frac{\epsilon^2}{12} c(u) u_x^2 \right] dx$$

Thm. (B.D., J.Ekstrand) Integrability *iff*

$$c(u) = (u - u_0)^{-3}$$

Lax operator

$$\epsilon \Psi' = U(\lambda)\Psi, \quad U(\lambda) = \begin{pmatrix} 0 & \frac{(u-u_0)^2}{u_0} + \lambda \\ \frac{1}{2} u_0 (u - u_0)^2 & 0 \end{pmatrix}$$

It gives only half of the first integrals H_f, for odd f

Back to general case

The main goal: to compare the properties of solutions to the perturbed system

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + \epsilon\, A_2(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}) + \epsilon^2 A_3(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}) + \ldots$$

with solutions to the "dispersionless limit $\epsilon \to 0$

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x$$

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- Hamiltonian

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$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x$$

- Hamiltonian
- completely integrable

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with solutions to the "dispersionless limit $\epsilon \to 0$

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x$$

- Hamiltonian
- completely integrable
- finite life span (nonlinearity!)

For the dispersionless system $\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x$

a *gradient catastrophe* takes place: the solution exists

for $t < t_0$, there exists the limit $\lim_{t \to t_0} \mathbf{u}(x,t)$

but, for some x_0

$\mathbf{u}_x(x,t), \quad \mathbf{u}_t(x,t) \quad \to \quad \infty$ for $(x,t) \to (x_0, t_0)$

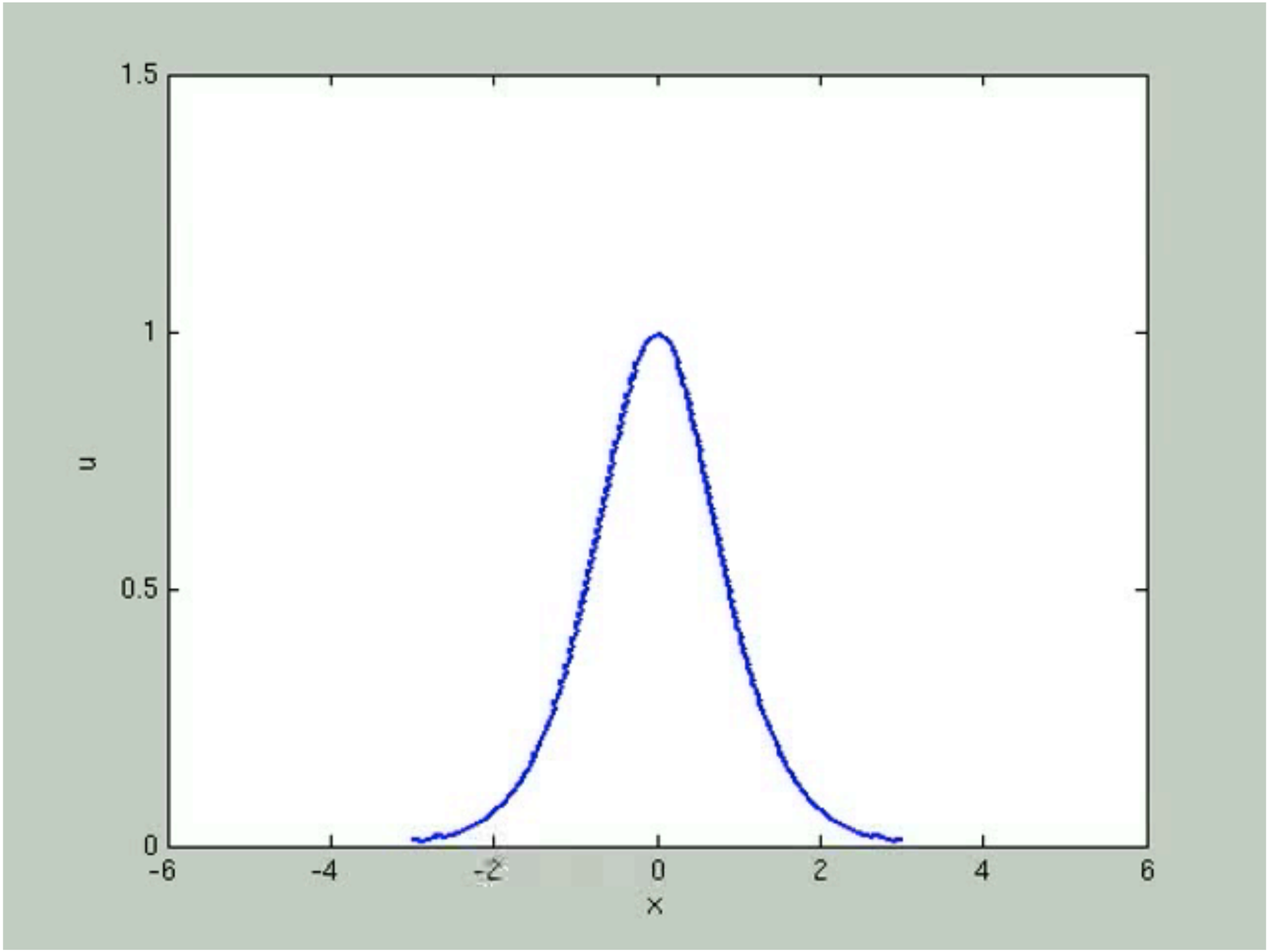
The problem: to describe the asymptotic behaviour of the generic solution $\mathbf{u}(x,t;\epsilon), \quad \mathbf{u}(x,0;\epsilon) = \mathbf{u}_0(x)$

to the perturbed system for $\epsilon \to 0$

in a neighborhood of the point of catastrophe (x_0, t_0)

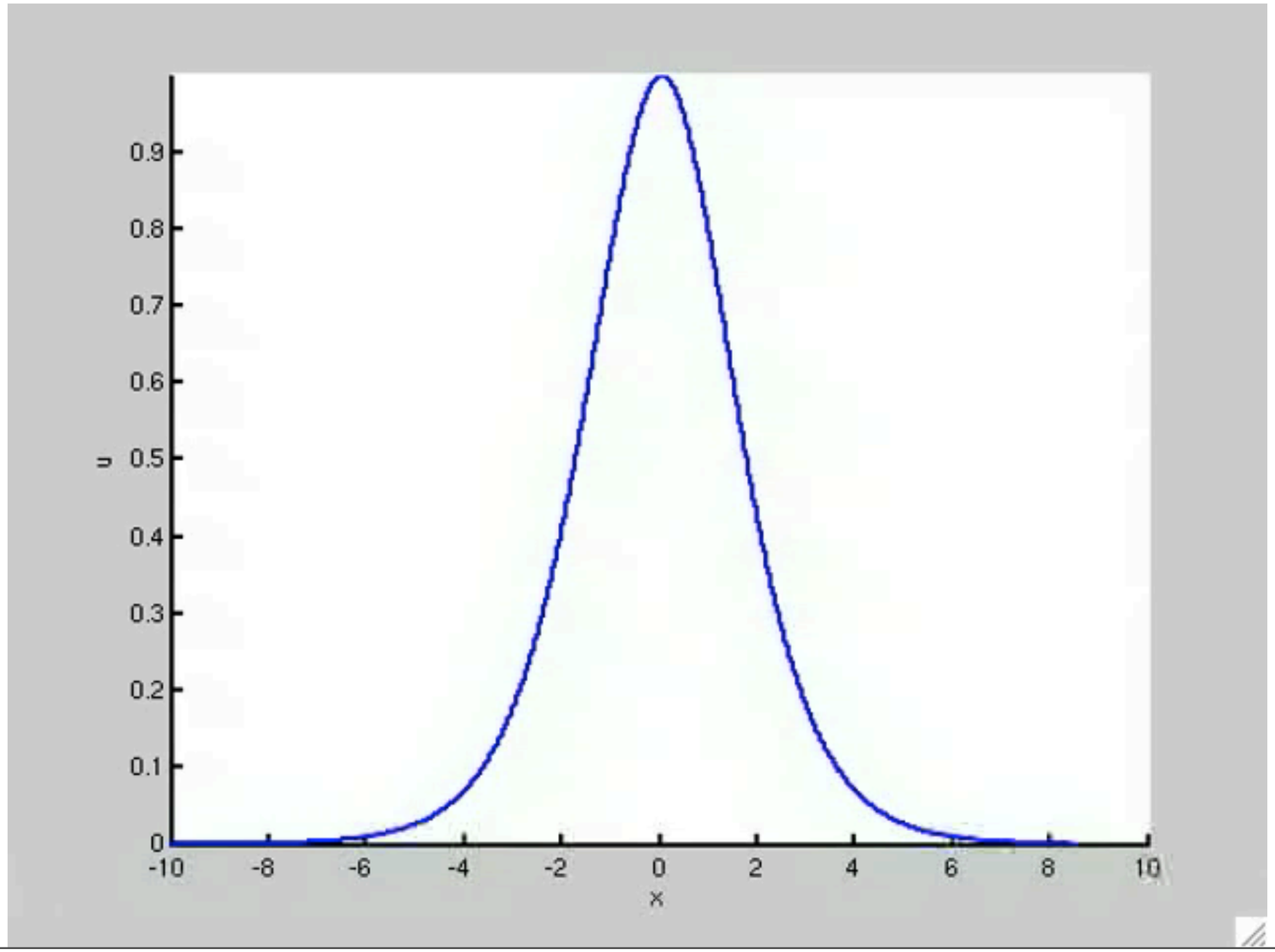
Gradient catastrophe for Hopf equation

$$u_t + u\,u_x = 0$$



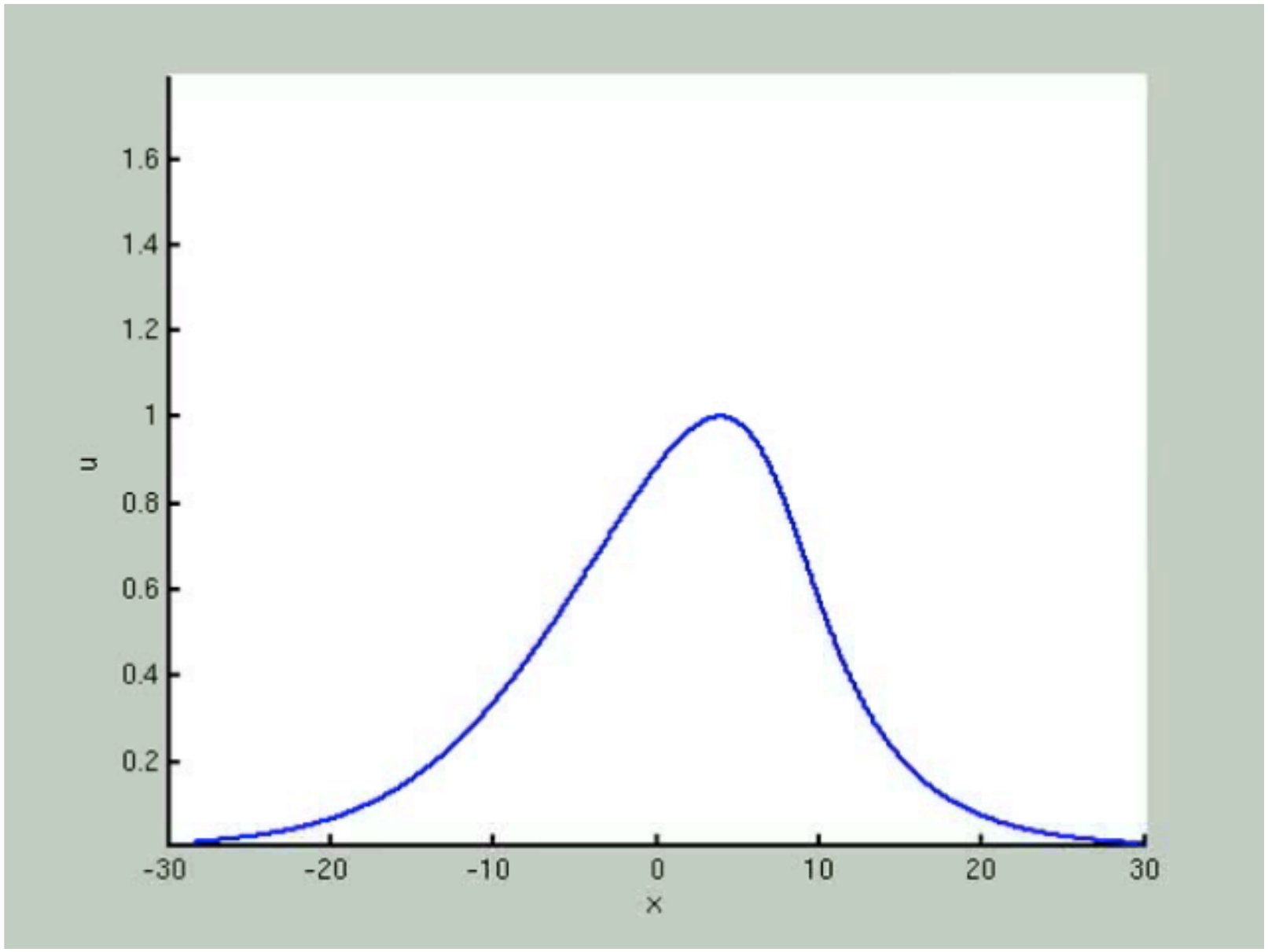
Perturbation: Burgers equation $u_t + u\,u_x = \epsilon\,u_{xx}$

(dissipative case)

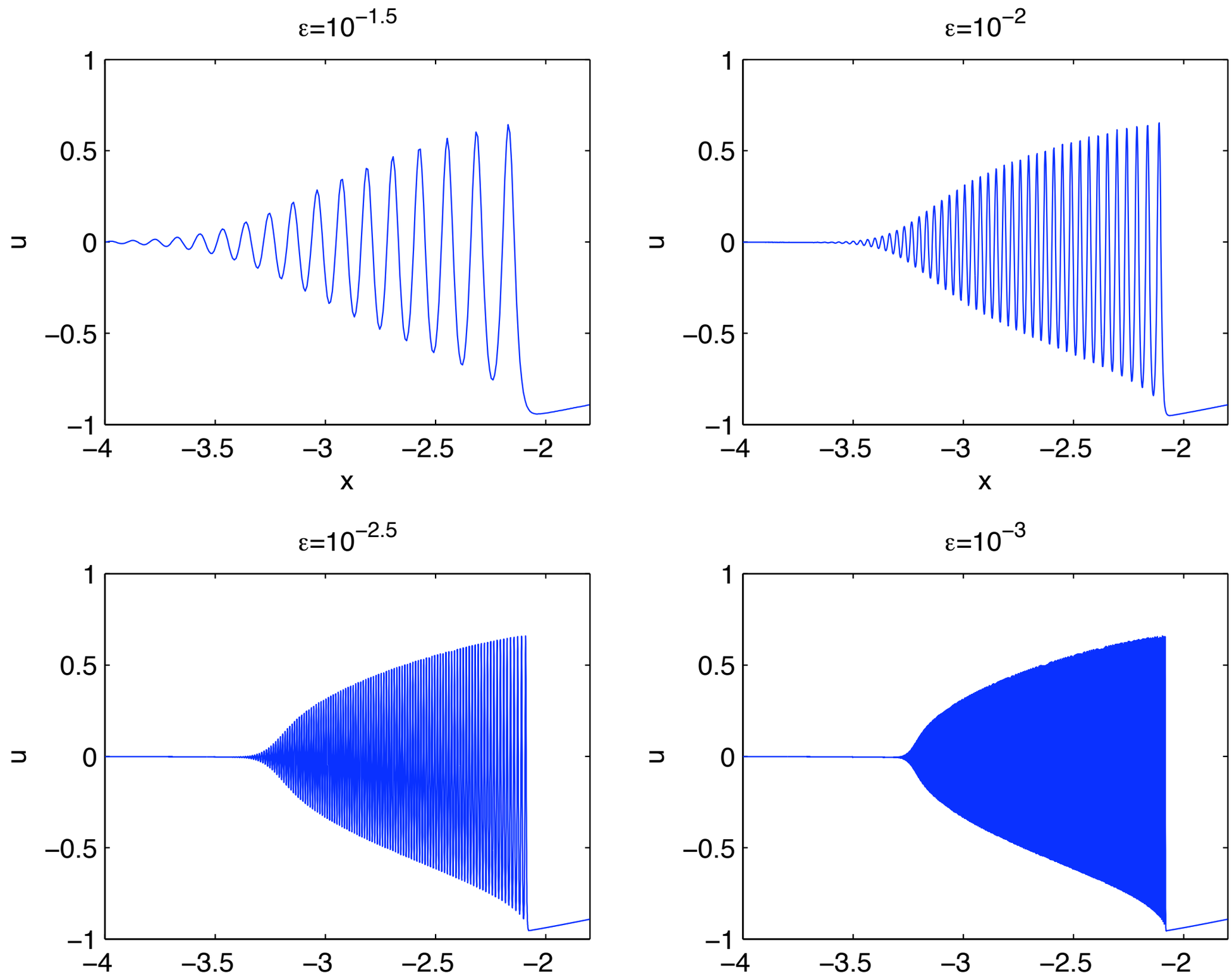


Perturbation: KdV equation $u_t + u\,u_x + \epsilon^2 u_{xxx} = 0$

(Hamiltonian case)

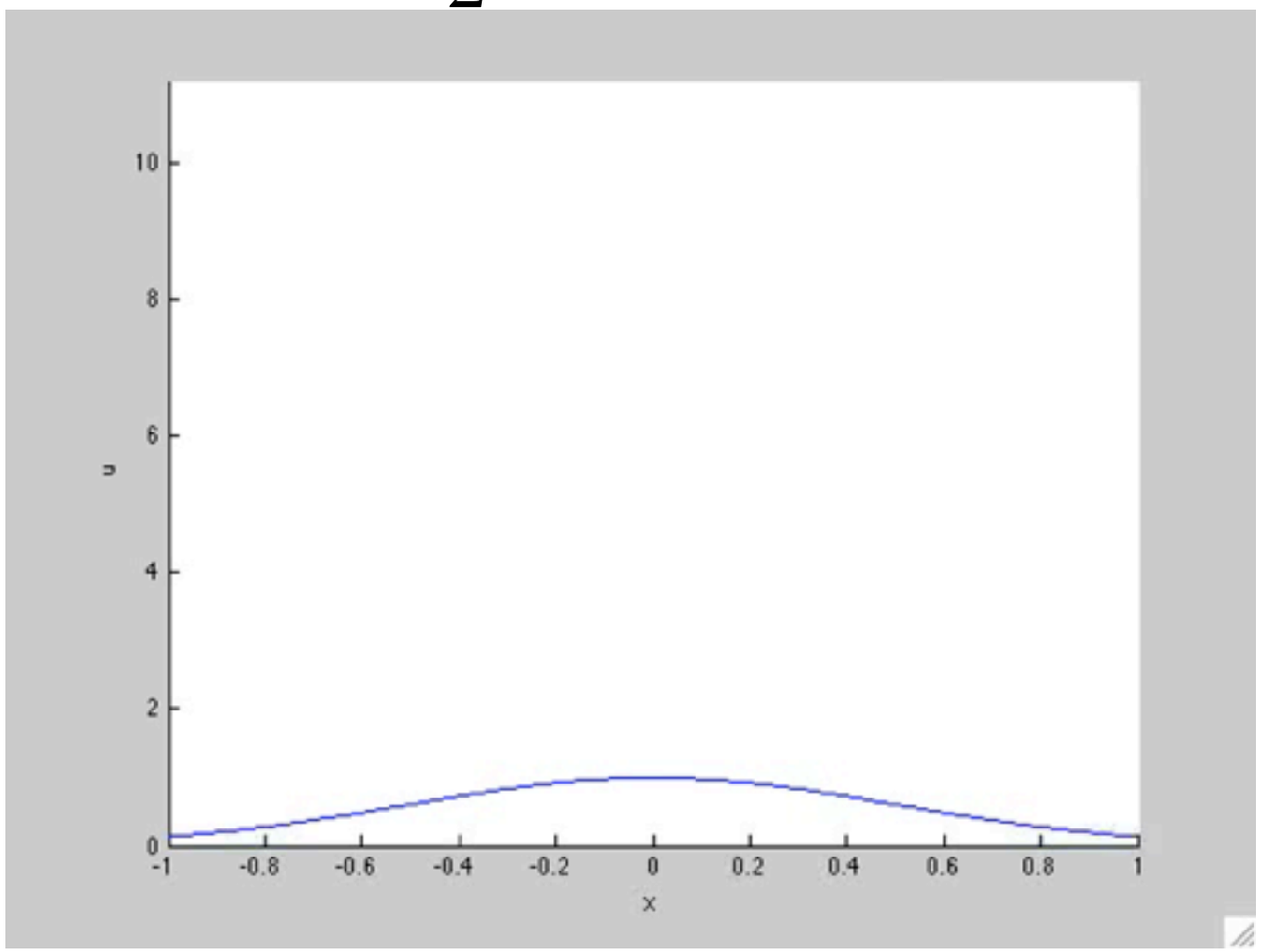


The smaller is ϵ, the faster are the oscillations



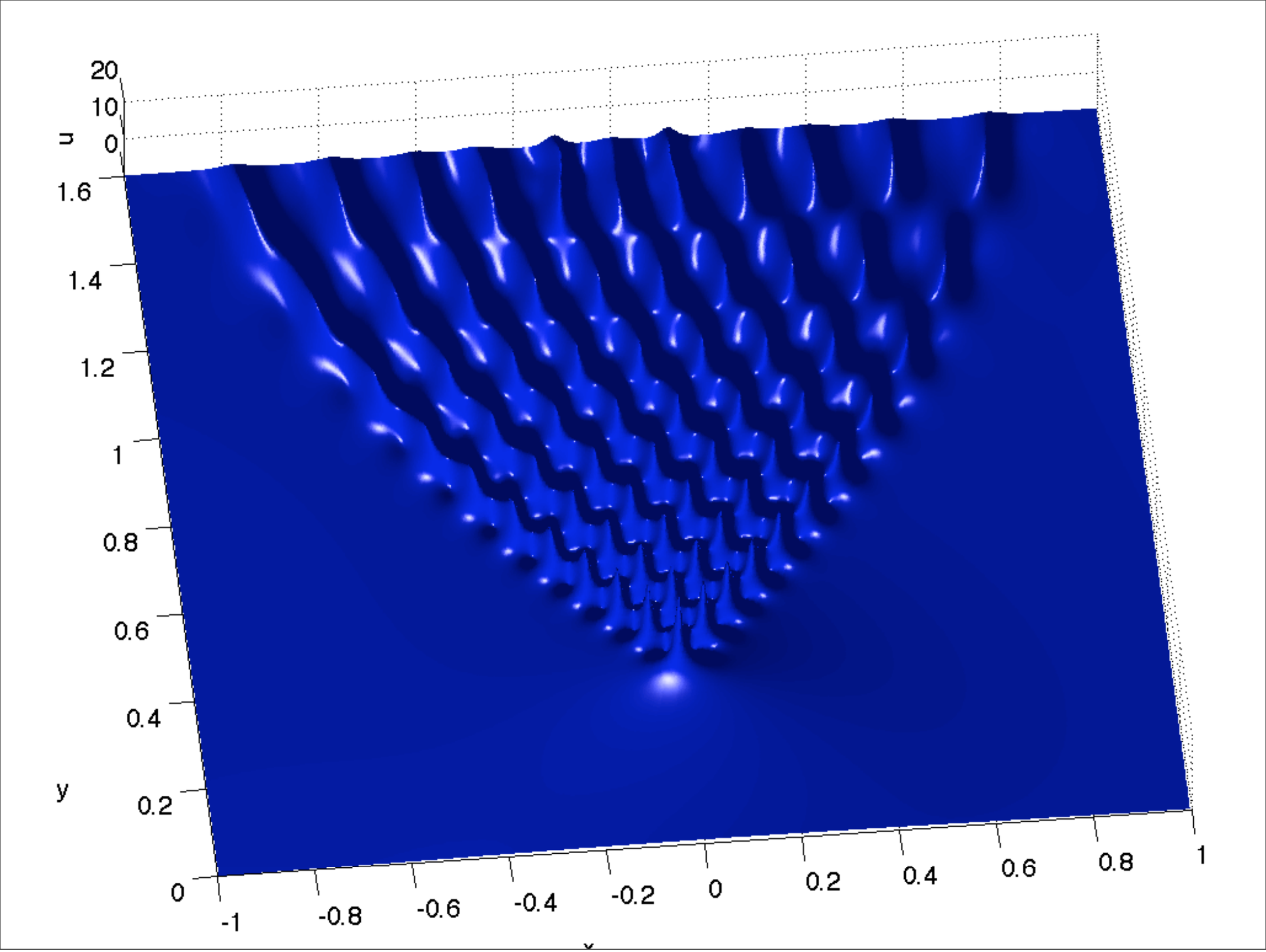
Nonlinear Schrödinger equation (the focusing case)

$$i\epsilon\,\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + |\psi|^2\psi = 0$$



NB: the dispersionless limit is a PDE of *elliptic type*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} + \begin{pmatrix} v & u \\ -1 & v \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = 0$$, eigenvalues $\lambda = v \pm i\sqrt{u}$


20
10
0
z
1.6
1.4
1.2
1
0.8
0.6
0.4
y
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Main Conjecture (B.D., 2005): a finite list of types of the critical behaviour

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(Universality)

Universality for the generalized Burgers equation (A.Il'in 1985)

$$u_t + a(u)u_x = \epsilon\, u_{xx}$$

$$u(x,t;\epsilon) = u_0 + \gamma\, \epsilon^{1/4}\Gamma\left(\frac{x - x_0 - a_0(t-t_0)}{\alpha\, \epsilon^{3/4}}, \frac{t-t_0}{\beta\, \epsilon^{1/2}}\right) + \mathcal{O}\left(\epsilon^{1/2}\right)$$

Here $\Gamma(\xi,\tau)$ is the logarithmic derivative of the Pearcey function

$$\Gamma(\xi,\tau) = -2\frac{\partial}{\partial\xi}\log\int_{-\infty}^{\infty} e^{-\frac{1}{8}\left(z^4 - 2z^2\tau + 4z\,\xi\right)}dz$$

What kind of special functions is needed for the Hamiltonian case?

Painlevé-I equation $U'' = 6U^2 - X$ $\qquad (P_I)$

and its 4th order analogue

$$X = T\,U - \left[\frac{1}{6}U^3 + \frac{1}{24}\left(U'^2 + 2U\,U''\right) + \frac{1}{240}U^{IV}\right] \qquad (P_I^2)$$

Painlevé property: any solution is a meromorphic function

in $X \in \mathbb{C}$

Isomonodromy representation for P_I^2

$$\frac{d\Psi}{d\lambda} = \mathbf{W}\Psi, \quad \frac{d\Psi}{dX} = \mathbf{U}\Psi, \quad \frac{d\Psi}{dT} = \mathbf{V}\Psi$$

$$\mathbf{W} = \begin{pmatrix} 12UU' + 8\lambda U' + U''' & 2(16\lambda^2 + 8\lambda\, U + 6U^2 + U'' - 60T) \\ 2\, w_{21} & -12\, UU' - 8\lambda U' - U''' \end{pmatrix}$$

$$w_{21} = 32\ \lambda^3 - 16\lambda^2 U - 2\lambda(2U^2 + U'' + 60\, T) + 8U^3 + 2U''U - U'^2 + 120X$$

$$\mathbf{U} = \begin{pmatrix} 0 & -1 \\ 2U - 2\lambda & 0 \end{pmatrix} \qquad \mathbf{V} = \frac{1}{6}\begin{pmatrix} U' & 2U + 4\lambda \\ 8\lambda^2 - 4\lambda U - 4U^2 - U'' & -U' \end{pmatrix}$$