Reducibility theory and its applications to Schrödinger operators

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Summary of the lecture

1. Introduction to quasi-periodic Schrödinger operator
2. Reducibility and Almost reducibility
3. Application 1: Kotani-Last Conjecture
4. Application 2: Dry Ten Martini Problem
5. Application 3: Aubry-André-Jitomirskaya conjecture

Based on joint works with A. Avila, X. Hou and Q. Zhou.

Lecture 1

1. Introduction to quasi-periodic Schrödinger operator
2. Reducibility and Almost reducibility

Background in Physics

- Conductivity of Quasi-crystal.
- Quantum Hall effect.
- Evolution of Bose-Einstein condensate in a disordered lattice nonlinear optics.
- Wave propagation in a nonlinear disordered media.

The mathematical modeling

The mathematical modeling for the above physical problems is the following quasi-periodic lattice nonlinear Schrödinger equations

$$\mathrm{i}\dot{q}_n + \epsilon(q_{n+1} + q_{n-1}) + V(n\alpha + x)q_n + \mu|q_n|^2 q_n = 0, \quad n \in \mathbb{Z} \qquad (1)$$

where

- $0 < \epsilon \ll 1, 0 \leq \mu \ll 1$,
- V is a non-constant real analytic function on $\mathbb{R}/\mathbb{Z}$,
- α satisfying the Diophantine condition.
- or the potential is independent identically distributed random variables.

To understand the dynamics of the above equation, understanding the corresponding operator $H = \triangle + V$ is a crucial step.

Introduction to one-frequency quasi-periodic Schrödinger operator

In this lecture, we are interested in the discrete one-frequency quasi-periodic Schrödinger operator

$$(H_\theta u)_n = u_{n+1} + u_{n-1} + v(\theta + n\alpha)u_n, u \in l^2(\mathbb{Z})$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\theta \in \mathbb{T}$, $v(\theta) \in C^\omega(\mathbb{T}, \mathbb{R})$.

Continuous case

$$H_\theta(u) = -\frac{d^2u}{dt^2} + v(\theta_1 + t, \theta_2 + t\alpha)u \quad u \in L^2(-\infty, \infty)$$

Almost Mathieu operators

Typical Example: Almost Mathieu operators

$$(H_{\lambda,\theta,\alpha}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos(n\alpha+\theta)u_n.$$

where λ is called the coupling constant, α is called the frequency, θ is called the phase.

Basic concepts

- Spectrum of H_θ:

$$\Sigma(H_\theta) = \{E \in \mathbb{C} \quad | \quad (H_\theta - E)^{-1}\text{doesn't exist}\},$$

since $\theta \to \theta + \alpha$ is minimal, then $\Sigma(H_\theta) = \Sigma$ for any $\theta \in \mathbb{T}$.

- Spectral measure $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$.

$$\langle (H_\theta - z)^{-1} f, f \rangle = \int_{\mathbb{R}} \frac{d\mu_\theta^f(t)}{t - z}, \qquad f \in l^2(\mathbb{Z}).$$

- IDS(Integrated density of states)

$$N(E) = \int_{\mathbb{R}/\mathbb{Z}} \mu_\theta^f(-\infty, E] d\theta.$$

Spectral Theory for quasi-periodic Schrödinger operators

Central goal: Understanding the spectrum Σ and the spectral measure of H_θ including

- Toplogy: toplogy of the spectrum Σ: For example, Ten Martini problem (Cantor set), Dry Ten Martini problem.

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- Toplogy: toplogy of the spectrum Σ: For example, Ten Martini problem (Cantor set), Dry Ten Martini problem.
- Measure: the property of the spectral measure: absolutely continuous? singular continuous? pure point? Phase transition?

Relation with dynamical systems

The eigenvalue equation

$$(H_\theta u)_n = u_{n+1} + u_{n-1} + v(n\alpha + \theta)u_n = Eu_n,$$

is equivalent to

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} E - v(n\alpha + \theta) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A^n \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$$

where

$$A^n = \begin{pmatrix} E - v(n\alpha + \theta) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - v(\theta) & -1 \\ 1 & 0 \end{pmatrix}$$

Schrödinger cocycle and $SL(2,\mathbb{R})$ cocycles

We shall call Schrödinger cocycle the map (α, S_E^v):

$$\begin{aligned} \mathbb{T}\times\mathbb{R}^2 &\to \mathbb{T}\times\mathbb{R}^2 \\ (\theta, w) &\mapsto (\theta+\alpha, S_E^v(\theta)\cdot w), \end{aligned}$$

where

$$S_E^v(\theta) = \begin{pmatrix} E - v(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Quasi-periodic $SL(2,\mathbb{R})$ cocycle (α, A) is more general: $A : \mathbb{T} \to SL(2,\mathbb{R})$.

Continuous Schrödinger operator

Continuous version Schrödinger equation

$$(\mathcal{L}_{\theta,q}u)(t) = -u''(t) + q(\theta + t\omega)u(t) = Eu(t).$$

can be rewritten as

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 1 \\ q(\theta) - E & 0 \end{pmatrix} x \\ \dot{\theta} = \omega \end{cases}$$

Quasi-periodic $sl(2,\mathbb{R})$ linear system (ω, A) is defined as

$$\begin{cases} \dot{x} = A(\theta)x \\ \dot{\theta} = \omega \end{cases}$$

Lyapunov exponent

Let $A(\cdot) : \mathbb{T} \to SL(2, \mathbb{R})$ be continuous, it is a theorem due to Kingman that for a.e. $\theta \in \mathbb{T}$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\theta)\|$$

exists and is same for a.e. $\theta \in \mathbb{T}$. We call this number Lyapunov exponent $L(\alpha, A)$. When $L(\alpha, A) > 0$ Oseledec Theorem tells us that for a.e. $\theta \in \mathbb{T}$,

- There exists a measurable decomposition $\mathbb{R}^2 = E_s(\theta) \bigoplus E_u(\theta)$.

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- There exists a measurable decomposition $\mathbb{R}^2 = E_s(\theta) \bigoplus E_u(\theta)$.
- There exists $\rho > 0$ and $C(\theta)$ depending measurably on θ such that for a.e. $\theta \in \mathbb{T}$, any $v \in \mathbb{R}^2$

$$\forall n \geq 0, \forall v \in E_s(\theta), \|A^n(\theta)v\| \leq Ce^{-n\rho}\|v\|$$

$$\forall n \geq 0, \forall v \in E_u(\theta), \|A^{-n}(\theta)v\| \leq Ce^{-n\rho}\|v\|$$

Uniformly hyperbolic cocycle

We say that the cocycle (α, A) is uniformly hyperbolic if

- There exists a decomposition $\mathbb{R}^2 = E_s(\theta) \bigoplus E_u(\theta)$ where $\theta \to E_{s,u}(\theta)$ are continuous.
- There exists C, ρ such that for any $\theta \in \mathbb{T}$, any $v \in \mathbb{R}^2$

$$\forall n \geq 0, \forall v \in E_s(\theta), \|A^n(\theta)v\| \leq Ce^{-n\rho}\|v\|$$

$$\forall n \geq 0, \forall v \in E_u(\theta), \|A^{-n}(\theta)v\| \leq Ce^{-n\rho}\|v\|$$

Notice that for all $\theta \in \mathbb{T}$, $\rho = \lim \frac{1}{n} \log \|A^n(\theta)v\|$.

(α, A) is called non-uniformly hyperbolic if $L(\alpha, A) > 0$ and (α, A) is not uniformly hyperbolic.

Fibred rotation number

The fibred rotation number measures how solutions wind around the origin in $\mathbb{R}^2$ in average. The continuous map

$$\begin{array}{rccc} F: & \mathbb{T}\times S^1 & \longrightarrow & \mathbb{T}\times S^1 \\ & (\theta, v) & \mapsto & \left(\theta+\alpha, \frac{A(\theta)v}{\|A(\theta)v\|}\right) \end{array} \tag{2}$$

admits a continuous lift $\tilde{F}:\mathbb{T}\times\mathbb{R}\to\mathbb{T}\times\mathbb{R}$ of the form:

$$\tilde{F}(\theta,t)=(\theta+\alpha, t+f(\theta,t))$$

such that

$$f(\theta+1,t+1)=f(\theta,t) \quad \text{and} \quad p\left(t+f(\theta,t)\right)=\frac{A(\theta)p(t)}{\|A(\theta)p(t)\|}$$

Define the fibred rotation number as

$$\rho(\alpha,A)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f\left(\tilde{F}^n(\theta,t)\right)$$

Links between the spectral and dynamical aspects

Links between the spectral and dynamical aspects

- Spectrum and uniformly hyperbolic
- IDS and Lyapunov exponent
- IDS and the fibred rotation number
- AC spectrum and zero Lyapunov exponent: Kotani's theory
- Spectrum and Reducibility

Spectrum and uniformly hyperbolic

Theorem

The spectrum of H is the set of $E \in \mathbb{R}$ for which the cocycle (α, S_E^v) is not uniformly hyperbolic.

IDS and Lyapunov exponent: Thouless formula

Theorem (Thouless formula)

$$L(\alpha, S_E^v) = \int_{\mathbb{R}} \ln|E-t| dN(t)$$

where $N(\cdot)$ is the IDS of the Schrödinger operator H.

IDS and the fibred rotation number

Theorem

$$N(E) = 1 - 2\rho(\alpha, S_E^v).$$

AC spectrum and zero Lyapunov exponent

Zero Lyapunov are highly related to the fact that the spectrum has some absolutely continuous part, which is known as Kotani's theory.

Theorem (Kotani's theory)

For Leb a.e. $\theta \in \mathbb{T}$, one has $\Sigma_{ac}(H_\theta) = \{E \in \mathbb{R} : L(E) = 0\}$ (where $\Sigma_{ac}(H_\theta)$ is the essential support of the a.c. part of the spectrum).

Reducibility

The cocycle (α, A) is called reducible: if it is conjugated to a constant cocycle in the following sense : there exist an analytic map $B : 2\mathbb{T} \to SL(2, \mathbb{R})$, $C \in SL(2, \mathbb{R})$ such that

$$(0, B) \circ (\alpha, A) \circ (0, B)^{-1} = (\alpha, C)$$

or equivalently

$$B(\cdot + \alpha)A(\cdot)B(\cdot)^{-1} = C.$$

Remark: Same concepts (LE, rotation number, reducibilty, $\cdots$)can be defined for quasi-periodic linear system.

Reducibility and the spectrum

Reducibility is a very important concept, it can be used to study Cantor spectrum, Dry Ten Martini Problem, absolutely continuous spectrum, pure point spectrum $\cdots$. In the rest of the lecture, we will first establish various reducibility results.

1. Local reducibility
   - Perturbative reducibility
   - Nonperturbative reducibility
   - Almost reducibility and rotations reducibility
2. Global reducibility
   - Renormalization
   - Avila's global theory, Almost reducibility conjecture

Arithmetic property of α

Arithmetic properties of α influence the spectrum and dynamics of the cocycle:

- Diophantine $DC(\gamma, \tau)$: $\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{|k|^\tau}, k \neq 0$

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- $DC \subset \{\alpha|\beta(\alpha) = 0\}$

Positive measue local reducibility results

Local reducibility: cocycle is close to constant

Theorem (Dinaburg-Sinai)

Let $\alpha \in DC(\gamma, \tau), \gamma, \tau, \kappa, \sigma > 0, A_0 \in SL(2, \mathbb{R}), h > 0$. *There exists* $\varepsilon_0 = \varepsilon(\gamma, \tau, \kappa, \sigma, A_0, h) > 0$, *s.t. for all* $F_0 \in C_h^\omega(\mathbb{T}, sl(2, \mathbb{R}))$ *satisfying*

(i) $\|F_0\|_h < \varepsilon_0$,

(ii) $2\rho(\alpha, e^{F_0} A_0) \in DC_\alpha(\kappa, \sigma) = \{\rho \in \mathbb{R} | \|\rho - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\kappa}{|k|^\sigma}, k \neq 0\}$,

the cocycles $(\alpha, e^{F_0(\cdot)} A_0)$ *is reducible on a strip of width* $h' = h/4$.

Outline of the proof

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- The philosophy of KAM theory is to construct a sequence of coordinate transformations making the perturbation smaller and smaller.
- Then, we have to prove that the compositions of all these transformations converge. This will finish the proof.

Outline of the proof

- Our aim is to find a conjugation $B_1(\cdot)$ "close to the identity": $B_1 = e^{Y_1}$ where Y_1 is small, $Y_1 \in C^{\omega}_{h_1}(\mathbb{T}, sl(2,\mathbb{R}))$ and a constant $A_1 \in SL(2,\mathbb{R})$, s.t.

$$e^{Y_1(\cdot+\alpha)} e^{F_0(\cdot)} A_0 e^{-Y_1(\cdot)} = e^{F_1(\cdot)} A_1,$$

with F_1 as small as possible.

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  with F_1 as small as possible.
- Using the fact $e^M = I + M + O(M^2)$, $(I+M)^{-1} = I - M + O(M^2)$, we see that if we are seeking for F_1 to be 0, one has

$$Y_1(\cdot+\alpha) - A_0 Y_1(\cdot) A_0^{-1} = -F_0 + A_1 A_0^{-1} - I + O_2(\|Y_1\|_{h_1}, \|F_0\|_h).$$

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- Conversely,

Linearized equation

Proposition

If one solves the linearized equation

$$Y_1(\cdot+\alpha) - A_0 Y_1(\cdot) A_0^{-1} = -F_0 + A_1 A_0^{-1} - I,$$

with $Y_1 \in C^{\omega}_{h_1}(\mathbb{T}, sl(2,\mathbb{R}))$*, then we will have*

$$e^{Y_1(\cdot+\alpha)}(e^{F_0(\cdot)}A_0)e^{-Y_1(\cdot)} = e^{F_1(\cdot)}A_1,$$

with $\|F_1\|_{h_1} = O_2(\|Y_1\|_{h_1}, \|F_0\|_h)$.

Linearized equation

Proposition

Assume that $A \in SL(2,\mathbb{R}), F \in C_h^\omega(\mathbb{T}, sl(2,\mathbb{R}))$, $\alpha \in DC(\gamma,\tau)$, $2\rho(\alpha, A) \in DC_\alpha(\kappa,\sigma)$. *Then the equation*

$$Y(\cdot + \alpha) - AY(\cdot)A^{-1} = F - \hat{F}(0)$$

has a unique analytic solution $Y \in C^\omega(\mathbb{T}, sl(2,\mathbb{R}))$ *defined on some strip of width* h' *for all* $h' < h$, *and s.t.*

$$\|Y\|_{h'} \leq C(\gamma,\kappa)\frac{\|F\|_h}{(h-h')^a},$$

with $a = 2 + \tau + \sigma$.

Solution of the linearized equation

The linearized equation (homological equation)

$$Y(\cdot+\alpha)-AY(\cdot)A^{-1}=F-\hat{F}(0)$$

is equivalent to

$$\begin{aligned} y_1(\theta+\alpha)-y_1(\theta) &= f_1(\theta) \\ y_2(\theta+\alpha)-e^{2\pi i\rho}y_2(\theta) &= f_2(\theta) \\ y_3(\theta+\alpha)-e^{-2\pi i\rho}y_3(\theta) &= f_3(\theta) \end{aligned}$$

checking the Fourier coefficients, one knows if $\alpha\in DC(\gamma,\tau)$, $2\rho\in DC_\alpha(\kappa,\sigma)$, then they have analytic solutions.

The iterative scheme

Let $h_n = h - \frac{h}{4} - \cdots - \frac{h}{2^{n+1}}$. We construct sequences A_n, F_n, Y_n, such that

(a) $A_n \in SL(2,\mathbb{R})$ and $Y_n, F_n \in C^{\omega}_{h_n}(\mathbb{T}, sl(2,\mathbb{R}))$;

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(b) $(0, e^{Y_n(\cdot)}) \circ (\alpha, e^{F_{n-1}(\cdot)} A_{n-1}) \circ (0, e^{Y_n(\cdot)})^{-1} = (\alpha, e^{F_n(\cdot)} A_n)$;

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(c) $2\rho(\alpha, A_n) \in DC_\alpha(\kappa, \sigma)$;

(d) If $\varepsilon_n := \|F_n\|_{h_n}$, then one has

$$\|Y_{n+1}\|_{h_n} \leq h_n^{-a} \varepsilon_n, \text{ and } \varepsilon_{n+1} \leq C h_n^{-a} \varepsilon_n^2;$$

The iterative scheme

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(c) $2\rho(\alpha, A_n) \in DC_\alpha(\kappa, \sigma)$;

(d) If $\varepsilon_n := \|F_n\|_{h_n}$, then one has

$$\|Y_{n+1}\|_{h_n} \leq h_n^{-a} \varepsilon_n, \text{ and } \varepsilon_{n+1} \leq C h_n^{-a} \varepsilon_n^2;$$

(e) $\|A_n - A_{n-1} e^{\hat{F}_{n-1}(0)}\| \leq \varepsilon_n$.

Invariance of the rotation number

Why point (c) holds?

Lemma

Assume $A \in SL(2, \mathbb{R})$, if $\|F\|$ is small enough, then we have

$$|\rho(\alpha, e^{F(\cdot)}A) - \rho(A)| \leq \|F\|.$$

Lemma

If the conjugation $B(\cdot)$ is close to the identity, then

$$\rho(\alpha, B(\cdot + \alpha)A(\cdot)B(\cdot)^{-1}) = \rho(\alpha, A).$$

The iterative scheme

Point (d) shows that ε_n goes to 0 very fast and then one has:

Proposition

The limits $A_* = \lim_{n\to\infty} A_n$, $B_*(\cdot) = \lim_{n\to\infty} e^{Y_n(\cdot)} \cdots e^{Y_1(\cdot)}$ *exist and* $B_* \in C^{\omega}_{h/4}(\mathbb{T}, SL(2,\mathbb{R}))$. *Thus one has*

$$(0, B_*) \circ (\alpha, e^{F_0(\cdot)} A_0) \circ (0, B_*)^{-1} = (\alpha, A_*).$$

Perturbative reducibility:

- The same method works for continuous quasi-periodic linear system, and the base frequency can be any dimension.

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- The same method works for continuous quasi-periodic linear system, and the base frequency can be any dimension.
- Eliasson's break through: ε_0 can be chosen independently of κ, which is a full measure result on the energy.
- The preceding results are called perturbative, i.e., ε_0 depends on γ.

Nonperturbative reducibility:

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- Shortcoming: using the structure of Schr$\ddot{\text{d}}$inger operator, thus only works for Schrödigner cocycle.
- Bourgain: If the base frequency $\alpha \in \mathbb{T}^2$, nonperturbative reducibility doesn't exist.

Non-perturbative reducibility for quasi-periodic linear system

Theorem (Hou-Y Invent of Math)

Let $\alpha \in DC(\gamma, \tau), \gamma, \tau, \kappa, \sigma > 0, A_0 \in sl(2, \mathbb{R}), h > 0$. *There exists* $\varepsilon_0 > 0$ *which depend on* A_0, h *but doesn't depend on* $\gamma, \tau, \kappa, \sigma$ *s.t. for all* $F_0 \in C^\omega(\mathbb{T}^2, sl(2, \mathbb{R}))$ *satisfying*

(i) $\|F_0\|_h < \varepsilon_0$,

(ii) $2\rho(\omega, A + F) \in DC_\alpha(\kappa, \sigma)$,

the quasi-periodic linear system

$$\begin{cases} \dot{x} = (A + F(\theta))x \\ \dot{\theta} = \omega = (\alpha, 1). \end{cases}$$

is reducible.

Obstructions to reducibility

Obstructions due to arithmetics of α : Even in the simplest case where $A = R_{\varphi(x)}$, where $\varphi(x+1) = \varphi(x)$ is analytic, the cocycle is not reducible. since in general

$$y(\theta + \alpha) - y(\theta) = f(\theta)$$

doesn't have analytic solution.

Almost reducibility and Rotations reducibility

Almost reducible: if there exist sequences of analytic matrix functions $B_n : 2\mathbb{T}^2 \to SL(2,\mathbb{R})$, $A_n \in sl(2,\mathbb{R})$ and a sequence of matrix functions $F_n : \mathbb{T}^2 \to sl(2,\mathbb{R})$ converging (analytically) to zero, such that the conjugacy B_n conjugate (ω, A) to $(\omega, A_n + F_n)$.

Rotations reducible: if there exist $B \in C^\omega(2\mathbb{T}^2, SL(2,\mathbb{R}))$ and $R_{\phi(\theta)} \in C^\omega(\mathbb{T}^2, so(2,\mathbb{R}))$ such that B conjugate (ω, A) to $(\omega, R_{\phi(\theta)})$.

These concepts can be defined similarly for quasi-periodic $SL(2,\mathbb{R})$ cocycles.

Almost reducibility for quasi-periodic linear system

Theorem (Hou-Y Invent of Math)

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}, A_0 \in sl(2,\mathbb{R}), h > 0$. *There exists* $\varepsilon_0 = \varepsilon_0(A_0, h) > 0$ *s.t. for all* $F_0 \in C^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$ *satisfying* $\|F_0\|_h < \varepsilon_0$, *the quasi-periodic linear system*

$$\begin{cases} \dot{x} = (A + F(\theta))x \\ \dot{\theta} = \omega = (\alpha, 1). \end{cases}$$

is almost reducible.

Linearized equation

As we did in Dinaburg-Sinai's result, one needs to find a constant matrix $\widetilde{A}$ and $sl(2,\mathbb{R})-$valued function $Y(\theta)$ which solves

$$\partial_\omega e^Y = (A+F)e^Y - e^Y\widetilde{A}.$$

in this case, the linearized equation is

$$\partial_\omega Y - [A,Y] = F$$

(Lie bracket $[\cdot,\cdot]$ is defined as $[X_1,X_2] = X_1X_2 - X_2X_1$).

Resonances and Non-resonance

One step in our proofs is to remove all the non-resonant components in the Fourier expansion of F. For any given $h > 0$, $\omega \in \mathbb{R}^2$ and $A \in sl(2, \mathbb{R})$, we decompose $\mathfrak{B}_h = \mathfrak{B}_h^{(nre)} \oplus \mathfrak{B}_h^{(re)}$ (the decomposition depends on A, ω, η) in such a way that for any $Y \in \mathfrak{B}_h^{(nre)}$

$$\partial_\omega Y, [A, Y] \in \mathfrak{B}_h^{(nre)}, \quad |\partial_\omega Y - [A, Y]|_h \geq \eta |Y|_h.$$

Let $\mathbb{P}_{nre}$ ($\mathbb{P}_{re}$) be the standard projection from $\mathfrak{B}_h$ onto $\mathfrak{B}_h^{(nre)}$ ($\mathfrak{B}_h^{(re)}$). We call $\mathfrak{B}_h^{(nre)}$ ($\mathfrak{B}_h^{(nre)}$) the η-nonresonant (η-resonant) subspace. With the above assumptions, one can prove the following Lemma.

Step 1: Eliminating the non-resonant terms

Let $$(h_n, \varepsilon_n) = (\tfrac{1}{8} h_{n-1}, \varepsilon_{n-1} e^{-c q_n h_{n-1}}),$$

Lemma

The system is conjugated to

$$\begin{cases} \dot{x} = (A + F^{(re)}(\theta))x \\ \dot{\theta} = \omega = (\alpha, 1) \end{cases}$$

with $F^{(re)}$ containing only resonant terms of the Fourier expansion. Moreover, $\|F^{(re)}\|_{h_n} < 2\varepsilon_n$.

Proof: Implicit Function Theorem.

The structure of Resonances

Let $\alpha \in (0,1)$ be irrational, $\frac{p_n}{q_n}$ be its continued fraction expansion. Let $\omega = (\alpha, 1)$. We know that

$$\frac{1}{q_{n+1} + q_n} < |q_n \alpha - p_n| < \frac{1}{q_{n+1}}.$$

Lemma (The structure of resonances)

Suppose that $k \in \mathbb{Z}^2$ with $|k| < \frac{1}{6} q_{n+1}$ satisfies $|\langle k, \omega \rangle| < \frac{1}{7q_n}$. Then k has the form $l(q_n, -p_n)$ for some $l \in \mathbb{Z}$.

The structure of Resonances

$$\mathcal{T}_{\frac{q_{n+1}}{6}} F^{(re)}(\theta) = \sum_{k\in\Lambda_1} \begin{pmatrix} \widehat{F}_{11}(k) & 0 \\ 0 & -\widehat{F}_{11}(k) \end{pmatrix} e^{i\langle k,\theta\rangle} + \begin{pmatrix} 0 & \sum_{k\in\Lambda_2} \widehat{F}_{12}(k) e^{i\langle k,\theta\rangle} \\ \sum_{k\in\Lambda_3} \widehat{F}_{12}(k) e^{i\langle k,\theta\rangle} & 0 \end{pmatrix}.$$

with

$$\begin{aligned} \Lambda_1 &= \{k \in \mathbb{Z}^2 | k = l(q_n, -p_n), l \in \mathbb{Z}\} \\ \Lambda_2 &= \{k \in \mathbb{Z}^2 | k = k_* + l(q_n, -p_n), l \in \mathbb{Z}\} \\ \Lambda_3 &= \{k \in \mathbb{Z}^2 | k = -k_* + l(q_n, -p_n), l \in \mathbb{Z}\} \end{aligned}$$

Step 2: Rotation

Lemma

There exists $R \in C^\omega(\mathbb{T}^2, SO(2,\mathbb{R}))$ *which conjugates the above system to*

$$\begin{cases} \dot{x} = (A_1 + F_1(\theta))x \\ \dot{\theta} = \omega = (\alpha, 1) \end{cases} \tag{3}$$

with $|A_1| \leq \varepsilon_n^{1/2}, \quad \|F_1\|_{h_n/4} < 2\varepsilon_n^{3/4}$. *and* F_1 *takes the following form*

$$\mathcal{T}_{\frac{q_{n+1}}{6}} F_1 = \sum_{k=l(q_n,-p_n),|k|<\frac{q_{n+1}}{6}} \hat{F}_1(k) e^{i\langle k,\theta\rangle}.$$

Remark Compared to Eliasson's perturbative result: If α is Diophantine, then $\mathcal{T}_{\frac{q_{n+1}}{6}} F_1 = 0$.

Step 3: Floquet

Lemma

There exists $B \in C^{\omega}_{h_{n+1}}(2\mathbb{T}^2, SL(2,\mathbb{R}))$ *which conjugates (**??**) to*

$$\begin{cases} \dot{x} = (A_2 + F_2(\theta))x \\ \dot{\theta} = \omega = (\alpha, 1) \end{cases}$$

with estimates $\|F_2\|_{h_{n+1}} \leq \varepsilon_{n+1}$

Proof:

Using the structure,

$$\begin{cases} \dot{x} = (A_1 + \mathcal{T}_{\frac{q_{n+1}}{6}} F_1(\theta))x \\ \dot{\theta} = \omega \end{cases}$$

is a periodic system, which can be conjugated to constant systems.

More discussion:

- Completely understanding of the local almost reducibility
- Shortcoming of the method: only works for continuous version
- Solution?

A local embedding theorem

Theorem (Y-Zhou,CMP)

Let $\mu \in \mathbb{T}^{d-1}$, $h > 0$, $G \in C_h^\omega(\mathbb{T}^{d-1}, sl(2,\mathbb{R}))$, A *is a constant. There exists a* $\varepsilon > 0$, *such that if* $\|G\|_h \leq \varepsilon$, *then there exists a* C^ω *quasi-periodic flow* $\Phi^t(\theta_1, \widetilde{\theta})$, *such that* $\Phi^1(0, \widetilde{\theta}) = e^A e^{G(\widetilde{\theta})}$. *More precisely, there exists a* $F \in C^\omega_{h/1+|\mu|}(\mathbb{T}^d, sl(2,\mathbb{R}))$ *such that* $(\mu, e^A e^{G(\cdot)})$ *can be embedded into*

$$\begin{cases} \dot{x} = (A + F(\theta))x \\ \dot{\theta} = (1, \mu) \end{cases}$$

with the estimate $\|F\|_{h/1+|\mu|} \leq c\varepsilon$.

It is true for any dimension!

Equivalence

With the embedding theorem, we prove the following equivalence of almost reducibility of linear systems and the corresponding Poincaré cocycles

Theorem (Y-Zhou, CMP)

Quasi-periodic linear systems (ω, A) is almost reducible (resp.rotations reducible) if and only if the corresponding Poincaré cocycle $(\alpha, \mathcal{A})$ is almost reducible (resp.rotations reducible).

Rotations reducibility for cocycles

Theorem (Hou-Y Invent of Math)

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $h > 0$. $F \in C_h^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$. *Assume that*

$$2\rho(\alpha, e^{F(\cdot)}R_{\rho_0}) \in DC_\alpha(\gamma, \tau).$$

Then there exists $\varepsilon_0 = \varepsilon_0(\gamma, \tau, h)$, *s.t. if* $\|F\|_h \leq \varepsilon_0$, *then the cocycle is rotations reducible.*

Remark Similar result was obtained by Avila-Fayad-Krikorian [GAFA], but using quite different method.

Step 1: Rotation

Let $L_n = \gamma(8q_n)^\tau$, $(h_n, \varepsilon_n) = (h_{n-1} - \frac{r_{n-1}}{2}, \varepsilon_{n-1} e^{-\frac{q_n r_{n-1}}{8L_{n-1}}})$.
We start with cocycle $(\alpha, e^{F(\cdot)} R_{\varphi(\cdot)})$ satisfying estimate $\|F\|_{h_n} \leq \varepsilon_n$.

Lemma

The cocycle $(\alpha, e^{F(\cdot)} R_{\varphi(\cdot)})$ *can be conjugated by* $(0, R_{\psi(\cdot)})$ *to* $(\alpha, e^{F_1(\cdot)} R_\rho)$, *where*

$$\psi(\cdot + \alpha) - \psi(\cdot) = -\mathcal{T}_{q_n} \varphi(\cdot) + \hat{\varphi}(0),$$

and $\rho = \hat{\varphi}(0) = \rho(\alpha, e^{F(\cdot)} R_{\varphi(\cdot)})$, $\|F_1\|_{h_n} \leq \varepsilon_n^{8/9}$.

Resonances

Lemma (The structure of resonances)

Let $\gamma > 0, \tau > 1$ *and* $\rho \in DC_\alpha(\gamma, \tau) \cap [-1, 1]$. *Then for any* $k \in \mathbb{Z}$ *satisfying* $|k| \leq q_{n+1}/L_n$, *we have*

$$\|k\alpha - 2\rho\|_{\mathbb{Z}} \geq \frac{1}{L_n}.$$

Step 2: Eliminating the non-resonant terms

Lemma

If $\rho \in DC_\alpha(\gamma,\tau)$, then there exists $Y_1 \in C^\omega_{h_n}(\mathbb{T}, sl(2,\mathbb{R}))$, s.t.

$$(0, e^{Y_1(\cdot)}) \circ (\alpha, e^{F_1(\cdot)} R_\rho) \circ (0, e^{Y_1(\cdot)})^{-1} = (\alpha, e^{F_2(\cdot)} R_{\rho+\tilde{\varphi}}).$$

with

$$\mathcal{T}_{q_{n+1}/L_n} F_2 = 0,$$

and

$$\|Y_1\|_{h_n} \le \varepsilon_n^{4/9}, \qquad \|F_2\|_{h_{n+1}} \le c\varepsilon_n^{7/9} e^{-\frac{q_{n+1} r_n}{4L_n}}.$$

Proof: Implicit Function Theorem

Step 3: Rotation back

Lemma

The cocycle $(\alpha, e^{F_2(\cdot)}R_{\rho+\tilde{\varphi}(\cdot)})$ *can be conjugated by* $(0, R_{-\psi(\cdot)})$ *to* $(\alpha, e^{F_+(\cdot)}R_{\varphi_+})$*, with*

$$\|\varphi_+ - \varphi\|_{h_{n+1}} \leq 2\varepsilon_n^{8/9}, \quad \|F_+\|_{h_{n+1}} \leq \varepsilon_{n+1}.$$

The iterative scheme

We construct sequences φ_n, F_n, Y_n, such that

(a) $\varphi_n \in C^\omega_{h_n}(\mathbb{T}, \mathbb{R})$ and $Y_n, F_n \in C^\omega_{h_n}(\mathbb{T}, sl(2, \mathbb{R}))$;

(b) $(0, e^{Y_n(\cdot)}) \circ (\alpha, e^{F_n(\cdot)} R_{\varphi_n(\cdot)} \circ (0, e^{Y_n(\cdot)})^{-1} = (\alpha, e^{F_{n+1}(\cdot)} R_{\varphi_{n+1}(\cdot)})$;

(c) $2\rho(\alpha, e^{F_n(\cdot)} R_{\varphi_n(\cdot)}) \in DC_\alpha(\kappa, \sigma)$;

(d) $\|Y_n\|_{h_n} \leq \varepsilon_n^{1/4}$, and $\|F_n\|_{h_{n+1}} \leq \varepsilon_{n+1}$.

(e) $\|\varphi_{n+1} - \varphi_n\|_{h_{n+1}} \leq \varepsilon_n^{8/9}$.

Global reducibility 1: Renormation

Avila-Krikorian: if α is recurrent Diophantine, then for Lebesgue $a.e.\varphi \in [0,1]$, $(\alpha, R_\varphi A(\cdot))$ is either C^ω reducible or non-uniformly hyperbolic.

Avila, Fayad and Krikorian: for any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, and for Lebesgue $a.e.\varphi \in [0,1]$, $(\alpha, R_\varphi A(\cdot))$ is either C^ω rotations reducible or non-uniformly hyperbolic.

Method: Renormalization. Global to local reduction

Global reducibiity 2: Avila's global theory

- Avila's global theory of analytic quasiperiodic $SL(2,\mathbb{R})$ cocycle: cocycles which are not uniflomly hyperbolic can be classfied into three cases: subcritical, supercritical, critical.

Global reducibiity 2: Avila's global theory

- Avila's global theory of analytic quasiperiodic $SL(2,\mathbb{R})$ cocycle: cocycles which are not uniflomly hyperbolic can be classfied into three cases: subcritical, supercritical, critical.
- Subcritical: the cocycle (α, A) is subcritical if there exists $\kappa > 0$, such that $L(\alpha, A(\cdot + i\varepsilon)) = 0$ for $|\varepsilon| < \kappa$.

Global reducibiity 2: Avila's global theory

- Avila's global theory of analytic quasiperiodic $SL(2,\mathbb{R})$ cocycle: cocycles which are not uniflomly hyperbolic can be classfied into three cases: subcritical, supercritical, critical.
- Subcritical: the cocycle (α, A) is subcritical if there exists $\kappa > 0$, such that $L(\alpha, A(\cdot + i\varepsilon)) = 0$ for $|\varepsilon| < \kappa$.
- Almost reducibility conjecture (ARC): subcritical implies almost reducibility.

Global reducibiity 2: Almost reducibility conjecture

Theorem (Avila)

For any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $A \in C^{\omega}(\mathbb{T}, SL(2,\mathbb{R}))$, (α, A) is almost reducible if it is subcritical.

Lecture 2: Applications

1. Application 1: Kotani-Last Conjecture
2. Application 2: Dry Ten Martini Problem
3. Application 3: Aubry-André-Jitomirskaya conjecture

Application 1: Counter-example to Kotani-Last conjecture

As the first application of reducibility theory, we will give a simple counterexample to Kotani-Last Conjecture.

Setting: Ergodic Schrödigner operator

Continuous ergodic family of Schrödinger operators

$$(\mathcal{L}_\theta y)(t) = -y''(t) + V_\theta(t)y(t).$$

with potentials

$$V_\theta(t) = v(T^t(\theta)), \quad \theta \in \Theta, \quad t \in \mathbb{R},$$

where $(\Theta, \mathcal{B}, \mu, T^t)$ is ergodic and $v : \Theta \to \mathbb{R}$ is measurable and bounded. The discrete counterparts are the ergodic Schrödinger operators

$$(H_\theta u)_n = u_{n+1} + u_{n-1} + V_\theta(n)u_n$$

in $l^2(\mathbb{Z})$, where $V_\theta(n) = v(F^n(\theta))$ and F is an ergodic map.

Examples

Take $(\Theta, \mathcal{B}, \mu, T^t)$ to be (T^d, m, R_ω) where ω is rational independent. This is the so called quasi-periodic Schödinger operators. By Dinaburg-Sinai and Eliasson's result, the operator might have ac spectrum.

On the other hand, if $(\Theta, \mathcal{B}, \mu, T^t)$ is very "chaotic", then Lyapunov exponent are always positive. Then there is no ac spectrum, which is due to Kotani's theory.

Question: if an operator has ac spectrum, how regular the potential $v(T^t\theta$ can be?

Kotani-Last Conjecture

Kotani-Last Conjecture, 1980' If $\Sigma_{ac} \neq \emptyset$, then the potential $V_\theta(\cdot)$ is almost periodic.

Simple counterexample to Kotani-Last conjecture

Theorem (Y-Zhou, IMRN, online 2014)

There exists non-almost periodic potential $V_\theta(t) = v(T^t(\theta))$ which is a composition by weak mixing flows $T^t(\theta)$ and non-constant analytic functions $v(\theta)$ on 2-torus, such that the Schrödigner operators $\mathcal{L}$ with $V_\theta(t)$ have absolutely continuous spectrum.

Almost periodic and weak-mixing

- Almost periodic: a function f is called almost periodic if every sequence $f(t+T_n)$ of translation of f has a subsequence converging. uniformly in $t \in \mathbb{R}$.
- Weak-mixing: For all $f, g \in L^2(\mu)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\int f(T^n x) g(x) d\mu - \int f(x) d\mu \int g(x) d\mu| = 0$$

- Basic facts from ergodic theory, if T is weaking mixing, then it is not almost periodic.

Subordinacy theory

Theorem (Gilbert-Pearson)

Let B be the set of $E \in \mathbb{R}$ such that the generalized eigenfunctions of the operator $\mathcal{L}$ with energy E are bounded, i.e.,

$$B = \{E : \text{ all solutions of } Lu = Eu \text{ are bounded}\}$$

If B has positive Lebesgue measure, then the operator has absolutely continuous spectrum.

Reducibility implies absolutely continuous spectrum:

- Subordinacy theory: bounded eigenfunctions implies absolutely continuous spectrum.

Reducibility implies absolutely continuous spectrum:

- Subordinacy theory: bounded eigenfunctions implies absolutely continuous spectrum.
- Positive measure reducibility implies absolutely continuous spectrum.

Reducibility implies absolutely continuous spectrum:

- Subordinacy theory: bounded eigenfunctions implies absolutely continuous spectrum.
- Positive measure reducibility implies absolutely continuous spectrum.
- Reducibility can be replaced by rotations reducibility.

Ac spectrum with quasi-periodic potential:

Quasi-periodic potential:

$$(\mathcal{L}_\theta y)(t) = -y''(t) + V(\theta + \omega t)y(t). \tag{4}$$

with $\omega = (1, \alpha)$.

Dinaburg-Sinai, Eliasson, Avila-Fayad-Krikorian, Hou-Y: If V is small, then $\mathcal{L}_\theta$ has absolutely continuous spectrum.

Reason: Positive measure reducibility (rotations reducibility).

Reparameterized linear flow

Base dynamics is given by the reparameterized linear flow:

$$\frac{d\theta}{dt} = \frac{1}{\phi(\theta)}(1, \alpha) \tag{5}$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\phi \in C^{\omega}(\mathbb{T}^2, \mathbb{R})$.

Fayad: If $\beta(\alpha) > 0$, for a dense G_δ subset $\mathcal{H}$ of functions ϕ in the space of analytic positive functions, the reparameterized flow T^t of $\frac{1}{\phi(\theta)}(1, \alpha)$ is weak mixing.

Let $v_\theta(t) = f(T^t\theta)$ with non-constant f, then the operator with potential $v_\theta(t)$ is not almost periodic.

Time-rescaling

The eigenvalue equation $\mathcal{L}_\theta y = Ey$ is equivalent to

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 1 \\ v(\theta) - E & 0 \end{pmatrix} x \\ \dot{\theta} = \frac{1}{\phi(\theta)}(1, \alpha). \end{cases} \tag{6}$$

and we rescale the time, and transform (**??**) into

$$\begin{cases} \frac{dx}{ds} = \begin{pmatrix} 0 & \phi(\theta) \\ \phi(\theta)(v(\theta) - E) & 0 \end{pmatrix} x \\ \frac{d\theta}{ds} = (1, \alpha) \end{cases}$$

Assume that $\phi(\theta) = 1 + f(\theta) \in \mathcal{H}$ with f being small enough.

Key point of the proof

Key point of the proof:

- Time-rescaling
- Positive measure rotations reducibility
- Subordinacy theory

Other Counter-examples

Counterexample to Kotani-Last conjecture:

- Avila (JAMS, 2014): both continuous and discrete potential
- Method: periodic approximation, direct construction, powerful but complicated, base dynamics: weak-mixing
- Volberg and Yuditskii(Inventiones Math, 2014): discrete potential
- Method: inverse spectral theory, indirect construction, complicated, the dynamics of the base is unclear.

Application 2: Dry Ten Martini Problem

In the rest of the lectures, we will focus on the application of reducibility theory to Almost Mathieu operators.

$$(H_{\lambda,\theta,\alpha}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos(n\alpha+\theta)u_n.$$

We first talk about Dry Ten Martini Problem.

Ten Martini problem

- Ten Martini problem: the spectrum of almost Mathieu operators is a Cantor set? or equivalently,
- Uniformly hyperbolic cocycles are the dense in the one parameter family $(\alpha, S_E^{2\lambda\cos})$ for all fixed $\lambda \neq 0$ and irrational α.
- Conjecture by Azbel 64;
- Partial solutions: Bellissard-Simon 82, Sinai 87, Helffer-Sjöstrand 89, Choi-Eliott-Yui 90, Last 94, Puig 04.

Ten Martini problem

Theorem (Avila-Krikorian, Ann. Math. 2006)

Ten Martini problem is true for $\lambda = 1$.

Theorem (Avila-Jitomirskaya, Ann. Math. 2009)

Ten Martini problem is true for all $\lambda \neq 1$.

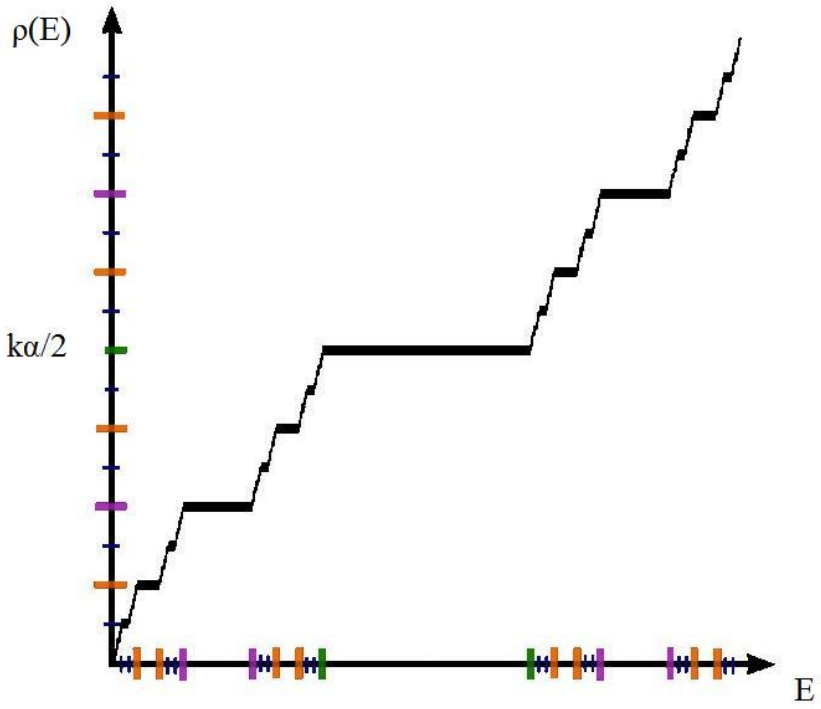
Gap Labeling Theorem

Spectral gaps: intervals in the complimentary of the spectrum set.

Theorem (Johnson-Moser, 1982)

In each spectral gap, the rotation number is a constant of the form $\frac{1}{2}\{n\alpha\}, n \in Z$.

Dry Ten Martin Problemi



Dry Ten Martini problem

- Dry Ten Martini problem: Do all possible gaps open? More precisely, if the set
$$\{E \in \mathbb{R}|, 2\rho(E) = \{n\alpha\}\}$$
is an open set for all n?
- Dry Ten Martini Problem: for all $\lambda \neq 0$, all irrational α, and all integers n_1, n_2 with $0 < n_1 + n_2\alpha < 1$, there is a gap for $H_{\lambda,\alpha,\theta}$ on which $N_{\lambda,\alpha}(E) = n_1 + n_2\alpha$.
- It is obvious that Dry Ten Martini problem implies Ten Martini problem.

Why called Dry Ten Martini problem?

In 1980's Kac hoped to know, if all possible gaps are open for almost Mathieu operators. He offered a reward of ten Martini (wine) to those people who can prove it.

Thereafter, Simon named the problems:

The weak version of the problem (Cantor spectrum) was called the Ten Martini problem;

The strong (original) version of the problem (all gaps open) was called the Dry Ten Martini problem.

Known results in Dry Ten Martini problem

- Choi-Elliott-Yui (Invent. Math. 1990): if $\beta = \infty$.
- Puig (CMP, 2006): Dry Ten Martini problem holds if α is Diophantine and λ small.
- Avila-Jitomirskaya (Ann. Math., 2009): if α is Diophantine and $\lambda \neq 0, \pm 1$ and $e^{-\beta} < \lambda < e^{\beta}$.

Dry Ten Martini problem

Theorem (Avila-Y-Zhou)

If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\lambda \neq 0$, then $H_{\lambda,\alpha,\theta}$ has all spectral gaps open for all $(\lambda, \beta(\alpha)) \neq (\pm 1, 0)$.

Remark: $\Sigma_\lambda = \Sigma_{\lambda^{-1}} = \Sigma_{-\lambda}$, therefore we only need to consider the case $0 < \lambda < 1$.

Aubry duality

Aubry duality from the point of view of reducibility

$$B(\theta+\alpha)^{-1}S_E^{2\lambda\cos}(\theta)B(\theta)=\begin{pmatrix} e^{2\pi i\phi} & 0 \\ 0 & e^{-2\pi i\phi} \end{pmatrix},$$

and write $B(\theta)=\begin{pmatrix} B_{11}(\theta) & B_{12}(\theta) \\ B_{21}(\theta) & B_{22}(\theta) \end{pmatrix}$, then we have

$$(E-2\lambda\cos(\theta))B_{11}(\theta)=B_{11}(\theta-\alpha)e^{-2\pi i\phi}+B_{11}(\theta+\alpha)e^{2\pi i\phi}. \tag{7}$$

Taking the Fourier transformation for (**??**), we have

$$\widehat{B}_{11}(n+1)+\widehat{B}_{11}(n-1)+2\lambda^{-1}\cos 2\pi(\phi+n\alpha)\widehat{B}_{11}(n)=\lambda^{-1}E\widehat{B}_{11}(n),$$

Say $B_{11}(\theta)$ is (analytic) block wave, and $\widehat{B}_{11}(n)$ are eigenfunction of the dual Mathieu operator.

Strategy of the proof in Diophantine case

- the cocycle $(\alpha, S_E^{2\lambda\cos})$ is reducible for $\rho(E) = \frac{1}{2}\{k\alpha\}$ when $0 < \lambda < 1$. When λ is small, this is a consequence of the famous result of Eliasson in his 1992 paper. For $0 < \lambda < 1$, it is proved by Avila-Jitomirskaya later.
- $(\alpha, S_E^{2\lambda\cos})$ for $\rho(E) = \frac{1}{2}\{k\alpha\}$ can not be reduced to the identity. Otherwise, using Aubry dual, it contradicts to the simplicity of eigenvalues.
- Thus the reduced one is parabolic. One can use Moser-Pöschel argument to prove the corresponding gap is open.

Limitation of the method

- If $\beta(\alpha) > 0$, Diophantine approach cann't cover all the rest parameter.
- Jitomirskaya: "even after pushing the existing approaches to absolute technique limits, there still exists parameter belongs to the arithmetically critical regime."
- Solution: replace reducibility by Almost reducibility

Quantitative Almost reducibility

Let $D_n = \begin{pmatrix} 1 & d_n \\ 0 & 1 \end{pmatrix}$.

Proposition

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $\beta(\alpha) > 0$, for any fixed $k_0 \in \mathbb{Z}$, if (α, A) is subcritical with $2\rho(\alpha, A) = k_0\alpha$ mod $\mathbb{Z}$, then there exist $n = n_{k_0}$, $B_n(\theta) \in C^{\omega}_{h_n}(2\mathbb{T}, SL(2,\mathbb{R}))$ with $\|B_n\|_{h_n} \leq e^{2q_{n+1}\varepsilon_n}$, such that

$$B_n(\theta+\alpha)^{-1}A(\theta)B_n(\theta) = A_n + F_n(\theta) \tag{8}$$

where $A_n = id$ (or $A_n = D_n$), $\|F_n\|_{h_n} \leq e^{-cq_{n+1}\delta_n}$, and $\varepsilon_n << \delta_n$.

Proof of almost reducibility

- Three main ingredients
- Avila's solution of almost reducibility conjecture: reduce to local regime.
- Y-Zhou [CMP,2013]: local embedding theorem, embed local cocycle into the flow.
- Hou-Y [Invention 2012]: local almost reducibility for continuous quasi-periodic linear system.
- Why we need local embedding theorem?

Almost reducibility conjecture in case $\beta(\alpha) > 0$

Theorem (Avila)

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $\beta(\alpha) > 0$, and (α, A) is subcritical, then there exists $C > 0$, such that if $\delta > 0$ is sufficiently small, then there exist $N = N(\alpha, \delta)$, $h = h(C, \delta) > 0$, $R_{\varphi_n} \in SO(2, \mathbb{R})$ such that

$$B_n(\theta + \alpha)^{-1} A(\theta) B_n(\theta) = R_{\varphi_n} + F_n(\theta) \tag{9}$$

where $n \geq N(\alpha, \delta)$, and $\|B_n\|_h \leq e^{Cq_n\delta}$, $\|F_n\|_h \leq e^{-q_n\delta}$.

Remark: The growth of conjugation B_n is much faster than the decay of the perturbation F_n.

Proof of Dry Ten Martini: Case 1

If $0 < \lambda < 1$, then for any $E \in \Sigma_\lambda$, $(\alpha, S_E^{2\lambda\cos})$ is subcritical. Suppose that $E_k \in \Sigma_\lambda$ such that $\rho(\alpha, S_E^{2\lambda\cos}) = k\alpha \mathrm{mod} \mathbb{Z}$. By quantitative almost reducibility, the cocycle is reduced to $A_n + F_n(\theta)$ by $B_n(\theta)$ with $\|B_n\|^{-1} \gg \|F_n\| \to 0$ and A_n being constants. Take N sufficiently large. There are two cases.

Case 1. $A_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we need the following:

Proof of Dry Ten Martini: Case 1

Proposition

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $0 < \lambda < 1$, $E \in \Sigma_\lambda$, *there exists* $C = C(h, \lambda)$, *such that for* ε *sufficiently small, there exists no* $B(\theta) \in C_h^\omega(2\mathbb{T}, SL(2, \mathbb{R}))$ *such that*

$$B(\theta + \alpha)^{-1} S_E^{\lambda^{-1}}(\theta) B(\theta) = id + F(\theta) \tag{10}$$

with estimate $\|B\|_h \leq \varepsilon^{-1}$, $\|F\|_h \leq \varepsilon^C$.

Proof: $B_{11}(\theta)$ and $B_{12}(\theta)$ are two approximating linearly independent Block waves. By Aubry dual, their coefficients are two approximating eigenvectors. This contradicts to the preservation of Wronskian.

Proof of Dry Ten Martini: Case 2

The second case:

Case 2. $A_N = \begin{pmatrix} 1 & c_N \\ 0 & 1 \end{pmatrix}$ with $|c_N| \neq 0$ relatively big.

In this case, we will prove that, there is a $\delta_k > 0$, such that the rotation number of $(\alpha, S_E^{2\lambda\cos})$ is same as that of $(\alpha, S_{E_k}^{2\lambda\cos})$ for $E \in (E_k + \delta_k, E_k + 2\delta_k)$ or for $E \in (E_k - 2\delta_k, E_k - \delta_k)$. This implies that the gap corresponding to $\rho(\alpha, S_E^{2\lambda\cos})$ is open.

Problem remained

All gaps are open in the critical case $\lambda = 1$. Then Dry Ten Martini problem is closed.

Application 3: Aubry-André-Jitomirskaya conjecture

The third application of the reducibility theory: Phase transition for almost Mathieu operators.

$$(H_{\lambda,\theta,\alpha}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos(n\alpha+\theta)u_n.$$

Aubry-André conjecture

In 1980, Aubry-André conjectured that the spectral measure of $H_{\lambda,\alpha,\theta}$ depends on λ in the following way:

- If $\lambda < 1$, then $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, and all $\theta \in \mathbb{R}$.
- If $\lambda > 1$, then $H_{\lambda,\alpha,\theta}$ has pure point spectrum for all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and almost all $\theta \in \mathbb{R}$.

Avron-Simon[Bull A.M.S 1982]: this conjecture is wrong, arithmetic property of α has to be taken into account.

Phase transition with Diophantine frequency

Jitomirskaya [Ann. of Math, 1999] settles Aubry-André's conjecture in the measure setting: if $\alpha \in DC$, then

- If $\lambda < 1$, then $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for a.e. $\theta \in \mathbb{R}$.
- If $\lambda > 1$, $H_{\lambda,\alpha,\theta}$ has Anderson Localization (purely point spectrum with exponentially decaying eigenfunctions) for a.e. $\theta \in \mathbb{R}$.

Thus $\lambda = 1$ is a transition point.

Jitomirskaya's conjecture:

- In 1995, Jitomirskaya conjectured the following:
  1. For $1 < \lambda < e^{\beta}$, the spectrum is purely singular continuous for all θ.
  2. For $\lambda > e^{\beta}$, the spectrum has AL for a.e. θ.
- Thus $\lambda = e^{\beta}$ is another transition point.

Phase transition with Liouvillean frequency

Theorem (Avila, online)

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, then $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for all θ if $|\lambda| < 1$.

Theorem (Avila-Y-Zhou)

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $0 < \beta(\alpha) < \infty$, then we have the following:

1. *If $1 \leq |\lambda| < e^{\beta}$, then $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for all θ.*
2. *If $|\lambda| > e^{\beta}$, then $H_{\lambda,\alpha,\theta}$ has Anderson Localization for a.e. θ.*

Phase transition at $\lambda = e^{\beta(\alpha)}$

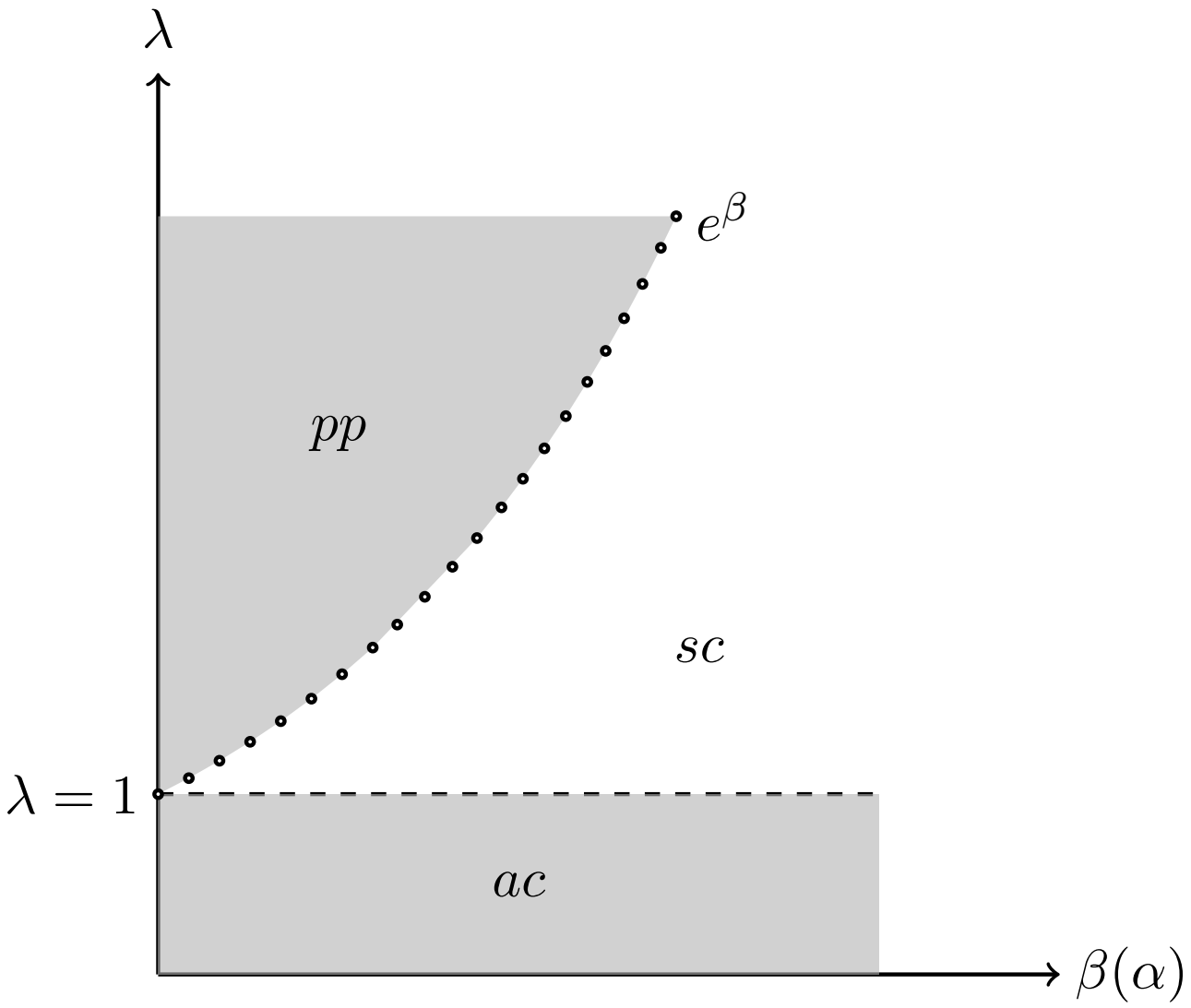


Figure :

Absolutely continuous spectrum

- In the subcritical regime $|\lambda| < 1$, the spectrum has absolutely continuous spectrum.
- Jitomirskaya [Ann of Math, 1999]: If $\alpha \in DC$, $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for *a.e.* $\theta \in \mathbb{R}$.
- Avila-Jitomirskaya [JEMS, 2010]: If α is Diophantine, then $H_{\lambda,\alpha,\theta}$ is purely absolutely continuous for *all* $\theta \in \mathbb{R}$.
- Avila-Damanik [Invent 2008]: If $\beta > 0$, $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for *a.e.* $\theta \in \mathbb{R}$.
- Avila: If $\alpha \in \mathbb{R}/\mathbb{Q}$, then $H_{\lambda,\alpha,\theta}$ is purely absolutely continuous for *all* $\theta \in \mathbb{R}$.

Anderson localization

- Jitomirskaya [Ann of Math, 1999]: If $\alpha \in DC$, $|\lambda| > 1$, then $H_{\lambda,\alpha,\theta}$ has AL for *a.e.* $\theta \in \mathbb{R}$.
- Avila-Jitomirskaya [Ann of Math, 2009]: If $\lambda > e^{\frac{16}{9}\beta}$, then $H_{\lambda,\alpha,\theta}$ has AL for *a.e.* $\theta \in \mathbb{R}$.
- Is there any new method to prove Anderson localization?

More on Aubry duality

- $H_{\lambda,\alpha,\theta}u = Eu$ $\Longleftrightarrow$ $\widehat{H}_{\lambda,\alpha,\phi}v = \frac{E}{\lambda}v$
- Bloch wave (reducibility) $\Longleftrightarrow$ eigenvalue
- a.c for a.e. θ $\Longleftrightarrow$ p.p for a.e. ϕ
- reducibility for "full measure" E $\Longleftarrow$ AL for a.e. ϕ
- reducibility for "full measure" E $\Longrightarrow$ AL for a.e. ϕ???

Full measure reducibility

Theorem (Avila-Y-Zhou)

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $\beta(\alpha) > 0$, if $\lambda > e^{\beta}$, $\rho(\alpha, S_E^{2\lambda^{-1}\cos})$ is Diophantine w.r.t. α, then $(\alpha, S_E^{2\lambda^{-1}\cos})$ is reducible.

- If $\lambda > 1$, $\rho(\alpha, S_E^{2\lambda^{-1}\cos})$ is Diophantine w.r.t. α, then $(\alpha, S_E^{2\lambda^{-1}\cos})$ is rotations reducible in any strip $|Imx| < \ln \lambda$.
- ARC + local rotations reducibility.
- If $\lambda > e^{\beta}$, the cocycle is in fact reducible.

Anderson localization

Theorem (Y-Zhou, 2013 CMP)

let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $\beta(\alpha) < \infty$, if $\lambda > Ce^{\beta}$, with C large enough, then for a.e. ϕ, the eigenvalues of $H_{\lambda,\alpha,\phi}$ is dense in the spectrum, moreover, the eigenfunctions decay exponentially.

1. Topological argument + Aubry duality
2. Check the proof carefully, one can already get $C = 1$.
3. How to get pure point spectrum: complete eigenfunctions in $l^2(Z)$?

Anderson localization

Theorem (Avila-Y-Zhou)

let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $\beta(\alpha) < \infty$, if $\lambda > e^{\beta}$, $H_{\lambda,\alpha,\phi}$ has Anderson localization for a.e. ϕ.

Proof: Study the distribution of the eigenfunctions, by Aubry duality, which can be analyzed by quantitative full measure reducibility.

Singular continuous spectrum

- Gordon's lemma(periodic approximation) help us to exclude the point spectrum.
- Avron-Simon [Bull A.M.S, 1982]: If $\beta(\alpha) = \infty$, then $H_{\lambda,\alpha,\theta}$ has no eigenvalues for any $\theta \in \mathbb{R}$.
- Gordon's lemma+ Kotani's theory: $1 < \lambda < e^{\frac{\beta}{2}}$, $H_{\lambda,\theta,\alpha}$ has purely singular continuous spectrum for any $\theta \in \mathbb{T}$.
- Reason to obtain $e^{\frac{\beta}{2}}$: approximate the solution by periodic ones along double periods.

Singular continuous spectrum

Theorem (Avila-Y-Zhou)

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $0 < \beta(\alpha) \leq \infty$. If $1 \leq \lambda < e^{\beta}$, then $H_{\lambda,\theta,\alpha}$ has purely singular continuous spectrum for any $\theta \in \mathbb{T}$.

Proof: Generalized Gordon Lemma.

Thanks

Thanks!