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Equivalence of three-dimensional Cauchy-Riemann manifolds and multisummability theory

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We apply the *multisummability theory* from Dynamical Systems to CR-geometry. As the main result, we show that two real-analytic hypersurfaces in $\mathbb{C}^2$ are formally equivalent, if and only if they are C^∞ CR-equivalent at the respective point. As a corollary, we prove that all formal equivalences between *real-algebraic* Levi-nonflat hypersurfaces in $\mathbb{C}^2$ are algebraic (and in particular convergent). By doing so, we solve a Conjecture due to N. Mir [29].

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Summability of divergent power series

Contents

1. Introduction

In his 1907 paper [32], H. Poincaré made the fundamental discovery that real-analytic hypersurfaces in complex Euclidean spaces possess nontrivial local invariants under the action of the pseudogroup of (local) biholomorphisms. Poincaré looked at local biholomorphisms as *complex power series maps*

$$(z,w) \longrightarrow \left(z + \sum_{k+l\geq 2} a_{kl} z^k w^l, w + \sum_{k+l\geq 2} b_{kl} z^k w^l\right), \quad (z,w) \in \mathbb{C}^2, \tag{1.1}$$

which in a natural sense act on real-analytic hypersurfaces. (Poincaré was not concerned with convergence of such maps, that is, he dealt with *formal* biholomorphic transformations of hypersurfaces.) He then showed that the action of formal biholomorphisms on the space of k-jets of defining functions of hypersurfaces is not transitive for sufficiently large k, and it implies the existence of the above biholomorphic invariants. This work of Poincaré is often considered as the starting point for studying the holomorphic geometry of real submanifolds in complex space, which (in an a bit more general setting) is referred to as *Cauchy-Riemann geometry* (shortly: CR-geometry).

The cited work of Poincaré led to numerous developments in the subject of CR-geometry and raised a large number of interesting problems, some of which are open till present. The main result of this paper completes one of those: precisely, it solves

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the long standing problem of constructing a *geometric* realization for *formal* biholomorphic transformations (1.1) between arbitrary real-analytic hypersurfaces in $\mathbb{C}^2$. The main tool which made such a realization of formal maps of *arbitrary* real-analytic hypersurfaces possible is the modern theory of summability for formal power series transformations of dynamical systems, developed in the work of Ramis, Sibuya, Ecale, Malgrange, Braaksma, and Balser in the 1990's and referred to as the *multisummability theory*.

Before stating our main result in detail, we shall briefly outline basics of CR-geometry. A hypersurface $M \subset \mathbb{C}^N$ gets endowed with a Cauchy-Riemann (CR) structure by its complex tangent bundle T^cM, whose fibers are the complex tangent planes $T^c_pM = T_pM \cap iT_pM$, $p \in M$. This is a complex vector bundle over M whose structure operator $J\colon T^cM \to T^cM$ is just multiplication by i in $\mathbb{C}^N$. The CR structure bundle is the subbundle $\mathcal{V} \subset \mathbb{C}TM$ whose fibers $\mathcal{V}_p$ consist of vectors of the form $X_p + iJX_p$ with $X_p \in T^c_pM$. A function f is said to satisfy the tangential Cauchy-Riemann equation or simply said to be CR if $\bar{L}f = 0$ for every section $\bar{L}$ of $\mathcal{V}$. Holomorphic functions in a neighborhood of a manifold give basic examples of CR-functions. For more on these, we refer the reader to [5].

Given two hypersurfaces $M, M^* \subset \mathbb{C}^N$, we say that a smooth map $H\colon M \to M^*$ is CR if the natural extension of its differential $dH\colon \mathbb{C}TM \to \mathbb{C}T(M^*)$ restricts to the CR-structure bundle: $dH\colon \mathcal{V} \to \mathcal{V}^*$. It turns out that this first-order system of PDEs on M is equivalent to requiring that if $H = (H_1, \dots, H_N)$, then the components H_j satisfy the tangential Cauchy-Riemann equations, i.e. $\bar{L}H_j = 0$ for every section $\bar{L}$ of $\mathcal{V}$ and $j = 1, \dots, N$. Biholomorphisms of the ambient space transforming two hypersurfaces into each other are basic examples of CR-maps.

Let us, for the rest of this paper, *consider only $\mathcal{C}^\infty$ smooth CR-functions and CR-maps*. Then, after a choice of holomorphic coordinates $Z \in \mathbb{C}^N$, the Taylor series T_pf of a CR-function f at a point p can be identified with a formal power series $T_pf \in \mathbb{C}[\![z-p]\!]^N$ (see e.g. [14,5]), and therefore a CR-map $H\colon M \to M^*$ gives rise to a formal power series map $T_pH = (T_pH_1, \dots, T_pH_N) \in \mathbb{C}[\![Z-p]\!]^N$ for every $p \in M$. (Here and below $\mathbb{C}[\![z-p]\!]^N$ denotes the ring of formal power series centered at p).

On the other hand, if $M = \{\varrho = 0\}$ and $M^* = \{\varrho^* = 0\}$ are given as the vanishing sets of real-analytic defining functions ϱ, ϱ^*, and $p \in M$, we define a formal CR-map $\hat{H}\colon (M,p) \to M^*$ as a formal power series map $\hat{H} \in \mathbb{C}[\![Z-p]\!]^N$ which in addition satisfies the formal condition $\varrho^* \circ \hat{H} = A\varrho$ for some $A \in \mathbb{C}[\![Z-p, \overline{Z-p}]\!]$. Formal CR-maps $\hat{H}\colon (M,p) \to M^*$ whose Jacobian matrix is invertible therefore encode the above mentioned formal obstructions to finding a smooth CR-diffeomorphism H between the real hypersurfaces $M \subset \mathbb{C}^N$ and $M^* \subset \mathbb{C}^N$ satisfying $H(p) = \hat{H}(p)$. We are going to say that M and M^* are *formally equivalent* at $p \in M$ and $p^* = H(p) \in M^*$ if there exists an invertible formal CR map $\hat{H}\colon (M,p) \to M^*$, and that they are CR equivalent if there exists a CR diffeomorphism $H\colon (M,p) \to M^*$. In short our discussion up to this point can be summarized as follows: if M and M^* are CR equivalent, they are also formally equivalent (at every point).

We therefore have the following natural problem: *does every formal CR-diffeomorphism* $\hat{H}\colon (M,p) \to (M^*,p^*)$ *arise as the Taylor series* T_pH *of a smooth CR map* $H\colon M \to M^*$? *Or more generally: if* M *and* M^* *are formally equivalent at some point, are they CR equivalent*?

Note that both questions make sense to ask even in the category of merely *smooth* hypersurfaces. However, the answer is then trivially negative: for example, the flat at the origin perturbation $M = \{\operatorname{Im} w = |z|^2 + e^{-1/|z|^2}\}$ of the quadric $Q = \{\operatorname{Im} w = |z|^2\}$ is formally equivalent to the quadric at the origin (their Taylor series simply coincide), however, it is not difficult to compute that M has a generically non-vanishing CR-curvature (e.g. [13]) and hence is not CR-diffeomorphic to Q.

Our main theorem answers both questions under discussion (in the real-analytic category) in the affirmative in $\mathbb{C}^2$. In order to state it, we also need to recall the notion of being Levi-flat: we say that a hypersurface $M \subset \mathbb{C}^2$ is Levi-flat if it is foliated by complex curves. Equivalently, we can either require that the distribution $T^cM \subset TM$ is integrable. If M is real-analytic, another equivalent condition is that in suitable local holomorphic coordinates (z,w) at p, M can (locally) be written as $\{\operatorname{Im} w = 0\}$.

Theorem 1. *Let* $M, M^* \subset \mathbb{C}^2$ *be two real-analytic hypersurfaces. Assume that* M *and* M^* *are formally equivalent at their reference points* $p \in M$, $p^* \in M^*$. *Then* M *and* M^* *are* (C^∞) *CR-equivalent at the respective points* p, p^*. *If* M, M^* *are in addition Levi-nonflat, then the given formal equivalence* $\widehat{H}$ *between them can be realized by a* (C^∞) *CR-diffeomorphism* $H : M \to M^*$, $H(p) = p^*$, *whose Taylor series at* p *is* $\widehat{H}$.

We shall particularly emphasize here that the proof of Theorem 1 exhibits an exciting and surprising application to CR-geometry of the *multisummability theory* from Dynamical Systems mentioned above (see Section 2.6 for details and references here).

We shall further note that the regularity asserted in Theorem 1 cannot be improved further! This is discussed below.

Finally, we shall explain that in the Levi-flat case a smooth realization of an arbitrary formal CR-diffeomorphism is not possible: for example, any map $z \mapsto f(z)$, $w \mapsto w$, $f(0) = 0$, $f'(0) \neq 0$ with a formal and divergent $f(z)$ is a formal CR-automorphism of the Levi-flat model $\Pi = \{\operatorname{Im} w = 0\}$ at the origin, which can not be realized by a smooth CR-automorphism. Indeed, a smooth CR-automorphism of Π has the form $(z,u) \mapsto H(z,u) = (F(z,u), g(u))$ with F, G smooth and F holomorphic in z. Thus F can be expanded as $\sum F_k(u)z^k$, and the series converges in z for each fixed u. But the Taylor series of H at 0 equals $(f(z), u)$, thus the series $\sum F_k(0)z^k$ and $f(z)$ coincide. Since the latter series is divergent, we arrive to a contradiction and this proves the desired claim.

The content of Theorem 1 is the endpoint (in $\mathbb{C}^2$) of a long development. The starting point was the case of Levi-nondegenerate hypersurfaces $M, M^* \subset \mathbb{C}^2$. We recall that Levi-nondegeneracy refers to the lowest order obstruction to being Levi-nonflat, i.e. T^cM being nonintegrable: one says that M is Levi-nondegenerate at p if for any two sections

X, Y of $T^c M$ with X_p and Y_p (real) linearly independent in $T^c M$, the commutator $[X, Y]_p \notin T^c_p M$. Levi-nondegenerate hypersurfaces in $\mathbb{C}^2$ have been classified up to local holomorphic equivalence, i.e. up to agreeing in suitable local holomorphic coordinates, by E. Cartan [12] (this was later generalized for Levi-nondegenerate hypersurfaces in $\mathbb{C}^N$, $N > 2$, by Tanaka [43] and Chern-Moser [13]). It turns out that the only obstruction to holomorphic equivalence of two Levi-nondegenerate hypersurfaces M and M^* in $\mathbb{C}^2$ is that they are formally equivalent, and that in fact every formal CR-equivalence between Levi-nondegenerate hypersurfaces converges.

Later on, it was shown by Baouendi, Ebenfelt, and Rothschild [6] that this convergence phenomenon persists in the case where the hypersurfaces are of finite type. Here one says that a hypersurface $M \subset \mathbb{C}^2$ is of finite type at p if for any two sections X, Y of $T^c M$ with X_p and Y_p (real) linearly independent in $T^c M$, some commutator $[X, [X, \dots, [X, Y] \dots]]_p \notin T^c_p M$.

The situation is quite different in the Levi-flat case: here, there are plenty of non-convergent formal maps, as any map of the form $H(z, w) = (f(z, w), g(w)) \in \mathbb{C}[\![z, w]\!]^2$ which satisfies $g(w) = \overline{g}(w)$ maps $\operatorname{Im} w = 0$ into itself.

The question what actually happens in the case of a Levi-nonflat, but *infinite type* hypersurface, has long remained open. We say that a hypersurface $M \subset \mathbb{C}^2$ is of infinite type at the point p if the distribution $T^c M$ has an integral submanifold through the point p (see e.g. [5]). It turns out that if M is real-analytic, then any such integral manifold necessarily is a nonsingular complex curve (contained in M). This complex curve is called the complex locus of (M, p).

Contrary to the finite type case, in the infinite type case, there can be *divergent* formal CR maps. To be exact, the first author and Shafikov [23] showed that there exist Levi-nonflat hypersurfaces $M, M^* \subset \mathbb{C}^2$ with M of infinite type at p such that M and M^* are formally equivalent at p, and such that every formal equivalence $\hat{H} \colon (M, p) \to M^*$ diverges. On the other hand, the first and the second author showed [21] that there exist hypersurfaces $M, M^* \subset \mathbb{C}^2$, and a point $p \in M$ of infinite type, such that M and M^* are CR equivalent, but not holomorphically equivalent. These phenomena came as a surprise to many specialists, and made the question of whether formal equivalence implies CR equivalence an interesting one.

We therefore see that the assertion of Theorem 1 cannot be further strengthened. In fact, Theorem 1 gives a complete answer to the nature of obstructions to CR equivalence of real-analytic hypersurfaces in $\mathbb{C}^2$: they are all purely formal.

Let us again emphasize that even though we are dealing with real-analytic hypersurfaces, we establish the existence of merely *smooth* CR-diffeomorphisms whose Taylor series agree with a given formal CR equivalence. The results of [23,21] mentioned above actually show that this is the best we can hope for. We also emphasize that our result does not require any assumptions besides analyticity of the manifolds. It contrasts with similar results in Dynamical Systems such as Chen-Sternberg theorem [40] concerning smooth classification of germs of vector fields at a fixed point, which require some *hyperbolicity* assumptions on the linear part of the vector field at the fixed point.

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A nice application of Theorem 1 was observed by Nordine Mir, resolving a conjecture (by Mir) stated in [29]. Before we state this result, let us recall that a power series map $\hat{H} \in \mathbb{C}[\![z-p]\!]^N$ is algebraic if there exist nontrivial polynomials $p_j(z,w)$ such that $p_j(z,\hat{H}_j(z)) = 0$ for every $j = 1,\dots,N$. Every algebraic power series map is actually convergent (one can see this from Artin's Approximation Theorem [1], see e.g. [38]). Let us also recall that a real-analytic hypersurface $M \subset \mathbb{C}^N$ is said to be real-algebraic if it is contained in the vanishing locus of a nontrivial real polynomial in the underlying real variables in $\mathbb{R}^{2N} = \mathbb{C}^N$.

Theorem 2. *Let $M, M^* \subset \mathbb{C}^2$ be two real-algebraic Levi-nonflat hypersurfaces, and $\widehat{H} : (M,p) \mapsto (M^*,p^*)$ a formal invertible CR-map. Then $\widehat{H}$ is algebraic, and in particular, $\widehat{H}$ is convergent.*

Proof of Theorem 2. By Theorem 1, we can find a CR-diffeomorphism $H\colon (M,p) \to (M^*,p^*)$, whose Taylor expansion at p coincides with $\widehat{H}$. We can now apply the algebraicity theorem of Baouendi, Huang and Rothschild [7] (see also Webster [46]) to conclude that H is an algebraic map (in particular, it is holomorphic). Since, again, $\widehat{H}$ is the Taylor expansion of H at p, this implies the assertion of the theorem. $\square$

As follows from the above mentioned theorem of Shafikov and the first author, the convergence phenomenon in Theorem 2 is a specific feature of *algebraic* (but not general analytic!) hypersurfaces, similarly to the theorem of Baouendi-Huang-Rothschild.

Before we describe our approach to the problem in more detail, we will state an additional result which our main theorem actually relies on. This result, furthermore, emphasizes the very analytic nature of Theorem 1. To formulate it, for a real-analytic Levi-nonflat hypersurface $M \subset \mathbb{C}^2$ of infinite type at a point p, we use coordinates of the kind

$$\operatorname{Im} w = (\operatorname{Re} w)^m \Phi(z,\bar{z},\operatorname{Re} w), \quad \Phi(z,0,\operatorname{Re} w) = \Phi(0,\bar{z},\operatorname{Re} w) = 0,\ \Phi \not\equiv 0 \tag{1.2}$$

(see Meylan [28]). The invariant integer $m \geq 1$ here is the *nonminimality order* of M at p, and Φ is holomorphic in all its variables.

Theorem 3. *Let $M, M^* \subset \mathbb{C}^2$ be two hypersurfaces (1.2), and $\widehat{H} : (M,0) \mapsto (M^*,0)$ a formal invertible CR-map. Then, there exist a disc $\Delta \subset \mathbb{C}$, sectors $S^{\pm} \subset (\mathbb{C},0)$ with vertex at 0 containing the directions $\mathbb{R}^{\pm}$, respectively, and holomorphic maps $H_{\pm}\colon \Delta \times S^{\pm} \to \mathbb{C}^2$ such that $\widehat{H}$ is the asymptotic expansion of $H_{\pm}$ in $\Delta \times S^{\pm}$ and $H_{\pm}(M \cap (\Delta \times S^{\pm})) \subset M^*$; in particular, $H_{\pm}|_M$ defines a CR diffeomorphism H of M onto M^*.*

An expanded version of Theorem 3 is given in the end of Section 4 (Theorem 5).

Remark 1.1. In fact, the proof of Theorem 3 shows that the formal map $\widehat{H}$ in Theorem 3 has actually a finite *(multi) Gevrey order* $(0,s)$ and, furthermore, has the *multisummabil-*

ity property, as stated in Theorem 5 below (see Section 2.6 for details of both concepts). The multisummability of $\widehat{H}$ plays the central role in the proof of the main result, as it gives the *uniqueness* of a holomorphic sectorial realization for the formal map $\widehat{H}$.

Remark 1.2. As follows from the counter-examples given in [23,21], the properties of formal CR-maps stated in Theorem 3 and Remark 1.1 are in general optimal and can't be strengthened further.

Remark 1.3. It can be verified from the proof of Theorem 3 that, for $m \geq 2$, the opening of the sectors $S^{\pm}$ in Theorem 3 can be chosen to be $\frac{\pi}{m-1}$ for a generic hypersurface M under consideration, and the Gevrey order s can be chosen to be $s = \frac{1}{m-1}$. For $m = 1$ one can take $s = 0$ (i.e., $\widehat{H}$ is *convergent*), as follows from the result of Juhlin and the second author [20].

We end this introduction by giving a short guide to the proof of Theorem 1. As discussed above, the main tool of the paper is the multisummability theory from Dynamical Systems, which meets CR-geometry via the recent CR $\longrightarrow$ DS (Cauchy-Riemann manifolds $\longrightarrow$ Dynamical Systems) technique developed by Shafikov and the first two authors in the recent works [23,24,21,22]. In this framework, we study maps of CR-submanifolds M, M^* with prescribed properties (such as being of infinite type) through symmetries of an associated holomorphic dynamical system $\mathcal{E}(M)$. The possibility to replace a real-analytic CR-manifold by a complex dynamical system is based on the fundamental parallel between CR-geometry and the geometry of completely integrable PDE systems. This parallel was first observed by E. Cartan and Segre [12,39] and was revisited and further developed in a recent series of publications by Sukhov ([41,42]). The "mediator" between a CR-manifold and the associated PDE system is the Segre family of the CR-manifold. Unlike the Levi-nondegenerate setting in the cited work [12,39,41,42], the CR - DS technique deals specifically with the *Levi-degenerate* setting, providing sort of a dictionary between CR-geometry and Dynamical Systems.

For the proof of Theorem 3 we need to develop the CR - DS technique further, extending it to the *entire* class of real-analytic hypersurfaces in $\mathbb{C}^2$. In Section 3, infinite type hypersurfaces satisfying a certain nondegeneracy assumption (generic infinite type case) are studied. To do so, we follow the approach in [24,22] and consider complex meromorphic differential equations associated with these hypersurfaces. Any formal map between real hypersurfaces has to transform the associated ODEs into each other, and working out the latter condition gives a certain *singular Cauchy problem* for the components of the map. We then apply the multisummability theory for formal power series solutions of nonlinear systems of ODE at an irregular singularity [10,37]. First, we show that the singular Cauchy problem has solutions, holomorphic in certain sectorial domains with Gevrey asymptotic expansion. Second, we show that these solutions have certain uniqueness properties giving the condition $H(M) \subset M^*$ for the arising CR-map defined on M.

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In Section 4, we have to extend the scheme in Section 3 to the exceptional (non-generic) case. For doing so, we introduce a new tool: *associated differential equations of high order*. In turns out that *any* Levi-nonflat real-analytic hypersurface M (in particular, a finite type hypersurface!) can be associated, in appropriate local holomorphic coordinates, with a system of singular ODEs of the kind (4.16). We achieve this by a sequence of coordinate changes and appropriate *blow-ups* (both in the initial space and in the space of parameters for Segre families). In this regard, the blow-up procedure of Mir and the second author from [25] is a key tool. The initial formal CR map is shown to be a transformation between the associated systems of singular ODE again. Working out the transformation rule here brings significant new difficulties, since we deal with jet prolongations of arbitrarily high order. After overcoming these difficulties, we are again able to apply the multisummability theory and obtain the desired regularity property for the formal CR map.

2. Preliminaries

2.1. Segre varieties

Let M be a generic smooth real-analytic submanifold in $\mathbb{C}^{n+k}$ of CR-dimension n and CR-codimension k, $n, k > 0$, $0 \in M$, and U a neighborhood of the origin where $M \cap U$ admits a real-analytic defining function $\phi(Z, \overline{Z})$ with the property that $\phi(Z, \zeta)$ is a holomorphic function for $(Z, \zeta) \in U \times \bar{U}$. For every point $\zeta \in U$ we associate its Segre variety in U by

$$Q_\zeta = \{Z \in U : \phi(Z, \overline{\zeta}) = 0\}.$$

Segre varieties depend holomorphically on the variable $\overline{\zeta}$, and for small enough neighborhoods U of 0, they are actually holomorphic submanifolds of U of codimension k.

One can choose coordinates $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^k$ and a neighborhood $U = U^z \times U^w \subset \mathbb{C}^n \times \mathbb{C}^k$ such that, for any $\zeta \in U$,

$$Q_\zeta = \left\{(z, w) \in U^z \times U^w : w = h(z, \overline{\zeta})\right\}$$

is a closed complex analytic graph. h is a holomorphic function on $U^z \times \bar{U}$. The anti-holomorphic $(n+k)$-parameter family of complex submanifolds $\{Q_\zeta\}_{\zeta \in U_1}$ is called *the Segre family* of M at the origin. The following basic properties of Segre varieties follow from the definition and the reality condition on the defining function:

$$\begin{aligned} &Z \in Q_\zeta \Leftrightarrow \zeta \in Q_Z, \\ &Z \in Q_Z \Leftrightarrow Z \in M, \\ &\zeta \in M \Leftrightarrow \{Z \in U : Q_\zeta = Q_Z\} \subset M. \end{aligned} \tag{2.1}$$

The fundamental role of Segre varieties for holomorphic maps is due to their *invariance property*: If $f : U \to U'$ is a holomorphic map which sends a smooth real-analytic submanifold $M \subset U$ into another such submanifold $M' \subset U'$, and U is chosen as above (with the analogous choices and notations for M'), then

$$f(Q_Z) \subset Q'_{f(Z)}.$$

For more details and other properties of Segre varieties we refer the reader to e.g. [46], [15], or [5].

The space of Segre varieties $\{Q_Z : Z \in U\}$, for appropriately chosen U, can be identified with a subset of $\mathbb{C}^K$ for some $K > 0$ in such a way that the so-called *Segre map* $\lambda : Z \to Q_Z$ is antiholomorphic. This can be seen from the fact that if we write

$$h(z, \bar{\zeta}) = \sum_{\alpha \in \mathbb{N}^n} h_\alpha(\bar{\zeta}) z^\alpha,$$

then $\lambda(Z)$ can be identified with $\big(h_\alpha(\bar{Z})\big)_{\alpha \in \mathbb{N}^n}$. After that the desired fact follows from the Noetherian property.

If M is a hypersurface, then its Segre map is one-to-one in a neighborhood of every point p where M is Levi nondegenerate. When such a real hypersurface M contains a complex hypersurface X, for any point $p \in X$ we have $Q_p = X$ and $Q_p \cap X \neq \emptyset \Leftrightarrow p \in X$, so that the Segre map λ sends the entire X to a unique point in $\mathbb{C}^N$ and, accordingly, λ is not even finite-to-one near each $p \in X$, i.e. M is *not essentially finite* at points $p \in X$. For the notion of essential finiteness, see e.g. [5].

2.2. Nonminimal real hypersurfaces

We recall that given a real-analytic Levi-nonflat hypersurface $M \subset \mathbb{C}^2$, for every $p \in M$ there exist so-called *normal coordinates* (z, w) centered at p, i.e. a local holomorphic coordinate system near p in which $p = 0$ and near 0, M is defined by an equation of the form

$$v = F(z, \bar{z}, u)$$

for some germ F of a holomorphic function on $\mathbb{C}^3$ which satisfies

$$F(z, 0, u) = F(0, \bar{z}, u) = 0$$

and the *reality condition* $F(z, \bar{z}, u) \in \mathbb{R}$ for $(z, u) \in \mathbb{C} \times \mathbb{R}$ close to 0 (see e.g. [5]).

We say that M is *nonminimal* at p if there exists a germ of a nontrivial complex curve $X \subset M$ through p. It turns out that in normal coordinates, such a curve X is necessarily defined by $w = 0$; in particular, any such X is nonsingular.

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Thus a Levi-nonflat hypersurface M is nonminimal if and only if with normal coordinates (z, w) and a defining function F as above, we have that $F(z, \bar{z}, 0) = 0$, or equivalently, if M can defined by an equation of the form (1.2). It turns out that the integer $m \geq 1$ in (1.2) is independent of the choice of normal coordinates (see [28]), and actually also of the choice of $p \in X$; we refer to m as the *nonminimality order* of a Levi-nonflat hypersurface M on X (or at p) and say that M is m-*nonminimal* along X (or at p).

Several other variants of defining functions for M are useful. Throughout this paper, we use the *complex defining function* Θ in which M is defined by

$$w = \Theta(z, \bar{z}, \bar{w});$$

it is obtained from F by solving the equation

$$\frac{w - \bar{w}}{2i} = F\left(z, \bar{z}, \frac{w + \bar{w}}{2}\right)$$

for w. The complex defining function satisfies the conditions

$$\Theta(z, 0, \eta) = \Theta(0, \xi, \eta) = \tau, \quad \Theta(z, \xi, \bar{\Theta}(\xi, z, w)) = w.$$

If M is m-nonminimal at p, then $\Theta(z, \xi, \eta) = \eta\theta(z, \xi, \eta)$ and thus M is defined by

$$\begin{gathered} w = \bar{w}\theta(z, \bar{z}, \bar{w}) = \bar{w}(1 + \bar{w}^{m-1}\tilde{\theta}(z, \bar{z}, \bar{w})), \\ \text{where } \tilde{\theta}(z, 0, \eta) = \tilde{\theta}(0, \xi, \eta) = 0 \text{ and } \tilde{\theta}(z, \xi, 0) \neq 0. \end{gathered}$$

The Segre family of M, where M is given in normal coordinates as above, with the complex defining function $\Theta\colon U_z \times \bar{U}_z \times \bar{U}_w = U_z \times \bar{U} \to U_w$ consists of the complex hypersurfaces $Q_\zeta \subset U$, defined for $\zeta \in U$ by

$$Q_\zeta = \{(z, w)\colon w = \Theta(z, \bar{\zeta})\}.$$

The real line

$$\Gamma = \{(z, w) \in M\colon z = 0\} = \{(0, u) \in M\colon u \in \mathbb{R}\} \subset M \tag{2.2}$$

has the property that

$$Q_{(0,u)} = \{w = u\}, \quad (0, u) \in \Gamma$$

for $u \in \mathbb{R}$, a property which actually is equivalent to the normality of the coordinates (z, w). More exactly, for any real-analytic curve γ through p one can find normal coordinates (z, w) in which γ corresponds to Γ in (2.2) (see e.g. [25]).

We finally have to point out that a real-analytic Levi-nonflat hypersurface $M \subset \mathbb{C}^2$ can exhibit nonminimal points of two kinds, which can be referred to as *generic* and *exceptional* nonminimal points, respectively. A generic point $p \in M$ is characterized by the condition that *the minimality locus* $M \setminus X$ *of* M *is Levi-nondegenerate locally near* p. At a generic nonminimal point, (1.2) is supplemented by the condition $\psi_{z\bar{z}}(0,0,0,) \neq 0$. In terms of the complex defining function, it gives the following useful representation for M:

$$w = \Theta(z, \bar{z}, \bar{w}) = \bar{w} + \bar{w}^m \sum_{k,l \geq 1} \Theta_{kl}(\bar{w}) z^k \bar{z}^l, \quad \Theta_{11}(0) \neq 0 \tag{2.3}$$

(see, e.g., [24]).

If, otherwise, the intersection of the minimal locus $M \setminus X$ of M with any neighborhood of p in M contains Levi-degenerate points, then such a point p is referred to as *exceptional*.

2.3. Real hypersurfaces and second order differential equations

To every Levi nondegenerate real hypersurface $M \subset \mathbb{C}^N$ we can associate a system of second order holomorphic PDEs with 1 dependent and $N-1$ independent variables, using the Segre family of the hypersurface. This remarkable construction goes back to E. Cartan [12] and Segre [39] (see also a remark by Webster [46]), and was recently revisited in the work of Sukhov [41], [42] in the nondegenerate setting, and in the work of Shafikov and the first two authors in the degenerate setting (see [23], [24], [21], [22]). We describe this procedure in the case $N = 2$ relevant for our purposes.

Let $M \subset \mathbb{C}^2$ be a smooth real-analytic hypersurface, passing through the origin, and $U = U_z \times U_w$ a sufficiently small neighborhood of the origin. In this case we associate a second order holomorphic ODE to M, which is uniquely determined by the condition that the equation is satisfied by all the graphing functions $h(z, \zeta) = w(z)$ of the Segre family $\{Q_\zeta\}_{\zeta \in U}$ of M in a neighborhood of the origin.

More precisely, since M is Levi-nondegenerate near the origin, the Segre map $\zeta \longrightarrow Q_\zeta$ is injective and the Segre family has the so-called transversality property: if two distinct Segre varieties intersect at a point $q \in U$, then their intersection at q is transverse. Thus, $\{Q_\zeta\}_{\zeta \in U}$ is a 2-parameter family of holomorphic curves in U with the transversality property, depending holomorphically on $\bar{\zeta}$. It follows from the holomorphic version of the fundamental ODE theorem (see, e.g., [19]) that there exists a unique second order holomorphic ODE $w'' = \Phi(z, w, w')$, satisfied by all the graphing functions of $\{Q_\zeta\}_{\zeta \in U}$.

To be more explicit we consider the complex defining equation $w = \rho(z, \bar{z}, \bar{w})$, as introduced above. The Segre variety Q_ζ of a point $\zeta = (a, b) \in U$ is now given as the graph

$$w(z) = \rho(z, \bar{a}, \bar{b}). \tag{2.4}$$

Differentiating (2.4) once, we obtain

$$w' = \rho_z(z, \bar{a}, \bar{b}). \tag{2.5}$$

Considering (2.4) and (2.5) as a holomorphic system of equations with the unknowns $\bar{a}, \bar{b}$, an application of the implicit function theorem yields holomorphic functions A, B such that

$$\bar{a} = A(z, w, w'), \, \bar{b} = B(z, w, w').$$

The implicit function theorem applies here because the Jacobian of the system coincides with the Levi determinant of M for $(z, w) \in M$ ([5]). Differentiating (2.5) once more and substituting for $\bar{a}, \bar{b}$ finally yields

$$w'' = \rho_{zz}(z, A(z, w, w'), B(z, w, w')) =: \Phi(z, w, w'). \tag{2.6}$$

Now (2.6) is the desired holomorphic second order ODE $\mathcal{E} = \mathcal{E}(M)$.

More generally, the association of a completely integrable PDE with a CR-manifold is possible for a wide range of CR-submanifolds (see [41,42]). The correspondence $M \longrightarrow \mathcal{E}(M)$ has the following fundamental properties:

(1) Every local holomorphic equivalence $F : (M, 0) \longrightarrow (M', 0)$ between CR-submanifolds is an equivalence between the corresponding PDE systems $\mathcal{E}(M), \mathcal{E}(M')$ (see subsection 2.4);
(2) The complexification of the infinitesimal automorphism algebra $\mathfrak{hol}^\omega(M, 0)$ of M at the origin coincides with the Lie symmetry algebra of the associated PDE system $\mathcal{E}(M)$ (see, e.g., [30] for the details of the concept).

Even though for a real hypersurface $M \subset \mathbb{C}^2$ which is nonminimal at the origin there is no a priori way to associate to M a second order ODE or even a more general PDE system near the origin, the Shafikov and the first author found an injective correspondence between nonminimal hypersurfaces $M \subset \mathbb{C}^2$ which are spherical outside the complex locus hypersurfaces and certain *singular* complex ODEs $\mathcal{E}(M)$ with an isolated singularity at the origin in [24]. It is possible to extend this construction to the non-spherical case, which we do in Section 3.

2.4. Equivalences and symmetries of ODEs

We start with a description of the jet prolongation approach to the equivalence problem (which is a simple interpretation of a more general approach in the context of *jet bundles*). We refer to the excellent sources [30], [8] for more details and collect the necessary prerequisites here. In what follows all variables are assumed to be complex, all mappings biholomorphic, and all ODEs to be defined near their zero solution $y(x) = 0$.

Consider two ODEs, $\mathcal{E}$ given by $y^{(k)} = \Phi(x, y, y', ..., y^{(k-1)})$ and $\tilde{\mathcal{E}}$ given by $y^{(k)} = \tilde{\Phi}(x, y, y', ..., y^{(k-1)})$, where the functions Φ and $\tilde{\Phi}$ are holomorphic in some neighborhood of the origin in $\mathbb{C}^{k+1}$. We say that a germ of a biholomorphism $H\colon (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ transforms $\mathcal{E}$ into $\tilde{\mathcal{E}}$, if it sends (locally) graphs of solutions of $\mathcal{E}$ into graphs of solutions of $\tilde{\mathcal{E}}$. We define the *k-jet space* $J^k(\mathbb{C}, \mathbb{C})$ to be the $(k+2)$-dimensional linear space with coordinates $x, y, y_1, ..., y_k$, which correspond to the independent variable x, the dependent variable y and its derivatives up to order k, so that we can naturally consider $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as complex submanifolds of $J^k(\mathbb{C}, \mathbb{C})$.

For any biholomorphism H as above one may consider its *k-jet prolongation* $H^{(k)}$, which is defined on a neighborhood of the origin in $\mathbb{C}^{k+2}$ as follows. The first two components of the mapping $H^{(k)}$ coincide with those of H. To obtain the remaining components we denote the coordinates in the preimage by (x, y) and in the target domain by (X, Y). Then the derivative $\frac{dY}{dX}$ can be symbolically recalculated, using the chain rule, in terms of x, y, y', so that the third coordinate Y_1 in the target jet space becomes a function of x, y, y_1. In the same manner one obtains the remaining components of the prolongation of the mapping H. Thus, for differential equations of order k, *a mapping H transforms the ODE $\mathcal{E}$ into $\tilde{\mathcal{E}}$ if and only if the prolonged mapping $H^{(k)}$ transforms $(\mathcal{E}, 0)$ into $(\tilde{\mathcal{E}}, 0)$ as submanifolds in the jet space $J^k(\mathbb{C}, \mathbb{C})$*. A similar statement can be formulated for systems of differential equations, as well as for certain singular differential equations, for example, the ones considered in the next subsection.

Some further details and properties of the jet prolongations $H^{(k)}$ are given in Section 4.

2.5. Tangential sectorial domains and smooth CR-mappings

Let $M \subset \mathbb{C}^2$ be a real-analytic Levi nonflat hypersurface, which is nonminimal at a point $p \in M$, and $X \ni p$ its complex locus. We choose for M local holomorphic coordinates (1.2) so that $p = 0$, $X = \{w = 0\}$. We next recall the following definition (see [22, Section 1.2]).

Definition 2.1. A set $D_p \subset \mathbb{C}^2$, $D_p \ni p$ is called a *tangential sectorial domain for M at p* if, in some local holomorphic coordinates (z, w) for M as above, the set D_p looks as

$$\Delta \times \Big(S^+ \cup \{0\} \cup S^-\Big). \tag{2.7}$$

Here $\Delta \subset \mathbb{C}$ is a disc of radius $r > 0$, centered at the origin, and $S^\pm \subset \mathbb{C}$ are sectors

$$S^+ = \big\{|w| < R,\, \alpha^+ < \arg w < \beta^+\big\}, \quad S^- = \big\{|w| < R,\, \alpha^- < \arg w < \beta^-\big\} \tag{2.8}$$

for appropriate $R > 0$ and such that $S^\pm$ contains the direction $\mathbb{R}^\pm$. We also denote by $D_p^\pm$ the domains $\Delta \times S^\pm \subset \mathbb{C}^2$ respectively.

As discussed in [22], *for any tangential sectorial domain D_p for M at p, the intersection of M with a sufficiently small neighborhood U_p of p in $\mathbb{C}^2$ is contained in D_p.*

Next, we recall the following classical notion.

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Definition 2.2. Let $f(w)$ be a function holomorphic in a sector $S \subset \mathbb{C}$. We say that a formal power series $\hat{f}(w) = \sum_{j \geq 0} c_j z^j$ is the *Poincaré asymptotic expansion of f in S*, if for any $n \geq 0$ we have:

$$\frac{1}{w^n}\left(f(w) - \sum_{j=0}^{n} c_j w^j\right) \to 0 \quad \text{when} \quad w \to 0,\ w \in S.$$

In the latter case, we write: $f(w) \sim \hat{f}(w)$.

For basic properties of the asymptotic expansion we refer to [45]. In particular, we recall that asymptotic expansion in a full punctured neighborhood of a point means the usual holomorphicity of a function.

The notion of Poincaré asymptotic expansion can be naturally extended to function holomorphic in products of sectors and the respective formal power series in several variables. This allows us to formulate the following

Definition 2.3. We say that a C^∞ CR-function f in a neighborhood of p in M is *sectorially extendable,* if for some (and then any sufficiently small) tangential sectorial domain D_p for M at p, there exist functions $f^\pm \in \mathcal{O}(D_p^\pm)$ such that

(i) each $f^\pm$ coincides with f on $D_p^\pm \cap M$, and

(ii) both $f^\pm$ admit the same Poincaré asymptotic representation

$$f^\pm \sim \sum_{k,l \geq 0} a_{kl} z^k w^l \in \mathbb{C}[\![z, w]\!]$$

in the respective domains $D_p^\pm$.

We can similarly define the sectorial extendability of CR-mappings or infinitesimal CR-automorphisms of real-analytic hypersurfaces. Crucially, it is not difficult to see (as discussed in [22]) that *restricting two holomorphic functions $f^\pm$, as in Definition 2.3, onto a nonminimal hypersurface M as above defines a C^∞ CR-function on M near* 0*, sectorially extendable into the initial tangential sectorial domain.* This observation will be the final ingredient for the proof of Theorem 1.

2.6. Summability of formal power series

In this section, we shall recall some known facts about multisummability of formal power series and we shall recall a key theorem due to Braaksma that says that any formal solution of a system of nonlinear differential equations at an irregular singularity is multisummable in any direction but a finite number of them. This means there are holomorphic solutions in some sectors with vertex at the singularity and having the formal solution as asymptotic power series. This has a long although recent history and we refer to [33,34,3,18,36] for more information.

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Definition 2.4. Let $s > 0$. A formal power series $\hat{f} = \sum_{n \geq 0} f_n z^n$ is said to be a *Gevrey series of order s* if there exist $A, B > 0$ such that $|f_n| \leq AB^n \Gamma(1 + sn)$ for all n. The space of such power series is denoted by $\mathbb{C}[\![z]\!]_s$.

In other words, we have $|f_n| \leq \tilde{A}\tilde{B}^n (n!)^s$ for some appropriate constants.

This notion can be in fact extended to functions of several variables, and then the Gevrey order is replaced by the *Gevrey (multi) order*. For example, a formal power series $\widehat{H}(z, w) = \sum_{k,l \geq 0} c_{kl} z^k w^l$ in the variables z, w (centered at the origin) is said to be *of the* (r, s) *Gevrey (multi) order,* $r, s > 0$, if there exist appropriate constants $A, B, C > 0$ such that the Taylor coefficients c_{kl}, $k, l \geq 0$ satisfy the bounds:

$$|c_{kl}| \leq A \cdot B^k \cdot C^l \cdot (k!)^r (l!)^s. \tag{2.9}$$

Let $I =]a, b[$ be an open interval of $\mathbb{R}$ and let $r > 0$. We denote by $\mathcal{S}_r(I)$ the open sector of $\mathbb{C}$ (or the Riemann surface of the Logarithm):

$$\mathcal{S}_r(I) := \{z \in \mathbb{C} | \quad a < \arg z < b, \quad 0 < |z| < r\}.$$

Now, we give

Definition 2.5. A holomorphic function $f \in \mathcal{O}(\mathcal{S}_r(I))$ is said to have an *s-Gevrey asymptotic expansion* at 0 if there exists a formal power series $\hat{f} = \sum_{j \geq 0} f_j z^j$ such that, for all $I' \subset\subset I$, there exist $C > 0$ and $0 < r' \leq r$ such that for all integer $n > 0$

$$\left| f(z) - \sum_{k=0}^{n-1} f_j z^j \right| \leq C^n \Gamma(1 + sn)|z|^n, \quad \forall z \in \mathcal{S}_{r'}(I').$$

We shall write $f \sim_s \hat{f}$. The space of these functions will be denoted by $\mathcal{A}_s(I)$.

Note that the above Gevrey asymptotic property strengthens the Poincaré asymptotic property introduced in the previous section. We also remark that asymptotic series $\hat{f}$ of such a function belongs to $\mathbb{C}[\![z]\!]_s$.

Definition 2.6. Let k be a positive real number. A formal power series $\hat{f} \in \mathbb{C}[\![z]\!]_{\frac{1}{k}}$ is said to be *k-summable* in the direction d if there exists a sector $\mathcal{S}_r(I)$, bisected by d and of opening $|I| > \frac{\pi}{k}$, and a holomorphic function $f \in \mathcal{O}(\mathcal{S}_r(I))$ such that $f \sim_{\frac{1}{k}} \hat{f}$. We also say that $\hat{f}$ is k-summable on I.

Such a holomorphic function f is unique (this is a consequence of Watson Lemma [26]) and called the *k-sum of* $\hat{f}$. We emphasize that a *k-summable power series is* $\frac{1}{k}$*-Gevrey*. In order to describe the properties of solutions of differential equations with irregular singularity, we need the more general notion of multi-summability.

Definition 2.7. Let $r \geq 1$ be an integer and let $\mathbf{k} := (k_1, \ldots, k_r) \in (\mathbb{R})^r$ with $0 < k_1 < \cdots < k_r$. For any $1 \leq j \leq r$, let $I_j :=]a_j, b_j[$ be an open interval of length $|I_j| = b_j - a_j > \frac{\pi}{k_j}$ such that $I_j \subset I_{j-1}$, $2 \leq j \leq r$. A formal power series $\hat{f} \in \mathbb{C}[\![z]\!]$ is said to be $\mathbf{k}$*-multisummable on* $\mathbf{I} = (I_1, \ldots, I_r)$ if there exist formal power series $\hat{f}_j$ such that $\hat{f} := \sum_{j=1}^{r} \hat{f}_j$ and such that each $\hat{f}_j$ is k_j-summable on I_j with sum f_j, $1 \leq j \leq r$. We shall also say that $\hat{f}$ *is* $\mathbf{k}$*-multisummable in the multidirection* $\mathbf{d} = (d_1, \ldots, d_r)$ where d_j bisects the sector $\{a_j < \arg z < b_j\}$.

In that case, we say that $\mathbf{f} = (f_1, \ldots, f_r)$ *is the multisum of* $\hat{f}$. Such a multisum is unique according the relative Watson lemma [26, Théorème 2.2.1.1]. From it, one can build the (unique) $\mathbf{k}$-sum of $\hat{f}$ on $\mathbf{I}$, denoted by $\mathbf{f}_{\mathbf{k},\mathbf{I}}$, that satisfies $\mathbf{f}_{\mathbf{k},\mathbf{I}} \sim_{\frac{1}{k_1}} \hat{f}$ on I_1 [10, p. 524]. Here we have used the definition of W. Balser [2] but there are other equivalent definitions due to Ecalle [16,27] and Malgrange-Ramis [35].

Next, we shall emphasize the following important property:

Proposition 2.8. *[35, Proposition 3.2,p. 358], [26, Théorème 2.2.3.1] Let Φ be a germ of holomorphic function at 0 of $\mathbb{C}^{p+1}$. Let $\hat{f}_i \in \mathbb{C}[\![z]\!]$ be a formal power series such that $\hat{f}_i(0) = 0$, $i = 1, \ldots, p$. Assume that $\hat{f}_i$ is $\mathbf{k}$-multisummable on $\mathbf{I} = (I_1, \ldots, I_r)$ with multisum $\mathbf{f_i} = (f_{i,1}, \ldots, f_{i,r})$. Then, $\Phi(z, \hat{f}_1(z), \ldots, \hat{f}_p(z))$ is also $\mathbf{k}$-multisummable on $\mathbf{I} = (I_1, \ldots, I_r)$ with multisum $\Phi(z, \mathbf{f}) = (\Phi(z, f_{1,1}, \ldots, f_{p,1}), \ldots, \Phi(z, f_{1,r}, \ldots, f_{p,r}))$.*

In particular, we conclude that the class of multisummable functions forms an algebra and is closed under the division operation, provided the denominator has no constant terms in its expansion.

The reason for introducing these notions is that these are the natural spaces to which solutions of nonlinear differential equations with irregular singularity must belong.

Let $r \in \mathbb{N}$, $k_j \in \mathbb{N}$, $j = 1, \ldots, r$, $0 < k_1 < \ldots < k_r$. We set $\mathbf{k} := (k_1, \ldots, k_r)$. Let $\mathbf{I} = (I_1, \ldots, I_r)$ where $I_j =]\alpha_j, \beta_j[$ is an open interval with $\beta_j - \alpha_j > \pi/k_j$. We also assume that $I_j \subset I_{j-1}, j = 1, \ldots, r$ where $I_0 = \mathbb{R}$. Consider

$$\operatorname{diag}\{x^{k_1} I^{(1)}, \ldots, x^{k_r} I^{(r)}\} x \frac{dy}{dx} = \Lambda y + x g(x, y) \tag{2.10}$$

where $I^{(j)}$ denotes the identity matrix of dimension $n_j \in \mathbb{N}$ and $n = n_1 + \ldots n_r$, $y \in \mathbb{C}^n$, $\Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, Λ is invertible and g is a $\mathbf{k}$-sum of some $\hat{g}(x, y) \in \mathbb{C}[[x, y]]$ on $\mathbf{I}$ uniformly in a neighborhood of $0 \in \mathbb{C}^n$ (g analytic at $(0, 0)$ in $\mathbb{C} \times \mathbb{C}^n$ is a special case). Let $\hat{y} = \sum_{h=1}^{\infty} c_h x^h$ be a formal solution of (2.10). This means that

$$\operatorname{diag}\{x^{k_1} I^{(1)}, \ldots, x^{k_r} I^{(r)}\} x \frac{d\hat{y}(x)}{dx} = \Lambda y + x \hat{g}(x, \hat{y}(x)).$$

Then the following holds (cf. [37,10,4]).

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Theorem 4. *[11] The formal solution $\hat{y}$ of (2.10) is* $\mathbf{k}$*-multisummable on* $\mathbf{I} = (I_1, \ldots, I_r)$ *if* $\arg\lambda_h \notin]\alpha_j + \pi/(2k_j), \beta_j - \pi/(2k_j)[$ *for all* $h \in [n_1 + \ldots + n_{j-1} + 1, n_1 + \ldots + n_j]$, $1 \le j \le r$.

Corollary 2.9. *[10, Corollary, p. 525] Consider an analytic nonlinear differential equation of the form*

$$z^{\nu+1}\frac{dy}{dz} = F(z, y) \tag{2.11}$$

where $z \in \mathbb{C}$, $y \in \mathbb{C}^n$, *and* F *is analytic in a neighborhood of the origin in* $\mathbb{C} \times \mathbb{C}^n$, $\nu > 0$. *Then, there exist positive integers* q *and* $0 < k_1 < \ldots < k_r$ *such that every formal power series solution* $\hat{y}$ *of (2.11) is* $(\frac{k_1}{q}, \ldots, \frac{k_r}{q})$*-multisummable.*

As shown in [9, p. 60], there exists an analytic transformation and a ramification $x = z^{1/q}$ which transforms (2.11) into (2.10). As a consequence, we can also apply Theorem 4 in the situation when the righthand side F is a $(\frac{k_1}{q}, \ldots, \frac{k_r}{q})$-sum, uniformly in a neighborhood of $0 \in \mathbb{C}^n$. The point for not stating this directly in the theorem is that both $\mathbf{k}$ and q need to be known and cannot be read off immediately on (2.11).

Remark 2.10. Let $\hat{y}$ be a formal power series solution of (2.11). Let $\mathbf{k}/\mathbf{q} := (\frac{k_1}{q}, \ldots, \frac{k_r}{q})$ as above. Then $\hat{y}$ has a $\mathbf{k}/\mathbf{q}$-sum $y^{\pm}$ defined in a sector containing the direction $\mathbb{R}^{\pm}$. Indeed, having done an appropriate analytic transformation and a ramification $x = z^{1/q}$, we consider (2.10). Let $\epsilon_j > 0$ and let $\tilde{I} := \cup_j \cup_{h \in [n_1 + \ldots + n_{j-1}+1, n_1 + \ldots + n_j]}]\arg\lambda_h - \epsilon_j, \arg\lambda_h + \epsilon_j[$. It is always possible to choose the ϵ_j's small enough so that the exists a $\tau_+ \notin \tilde{I}$ and so that $|\tau_+| < \frac{\pi}{2k_r} + \frac{1}{2}\min\frac{\epsilon_j}{2}$. Therefore, for all j, $-\tau_+ - \frac{\pi}{2k_j} - \frac{\epsilon_j}{2} < 0 < -\tau_+ + \frac{\pi}{2k_j} + \frac{\epsilon_j}{2}$. This means that $\mathbb{R}^+$ belongs to the sector I_j^+ bisected by τ_+ and of opening $\frac{\pi}{k_j} + \epsilon_j$, for all j. Setting $\tau_- = \tau_+ + \pi$, then $\mathbb{R}^-$ belongs to the sector I_j^- bisected by τ_- and of opening $\frac{\pi}{k_j} + \epsilon_j$, for all j. According to Theorem 4, $\hat{y}$ is $\mathbf{k}$-multisummable on $\mathbf{I}^{\pm}$ and its $\mathbf{k}$-sum $\mathbf{y}_{\mathbf{k},\mathbf{I}^{\pm}}$ is defined on $\mathbb{R}^{\pm}$. To obtain the same result for (2.11), one has to divide τ_+ by q and set $\tau_- = \tau_+ + \pi/q$.

3. Complete system for a generic nonminimal hypersurface

We start with the proof of Theorem 1. We assume both reference points p, p^* to be the origin. As was discussed in the Introduction, in the finite type case the assertion of Theorem 1 follows from [6]. In the Levi-flat case the assertion is obvious. Hence, we assume in what follows that *both* M, M^* *are nonminimal at the reference point* 0 *but are Levi-nonflat*.

In this section, we prove Theorem 1 for the class of m-nonminimal at the origin hypersurfaces, satisfying the generic assumption that *the minimal part* $M \setminus X$ *of* M *is Levi-nondegenerate* (thus the origin is a generic nonminimal points, in the terminology of Section 2). As was explained in Section 2, any such hypersurface can be written in

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appropriate local holomorphic coordinates by an equation (2.3). For the convenience of the reader, we shall first show how the individual results proved in this section combine to yield the proof of Theorem 1.

3.1. Outline of the results and conclusion of the proof in the generic case

In order to state the (technical) results needed, we first need a convenient prenormalization of formal power series maps. Let us observe that given two hypersurfaces $M, M^* \subset \mathbb{C}^2$, given near the origin by (2.3) then any formal power series map

$$H = (F, G): \quad (M, 0) \mapsto (M^*, 0)$$

between them has the following specific form.

Lemma 3.1. *Any formal power series map*

$$(z, w) \mapsto \big(F(z, w), G(z, w)\big)$$

between germs at the origin of two hypersurfaces of the form (2.3) *satisfies:*

$$G = O(w), \quad G_z = O(w^{m+1}). \tag{3.1}$$

Proof. We interpret (2.3) as:

$$w = \bar{w} + \bar{w}^m \cdot z\bar{z} \cdot O(1).$$

Then the basic identity gives:

$$G(z, w) = \bar{G}(\bar{z}, \bar{w}) + \bar{G}^m(\bar{z}, \bar{w}) \cdot \bar{F}(\bar{z}, \bar{w}) \cdot F(z, w) \cdot O(1), \ \text{ where } \ w = \bar{w} + \bar{w}^m \cdot z\bar{z} \cdot O(1).$$

Putting in the latter identity $\bar{z} = \bar{w} = 0$, we get $G(z, 0) \equiv 0$. Further, differentiating with respect to z, evaluating at $\bar{z} = 0$ at which one has $w = \bar{w}$, we get:

$$G_z(z, \bar{w}) = \bar{G}(0, \bar{w})^m \cdot \bar{F}(0, \bar{w}) \cdot O(1),$$

which already implies the assertion of the lemma. □

Lemma 3.1 immediately implies that, when considering formal invertible mappings between hypersurfaces of the form (2.3), *we can restrict to transformations of the form:*

$$z \mapsto z + f(z, w), \quad w \mapsto w + wg_0(w) + w^m g(z, w)$$

with

$$f_z(0,0)=0, \quad g_0(0)=0, \quad g(z,w)=O(zw) \tag{3.2}$$

(normalizing the coefficients of z, w for F, G respectively is possible by means of a linear scaling applied to the source hypersurface). We now expand f, g as:

$$f(z,w)=\sum_{j=0}^{\infty} f_j(w)z^j, \quad g(z,w)=\sum_{j=1}^{\infty} g_j(w)z^j \tag{3.3}$$

(we point out that the function $g_0(w)$, as in (3.2), is *not* present in the expansion (3.3)!). In view of (3.2) we have

$$f_1(0)=g_1(0)=0. \tag{3.4}$$

We also introduce the new functions

$$\begin{gathered} y_1 := f_0, \quad y_2 := g_0, \quad y_3 := f_1, \quad y_4 := g_1, \quad y_5 := w^m f_0', \\ y_6 := w g_0', \quad y_7 := w^m f_1', \quad y_8 := w^m g_1'. \end{gathered} \tag{3.5}$$

It is *important* that all the y_j do not have a constant term, as follows from (3.2), (3.4) and the fact that our transformation maps the origin to itself. We clearly have

$$w^m y_1' = y_5, \quad w y_2' = y_6, \quad w^m y_3' = y_7, \quad w^m y_4' = y_8. \tag{3.6}$$

We can now state the first main technical result of this section:

Proposition 3.2. *The formal vector function $Y_0(w) := (y_1(w), \dots, y_8(w))$ satisfies a meromorphic differential equation*

$$w^m \frac{dY}{dw} = A(w, Y), \tag{3.7}$$

where $A(w, Y)$ is a holomorphic at the origin function.

Applying now the fundamental Theorem 4 on the multisummability of formal solutions of nonlinear differential equation at an irregular singularity, as well as Remark 2.10 (see Section 2.6), we immediately obtain

Corollary 3.3. *There exist sectors $S^+, S^- \subset \mathbb{C}$, containing the positive and the negative real lines, directions $d^\pm$, functions $f_0^\pm(w), g_0^\pm(w), f_1^\pm(w), g_1^\pm(w)$ holomorphic in the respective sectors, and a multi-order $\mathbf{k} = (k_1, \dots, k_l)$ such that the following holds.*

(i) The functions $f_0^\pm(w), g_0^\pm(w), f_1^\pm(w), g_1^\pm(w)$ are the $\mathbf{k}$-multisums of f_0, g_0, f_1, g_1 in the directions $d^\pm$, respectively;

(ii) The holomorphic in respectively $S^\pm$ functions $Y^\pm(w)$, constructed via $f_0^\pm(w)$, $g_0^\pm(w), f_1^\pm(w), g_1^\pm(w)$ by using formulas (3.5), satisfy the ODE (3.7).

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The last point is a consequence of uniqueness of multisummable functions. Since Y is $\mathbf{k}$-multisommable on some multisectors, so are functions $w^m f_0', wg_0', w^m f_1', w^m g_1'$. Thus equalities (3.5) hold. We will also need that Corollary 3.3 also holds with f_0, f_1, g_0, g_1 replaced by their conjugates, with the same multi-order $\mathbf{k}$:

Corollary 3.4. *There exist functions*

$$\overline{f_0^\pm}(w), \overline{g_0^\pm}(w), \overline{f_1^\pm}(w), \overline{g_1^\pm}(w), \tag{3.8}$$

holomorphic in the respective sectors $S^\pm$, which are the $\mathbf{k}$-multisums in the directions $d^\pm$ of $\bar f_0(w), \bar g_0(w), \bar f_1(w), \bar g_1(w)$. Furthermore, the corresponding maps $\overline{Y^\pm}$ defined as in (3.5), satisfy the meromorphic ODE $w^m \overline{Y^\pm} = \bar A(w, \overline{Y^\pm})$.

The second main technical result of this section shows that f and g can be reproduced from f_0, f_1, g_0, g_1.

Proposition 3.5. *There exist holomorphic functions φ and ψ defined in a neighborhood of the origin in $\mathbb{C}^7$ such that the equality*

$$\begin{aligned} f(z,w) &= \varphi\big(z, w, g_0(w), wg_0'(w), f_0(w), f_1(w), g_1(w)\big), \\ g(z,w) &= \psi\big(z, w, g_0(w), wg_0'(w), f_0(w), f_1(w), g_1(w)\big) \end{aligned} \tag{3.9}$$

holds for every (f, g) as above.

Before we turn to the proofs of the technical statements above, we show how these statements imply Theorem 1 in the generic case.

Proof of Theorem 1 in the generic setting. Let us introduce the functions

$$\begin{aligned} f^\pm(z,w) &= \varphi\big(z, w, g_0^\pm(w), w\cdot(g_0^\pm)'(w), f_0^\pm(w), f_1^\pm(w), g_1^\pm(w)\big), \\ g^\pm(z,w) &= \psi\big(z, w, g_0^\pm(w), w\cdot(g_0^\pm)'(w), f_0^\pm(w), f_1^\pm(w), g_1^\pm(w)\big), \end{aligned} \tag{3.10}$$

well defined in the product of a disc Δ in z centered at the origin and the sectors $S^\pm$ in w (this product forms a *tangential sectorial domain*, as described in Section 2. $f^\pm(z,w), g^\pm(z,w)$ are asymptotically represented in their domains by $f(z,w), g(z,w)$ respectively, as follows from (3.9). Based on (3.8), we similarly introduce $\overline{f^\pm}(z,w)$, $\overline{g^\pm}(z,w)$, asymptotically representing $\bar f(z,w), \bar g(z,w)$, respectively.

Let us now consider the (complexified) basic identity

$$G(z,w) - \rho^*\big(F(z,w), \overline{F}(\xi,\eta), \overline{G}(\xi,\eta)\big)|_{w=\rho(z,\xi,\eta)} = 0 \tag{3.11}$$

for the map (F, G) between the germs at the origin of the initial hypersurfaces $M = \{w = \rho(z, \bar z, \bar w)\}$ and $M^* = \{w = \rho^*(z, \bar z, \bar w)\}$. We claim that the sectorial map

$(F^{\pm}(z,w), G^{\pm}(z,w))$ constructed via $f^{\pm}, g^{\pm}$ by the formula (3.2) satisfies the basic identity (3.11) as well, i.e.

$$G^{\pm}(z,w) - \rho^*\big(F^{\pm}(z,w), \overline{F^{\pm}}(\xi,\eta), \overline{G^{\pm}}(\xi,\eta)\big)|_{w=\rho(z,\xi,\eta)} = 0, \quad (z,\xi,\eta) \in \Delta\times\Delta\times S^{\pm}. \tag{3.12}$$

To prove the claim, let us analyze the identity (3.12). The left hand side of it, which we denote by

$$\chi(z,\xi,\eta),$$

is holomorphic in $\Delta\times\Delta\times S^{\pm}$, respectively. Accordingly, the identity (3.12) holds if and only if we have:

$$\frac{\partial^{p+q}}{\partial z^p \partial \xi^q}\chi(z,\xi,\eta)\bigg|_{z=\xi=0} \equiv 0, \quad p,q \geq 0. \tag{3.13}$$

However, it is not difficult to verify (by applying the chain rule) that for each fixed $p,q \geq 0$ the left hand side in (3.13) is an analytic function $R_{p,q}$ in η, the sectorial functions

$$f_0^{\pm}(\eta), g_0^{\pm}(\eta), f_1^{\pm}(\eta), g_1^{\pm}(\eta), \overline{f_0^{\pm}}(\eta), \overline{g_0^{\pm}}(\eta), \overline{f_1^{\pm}}(\eta), \overline{g_1^{\pm}}(\eta),$$

and their derivatives of order $\leq p+q$. Hence, each left hand side in (3.13) is the **k**-multisum of the identical analytic expressions $R_{p,q}$ in formal series, where $f_0^{\pm}(\eta), g_0^{\pm}(\eta), f_1^{\pm}(\eta), g_1^{\pm}(\eta)$ are replaced by the asymptotic expansions f_0, g_0, f_1, g_1, respectively, and $\overline{f_0^{\pm}}(\eta), \overline{g_0^{\pm}}(\eta), \overline{f_1^{\pm}}(\eta), \overline{g_1^{\pm}}(\eta)$ by their asymptotic expansions $\bar f_0(\eta), \bar g_0(\eta), \bar f_1(\eta), \bar g_1(\eta)$, respectively. In view of the (valid!) formal basic identity (3.11), the latter formal series in η vanish identically for any $p,q \geq 0$. The uniqueness property within the class of **k**-multisummable series in the directions $d^{\pm}$ implies now that all the left hand sides in (3.13) *all vanish identically.*

As was explained in Section 2, the property (3.12) for a sectorial map defined in a tangential sectorial domain implies that the restriction of the map onto the source manifold is a C^∞ CR-map onto the target. Thus, the claim under discussion implies the assertion of the theorem. $\square$

The rest of this section is devoted to the proofs of the Propositions above.

3.2. Associated complete system

We show the following:

Proposition 3.6. *Associated with a hypersurface* (2.3) *is a second order singular holomorphic ODE* $\mathcal{E}(M)$ *given by*

$$w'' = w^m \Phi\left(z, w, \frac{w'}{w^m}\right), \tag{3.14}$$

where $\Phi(z,w,\zeta)$ is a holomorphic near the origin in $\mathbb{C}^3$ function with $\Phi = O(\zeta)$. The latter means that all Segre varieties of M (besides the complex locus $X = \{w = 0\}$ itself), considered as graphs $w = w_p(z)$, satisfy the ODE (3.14).

Proof. The argument of the proof very closely follows the one given in the proof of an analogues statement in [24], [22] for the case of *m-admissible hypersurfaces*, and we leave the details of the proof to the reader. □

Based on the connection between mappings of hypersurfaces and that of the associated ODEs discussed in Section 2 and Lemma 3.1, we come to the consideration of ODEs (3.14) and formal power series mappings (3.2) between them. We further recall that the fact that a mapping $(F(z,w), G(z,w))$ transforms an ODE $\mathcal{E}$ into an ODE $\mathcal{E}^*$ is equivalent to the fact that the second jet prolongation $(F^{(2)}, G^{(2)})$ transforms the ODEs $\mathcal{E}, \mathcal{E}^*$ into each other, where the ODEs are considered as submanifolds in $J^2(\mathbb{C}, \mathbb{C})$. Applying this to two nonsingular ODEs $\mathcal{E} = \{w'' = \Psi(z,w,w')\}$, $\mathcal{E}^* = \{w'' = \Psi^*(z,w,w')\}$ and employing the classical jet prolongation formulas (e.g., [8]), we obtain:

$$\begin{aligned}\Psi(z,w,w') = \frac{1}{J}\Bigg((F_z + w'F_w)^3\Psi^*\Big(F(z,w), G(z,w), \frac{G_z + w'G_w}{F_z + w'F_w}\Big) + \\ + I_0(z,w) + I_1(z,w)w' + I_2(z,w)(w')^2 + I_3(z,w)(w')^3\Bigg),\end{aligned} \tag{3.15}$$

where $J := F_zG_w - F_wG_z$ is the Jacobian determinant of the transformation and

$$\begin{aligned}
I_0 &= G_zF_{zz} - F_zG_{zz}\\
I_1 &= G_wF_{zz} - F_wG_{zz} - 2F_zG_{zw} + 2G_zF_{zw}\\
I_2 &= G_zF_{ww} - F_zG_{ww} - 2F_wG_{zw} + 2G_wF_{zw}\\
I_3 &= G_wF_{ww} - F_wG_{ww}.
\end{aligned} \tag{3.16}$$

Setting then $\Psi(z,w,w') := w^m\Phi\left(z, w, \frac{w'}{w^m}\right)$ (and similarly for Φ^*) and switching to the notations in (3.2), we obtain the transformation rule for the class of ODEs (3.14) and mappings (3.2) between them:

$$\begin{aligned}w^m\Phi\left(z,w,\frac{w'}{w^m}\right) = \frac{1}{J}\Big[(1+f_z+w'f_w)^3(1+g_0(w)+w^{m-1}g)^m\cdot \\ \cdot w^m\Phi^*\Big(z+f, w+wg_0(w)+w^mg, \frac{w^mg_z + w'(1+wg_0'+g_0+mw^{m-1}g+w^mg_w)}{w^m(1+g_0(w)+w^{m-1}g)^m(1+f_z+w'f_w)}\Big) + \\ + I_0(z,w) + I_1(z,w)w' + I_2(z,w)(w')^2 + I_3(z,w)(w')^3\Big],\end{aligned} \tag{3.17}$$

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where

$$\begin{aligned}
J &= (1+f_z)(1+g_0+wg_0'+w^mg_w+mw^{m-1}g)-w^mf_wg_z,\\
I_0 &= w^m\big(g_zf_{zz}-(1+f_z)g_{zz}\big),\\
I_1 &= \big(1+wg_0'+g_0+mw^{m-1}g+w^mg_w\big)f_{zz}-w^mf_wg_{zz}-\\
&\quad -2(1+f_z)(mw^{m-1}g_z+w^mg_{zw})+2w^mg_zf_{zw},\\
I_2 &= w^mg_zf_{ww}-(1+f_z)(wg_0''+2g_0'+m(m-1)w^{m-2}g+2mw^{m-1}g_w+w^mg_{ww})-\\
&\quad -2f_w(mw^{m-1}g_z+w^mg_{zw})+2(1+wg_0'+g_0+mw^{m-1}g+w^mg_w)f_{zw},\\
I_3 &= (1+wg_0'+g_0+mw^{m-1}g+w^mg_w)f_{ww}-\\
&\quad -f_w(wg_0''+2g_0'+m(m-1)w^{m-2}g+2mw^{m-1}g_w+w^mg_{ww}).
\end{aligned} \tag{3.18}$$

Importantly, after putting $w'=\zeta w^m$, (3.17) becomes an identity of formal power series in the *independent* variables z,w,ζ.

We now extract from (3.18) four identities of power series *in* z,w *only*, in the following way. For the first identity, we extract in (3.18) terms with $(w')^0$ and divide the resulting identity by w^m. For the second identity, we extract in (3.18) terms with $(w')^1$. For the third identity, we extract in (3.18) terms with $(w')^2$ and multiply the resulting identity (which has a pole in w of order m) by w^m. For the last identity, we extract in (3.18) terms with $(w')^3$ and multiply the resulting identity (which has a pole in w of order $2m$) by w^{2m}. The four resulting identities of formal power series in z,w can be written as:

$$\begin{aligned}
&I_0=w^mT_0(z,w,j^1(f,g,g_0)), \quad I_1=T_1(z,w,j^1(f,g,g_0)),\\
&w^mI_2=T_2(z,w,j^1(f,g,g_0)), \quad w^{2m}I_3=T_3(z,w,j^1(f,g,g_0)),
\end{aligned} \tag{3.19}$$

where $j^1(f,g.g_0)$ denotes the 1-jet of f,g,g_0 (the collection of derivatives of order ≤ 1), and $T_k(\cdot,z,w)$ are four precise holomorphic at the origin functions, exact form of which is of no interest to us. We though emphasize two important properties of the identities (3.19):

(a) the derivatives f_w,g_w come in each T_k with the factor w^m, and the derivative g_0' comes in each T_k with the factor w;

(b) the derivatives f_w,g_w,f_{zw},g_{zw} all come in all the left hand sides in (3.19) with the factor w^m, the derivatives f_{ww},g_{ww} all come in all the left hand sides in (3.19) with the factor w^{2m}, and the derivatives g_0',g_0'' come in all the left hand sides in (3.19) with the factor w.

It is also not difficult to verify that the identities (3.19) are well defined, i.e. the formal power series under considerations all come into the right hand side in (3.19) with the zero constant term.

We can now prove Proposition 3.2.

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Proof of Proposition 3.2. We consider in the last two identities in (3.19) terms with z^0, z^1, respectively. This gives us *four second order singular ODEs for the functions* f_0, f_1, g_0, g_1. In the two identities with z^0, only the second order derivatives f_0'', g_0'' participate (the other derivatives have order ≤ 1). It is not difficult to solve the latter identities for $w^{2m}f_0'', w^{m+1}g_0''$ (by applying the Cramer rule to the nondegenerate linear system). We obtain, by combining the information in (3.18), (3.6) and the observations (a), (b) above:

$$w^{2m}f_0'' = U(y_1, y_2, ..., y_8, w), \quad w^{m+1}g_0'' = U(y_1, y_2, ..., y_8, w), \tag{3.20}$$

where U and V are two holomorphic at the origin functions in all their variables, exact form of which is of no interest to us. Using the y-notations and (3.6), the equations (3.20) give:

$$w^m y_5' = \tilde{U}(y_1, y_2, ..., y_8, w), \quad w^m y_6' = \tilde{V}(y_1, y_2, ..., y_8, w), \tag{3.21}$$

where, again, $\tilde{U}$ and $\tilde{V}$ are two holomorphic at the origin functions in all their variables, exact form of which is of no interest to us.

To obtain the missing conditions for y_7', y_8', we use the system of two second order ODEs obtained by collecting in the last two identities of (3.19) terms with z^1. Considering this system as a (nondegenerate) linear system in $w^{2m}f_1'', w^{2m}g_1''$ and solving by Cramer rule, we get:

$$w^{2m}f_1'' = X(y_1, y_2, ..., y_8, w^{2m}f_0'', w), \quad w^{2m}g_1'' = Y(y_1, y_2, ..., y_8, w^{m+1}g_0'', w), \tag{3.22}$$

where X and Y are two holomorphic at the origin functions in all their variables, exact form of which is of no interest to us. Combining this with (3.20) and using (3.6), we finally obtain

$$w^m y_7' = \tilde{X}(y_1, y_2, ..., y_8, w), \quad w^m y_8' = \tilde{Y}(y_1, y_2, ..., y_8, w). \tag{3.23}$$

By putting (3.6), (3.21), (3.23) together, we obtain the Proposition. □

We now turn to the proof of Corollary 3.4.

Proof of Corollary 3.4. We first want to show that the "barred" power series $\bar{f}_0(w)$, $\bar{g}_0(w), \bar{f}_1(w), \bar{g}_1(w)$ belong to the same summability class as the original series. For doing so, let us consider the associated with (3.7) ODE

$$w^m \frac{dZ}{dw} = \bar{A}(w, Z), \tag{3.24}$$

where $A(w, Y)$ is as in (3.7). We first note that the "barred" power series $\bar{Y}_0(w)$ satisfies the ODE (3.24). Now, let us write $\mathbf{Y} := (Y, Z)$ and

$$\mathbf{A}(w, \mathbf{Y}) := \begin{pmatrix} A(w, Y) & 0 \\ 0 & \bar{A}(w, Z) \end{pmatrix},$$

and then consider the system

$$w^m \frac{d\mathbf{Y}}{dw} = \mathbf{A}(w, \mathbf{Y}). \tag{3.25}$$

Applying Theorem 4 and Remark 2.10 for the "decoupled" system (3.25), we find sectors $S^+, S^- \subset \mathbb{C}$ containing the positive and the negative real lines (which we without loss of generality assume to be equal to the ones in Corollary 3.3), direction $d^\pm$ (which we without loss of generality assume to be equal to the ones in Corollary 3.3), and functions

$$\overline{f_0^\pm}(w), \overline{g_0^\pm}(w), \overline{f_1^\pm}(w), \overline{g_1^\pm}(w), \tag{3.26}$$

holomorphic in the respective sectors $S^\pm$, which are the $\mathbf{k}$-multisums in the directions $d^\pm$ of $\bar{f}_0(w), \bar{g}_0(w), \bar{f}_1(w), \bar{g}_1(w)$, respectively (we, again, assume without loss of generality that the multi-order $\mathbf{k}$ equals to the one in Corollary 3.3). In addition, the holomorphic in respectively $S^\pm$ function $\overline{Y^\pm}(w)$, constructed via $\overline{f_0^\pm}(w), \overline{g_0^\pm}(w), \overline{f_1^\pm}(w), \overline{g_1^\pm}(w)$ by using formulas (3.5), satisfies the ODE (3.7). $\square$

The last remaining piece is now the proof of Proposition 3.5.

Proof of Proposition 3.5. We now consider the first two equations in (3.19). Read together, they can be treated as a system of linear equations in f_{zz}, g_{zz} determinant of which at the origin is non-vanishing. Applying the Cramer rule, we obtain the following system of equations:

$$f_{zz} = P(z, w, j^1(f, g), g_0, wg_0', f_{zw}, g_{zw}), \quad g_{zz} = Q(z, w, j^1(f, g), g_0, wg_0', f_{zw}, g_{zw}), \tag{3.27}$$

where P, Q are appropriate functions holomorphic in their arguments. We now consider the intimately related Cauchy problem

$$f_{zz} = P(z, w, j^1(f, g), \alpha_0, \alpha_1, f_{zw}, g_{zw}), \quad g_{zz} = Q(z, w, j^1(f, g), \alpha_0, \alpha_1, f_{zw}, g_{zw}) \tag{3.28}$$

with the Cauchy data

$$f(0, w) = \beta_0, \quad f_z(0, w) = \beta_1, \quad g(0, w) = 0, \quad g_z(0, w) = \beta_2, \tag{3.29}$$

where α_i, β_j are additional parameters. By the parametric version of the Cauchy-Kowalevski theorem, namely the Ovcyannikov's theorem [31,44], the latter Cauchy problem has a unique analytic solutions

$$f = \varphi(z, w, \alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2), \quad g = \psi(z, w, \alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2),$$

where φ and ψ depend analytically on all their arguments. Hence, taking into account (3.2), (3.3), we finally have the identities (3.9). (we emphasize that the substitution of formal power series into φ, ψ is well defined here, since *all* the formal data being substituted has no constant term!). $\square$

4. The exceptional case

In this section, we prove Theorem 1 in full generality. For that, we have to consider the case when, for an m-nonminimal at the origin hypersurface $M \subset \mathbb{C}^2$, the minimal part $M \setminus X$ contains Levi degenerate points. In this case, M can *not* be associated to an ODE (3.14). We overcome this difficulty by introducing *associated ODEs of high order.*

The proof of Theorem 1 in the general case has several ingredients, each of which we put in a separate subsection below. While we follow closely the structure of Section 3, the tools which we need to introduce in this section are considerably harder: We treat the multisummability of what will later be "initial terms" in certain Cauchy problems in 4.1, then discuss the associated ODEs of higher order in 4.2. We can then prove Theorem 1 under an additional technical condition in 4.3, and in 4.4 introduce the necessary geometrical concept to use this technical condition in full generality to complete the proof in 4.5.

4.1. **k**-*Summability of initial terms*

In what follows, for hypersurfaces under consideration we consider the defining equation (1.2). Since M is strictly pseudoconvex at generic points, a generic real-analytic curve $\Gamma \subset M$ through 0, transverse to $T^c_0 M$, will not contain any Levi-degenerate point except for 0. If we choose normal coordinates for which $\Gamma = \{(z, w) \in M : z = 0\}$ (which is possible, see e.g. [25, Lemma 4.1], then its the complex defining equation

$$w = \Theta(z, \bar{z}, \bar{w}) \quad \Theta(z, \bar{z}, \bar{w}) = \bar{w} + \sum_{j,k \geq 1} \Theta_{jk}(\bar{w}) z^k \bar{z}^l, \quad \Theta \not\equiv 0 \tag{4.1}$$

satisfies the additional condition

$$\Theta_{11}(\bar{w}) \not\equiv 0. \tag{4.2}$$

The fact that the minimal part $M \setminus X$ contains Levi degenerate points reads as

$$\operatorname{ord}_0 \Theta_{11}(\bar{w}) > m. \tag{4.3}$$

We start by considering for a formal power series map (F, G) between germs at the origin of hypersurfaces (4.1) the expansion:

$$F = \sum_{j \geq 0} F_j(w) z^j, \quad G = \sum_{j \geq 0} G_j(w) z^j. \tag{4.4}$$

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Arguing similarly to the proof of Lemma 3.1, it is not difficult to prove

Lemma 4.1. *The components of the formal map* (F, G) *satisfy:*

$$F_z(0,0) = F_1(0) \neq 0, \quad G_w(0,0) = G_0'(0) \neq 0, \quad G(z,w) = O(w), \quad G_z(z,w) = O(w^m). \tag{4.5}$$

Thus, in suitable coordinates, we may assume

$$F_z(0,0) = F_1(0) = 1, \quad G_w(0,0) = G_0'(0) = 1.$$

Our goal in this subsection is to prove the following

Proposition 4.2. *There exist sectors* $S^+, S^- \subset \mathbb{C}$*, containing the positive and the negative real lines respectively, directions* $d^\pm \subset S^\pm$*, a multi-order* $\mathbf{k} = (k_1, ..., k_l)$*, and functions* $F_j^\pm(w), G_j^\pm(w)$ *holomorphic in the respective sectors, such that for each* $j \geq 0$*, the functions* $F_j^\pm(w), G_j^\pm(w)$ *are the* $\mathbf{k}$*-multisums of* F_j, G_j *in the directions* $d^\pm$*, respectively.*

Proposition 4.2 is proved in several steps.

Step I. We first observe that the assertion of Proposition 4.2 is invariant under *biholomorphic* transformations of the target. Indeed, a holomorphic coordinate change

$$z \mapsto U(z,w), \quad w \mapsto V(z,w)$$

in the target changes the components of the map as follows:

$$\widetilde{F} = U(F(z,w), G(z,w)), \quad \widetilde{G} = V((F(z,w), G(z,w)). \tag{4.6}$$

The new coefficient functions $\widetilde{F}_j, \widetilde{G}_j$ can be computed by differentiating (4.6) in z sufficiently many times and evaluating at $z = 0$, i.e. for some germs of holomorphic functions $C_{j_1 \ldots j_r, \ell_1 \ldots \ell_s}(z,w)$, $D_{j_1 \ldots j_r, \ell_1 \ldots \ell_s}(z,w)$ we can write

$$\widetilde{F}_j(w) = \sum_{j_1 + \cdots + \ell_s = j} C_{j_1 \ldots j_r, \ell_1 \ldots \ell_s}(F_0(w), G_0(w)) F_{j_1} \ldots F_{j_r} G_{\ell_1} \ldots G_{\ell_s},$$
$$\widetilde{G}_j(w) = \sum_{j_1 + \cdots + \ell_s = j} D_{j_1 \ldots j_r, \ell_1 \ldots \ell_s}(F_0(w), G_0(w)) F_{j_1} \ldots F_{j_r} G_{\ell_1} \ldots G_{\ell_s}.$$

Thus the desired invariance property follows from an application of Corollary 2.8 (see the discussion of the properties of multisummable functions in Section 2).

Step II. In this step, we make use of the following efficient blow-up procedure introduced in [25] by Mir and the second author.

Lemma 4.3 *(Blow-up Lemma, see [25]). Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface, which is Levi-degenerate at the origin and Levi-nonflat. Assume that M is given in coordinates (4.1) and that the distinguished curve*

$$\Gamma = \{(z, w) \in M : \ z = 0\} \subset M \tag{4.7}$$

does not contain Levi-degenerate points of M other than the origin. Then there exists a blow-down map

$$B(\xi, \eta): \quad (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0), \quad B(\xi, \eta) = (\xi\eta^s, \eta), \quad s \in \mathbb{Z}, \quad s \geq 2, \tag{4.8}$$

and a real-analytic nonminimal at the origin hypersurface $M_B \subset \mathbb{C}^2_{(\xi,\eta)}$ with the complex locus $X = \{\eta = 0\}$ such that:

(i) $B(M_B) \subset M, \quad B(X) = \{0\}$;

(ii) $M_B \setminus X$ is Levi-nondegenerate, and M_B is given by an equation of the kind (2.3).

We note at this point that the condition for Γ in Lemma 4.3 is precisely equivalent to (4.2).

We will need some control of the integer s from the proof of the Blow-up Lemma. We quickly recall the needed details. For an m-nonminimal hypersurface, transformations bringing to coordinates of the kind (4.1) are associated with curves $\gamma \subset M$ passing through 0 and transverse to the complex tangent at 0. Such a curve γ is being transformed into the distinguished (2.2) in the new coordinates (1.2).

We then choose γ in such a way that $\gamma \cap \Sigma = \{0\}$ for the Levi degeneracy set $\Sigma \subset M$, and bring to coordinates (4.1). This means that for the resulting hypersurface (4.1) we have $\Theta_{11} \not\equiv 0$. For each $k \geq 2$, let us denote

$$m(k) := \min_{p+q=k} \mathrm{ord}_0 \Theta_{pq}.$$

We have $m(j) \geq m$ for all $j \geq 2$. After that, an integer s in (4.8) is determined as any integer satisfying all the inequalities

$$2s + m(2) \leq ks + m(k), \quad k \geq 3. \tag{4.9}$$

In fact, one can require the unique (stronger) inequality

$$m(2) < s, \tag{4.10}$$

and thus avoid considering $m(k)$, $k \geq 3$.

We now proceed as follows. We may assume that both M and M^* are given by coordinates (1.2) with $\Theta_{11} \not\equiv 0$. We then fix an integer s, which satisfies (4.10) for both M and M^*. Next, we consider the formal curve $\gamma \subset M$ - the pre-image of (2.2) under the given formal map $H = (F, G)$. Let us choose an *analytic* curve $\tilde{\gamma} \subset M$ tangent to

γ to order $s+1$, and a *biholomorphic* map H_1 transforming $\tilde{\gamma}$ into (2.2) and M into a hypersurface $\widetilde{M}$ of the kind (1.2). Put $H_2 := H \circ H_1^{-1}$, so that $H = H_2 \circ H_1$. Finally, put

$$\tilde{\Gamma} := H(\tilde{\gamma}).$$

Note that, since γ and $\tilde{\gamma}$ are tangent to order $s+1$, the same is true for (2.2) and $\tilde{\Gamma}$.

We then can decompose H^{-1} as a product

$$H^{-1} = H_1^{-1} \circ H_2^{-1}, \tag{4.11}$$

where H_1^{-1} is a *biholomorphic* map transforming (2.2) into $\tilde{\gamma}$ and $\widetilde{M}$ into M, and H_2^{-1} is a *formal* invertible map transforming $\tilde{\Gamma}$ into (2.2) and M^* into the real-analytic hypersurface $\widetilde{M}$. Importantly, in view of the tangency condition, the formal map H_2^{-1} satisfies

$$\operatorname{ord}_0 F_0(w) \geq s+1, \tag{4.12}$$

where F_0 is as in (4.4). Moreover, the blow up integer s can be kept the same as before for the hypersurface $\widetilde{M}$ as well. Indeed, a transformation satisfying (4.12) clearly preserves the corresponding integer $m^*(2)$ in (4.10) (as we choose $s > m^*(2)$), so that the inequalities (4.10) still hold true for the same s and the hypersurface $\widetilde{M}$.

Finally, we recall that, in view of the considerations of Step I, the assertion of Proposition 4.2 applied for H_2^{-1} is equivalent to that for H^{-1}.

We summarize the considerations of Step II as follows: in view of the decomposition (4.11) and the subsequent properties of H_2^{-1},

it is sufficient to prove Proposition 4.2 for maps (F, G) *satisfying, in addition, the inequality* (4.12).

Step III. In this step, we are finally able to reduce Proposition 4.2 to the results already proved in the generic case. For that, we use the above blow up procedure.

In accordance with the outcome of the previous step, we consider a map $(F, G) : (M, 0) \mapsto (M^*, 0)$ satisfying, in addition, (4.12). Here the integer s in (4.12) is an admissible integer for the blow down map (4.8) both in the source and in the target. After performing the blow ups (with the integer s in (4.8)), we obtain real-analytic hypersurfaces M_B, M_B^*, respectively.

Re-calculating the map (F, G) in the "blown up" coordinates (ξ, η) gives:

$$G_B(\xi, \eta) = G(\xi\eta^s, \eta), \quad F_B(\xi, \eta) = \frac{F_0(\eta)}{\eta^s} + F_1(\eta)\xi + \cdots, \tag{4.13}$$

where dots stand for a power series in ξ, η of the kind $O(\xi^2)$. In view of (4.12), $F_B(\xi, \eta)$, $G_B(\xi, \eta)$ are *well defined power series*. It is immediate then that the formal map

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$$H_B(\xi,\eta) := \big(F_B(\xi,\eta),\, G_B(\xi,\eta)\big)$$

transforms $(M_B, 0)$ into $(M_B^*, 0)$. Furthermore, in view of (4.5), the formal map $H_B(\xi,\eta)$ is *invertible*, so that the results of Section 3 are applicable to it. Expanding now

$$F_B(\xi,\eta) = \sum_{j\geq 0} F_j^B(\eta)\xi^j, \quad G_B(\xi,\eta) = \sum_{j\geq 0} G_j^B(\eta)\xi^j,$$

and applying to H_B the assertion of Corollary 3.3 and the formulas (3.9), we immediately obtain for the components F_j^B, G_j^B the desired $\mathbf{k}$-summability property (identical to the one stated in Proposition 4.2). At the same time, the relations (4.13) show that

$$G_j^B(\eta) = \eta^{sj} G_j(\eta). \tag{4.14}$$

We immediately obtain from (4.14) the assertion of Proposition 4.2 for the components G_j (with the same sectors, multi-directions and multi-order $\mathbf{k}$ as for F_B, G_B). Finally, since we have

$$F(\xi\eta^s,\eta) = \big(G(\xi\eta^2,\eta)\big)^s \cdot F_B(\xi,\eta),$$

the chain rule and the multisummability property for F_j^B, G_j imply the assertion of Proposition 4.2 for the components F_j. This finally proves Proposition 4.2. □

4.2. Associated ODEs of high order

In this section we consider the case when the source and the target m-nonminimal hypersurfaces satisfy the additional *k-nondegeneracy* condition. The latter means that for some $k \geq 1$ we have

$$\operatorname{ord}_0 \Theta_{k1} = m \tag{4.15}$$

for the defining function (4.1). As a well known fact (e.g. [28]) the property of being m-nonminimal k-nondegenerate is invariant under (formal) invertible transformations. In view of (4.3), we may assume that $k \geq 2$ in our setting.

The main goal of this section is to show that we can associate a *system $\mathcal{E}(M)$ of k singular ODEs of orders* $\leq k+1$ to an m-nonminimal k-nondegenerate hypersurface M. By the latter we mean (as in the generic case) that all the Segre varieties Q_p of M for $p \notin X$ considered as graphs $w = w_p(z)$ satisfy the system of ODEs $\mathcal{E}(M)$ as follows:

Proposition 4.4. *Let $M \subset \mathbb{C}^2$ be an m-nonminimal k-nondegenerate hypersurface. Then there exists a system of holomorphic ODEs $\mathcal{E}(M)$, called the associated system to M, of the form*

$$w' = \Phi_1\left(z, w, \frac{w^{(k)}}{w^m}\right), \cdots, w^{(k-1)} = \Phi_{k-1}\left(z, w, \frac{w^{(k)}}{w^m}\right), \;\; w^{(k+1)} = \Phi\left(z, w, \frac{w^{(k)}}{w^m}\right) \tag{4.16}$$

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such that $w = w_p(z)$ is a solution of $\mathcal{E}(M)$ for $p \notin X$ and

$$\Phi_j = O(w^m \zeta), j = 1, \dots, k. \tag{4.17}$$

Furthermore, we can assume that $\Phi_1(z, w, \zeta)$ satisfies

$$\text{Property } (*) : \quad \frac{\partial^{k+s}\Phi_1}{\partial w^m \zeta^k}(0) = k!s!(\pm i).$$

If M^* is another such hypersurface, than any formal map (F, G) taking M into M^* satisfies (4.5) and transforms the system $\mathcal{E}(M)$ into $\mathcal{E}(M^*)$.

For producing the associated ODEs, we consider the Segre family of an m-nonminimal hypersurface (4.1) satisfying the additional k-nondegeneracy condition, and produce for it an elimination procedure, in the spirit of that discussed in Section 2. This Segre family looks as:

$$w = b + O(ab^m z) \tag{4.18}$$

(we use the notation $p = (\bar{a}, \bar{b})$). Differentiating (4.18) k times in z and using (4.15), we obtain:

$$w^{(k)} = ab^m(\alpha + o(1)), \quad \alpha \neq 0 \tag{4.19}$$

(here α is a fixed constant). Dividing (4.19) by the m-th power of (4.18) gives:

$$\frac{w^{(k)}}{w^m} = \alpha a + o(a). \tag{4.20}$$

Solving the system (4.20), (4.18) for a, b by the implicit function theorem yields

$$a = A\left(z, w, \frac{w^{(k)}}{w^m}\right), \quad b = B\left(z, w, \frac{w^{(k)}}{w^m}\right) \tag{4.21}$$

for two holomorphic near the origin in $\mathbb{C}^3$ functions $A(z, w, \zeta), B(z, w, \zeta)$ with $A = O(\zeta)$ and $B = O(w)$. Differentiating then (4.18) j times for each $j = 1, ..., k-1, k+1$ and substituting (4.21) into the results finally gives us (4.16). Note that $\Phi_1(z, w, \zeta), ..., \Phi_{k-1}(z, w, \zeta), \Phi(z, w, \zeta)$ are all holomorphic near the origin in $\mathbb{C}^3$ functions which satisfy (4.17) (as follows from the elimination procedure). It is immediate, in the same way as in the nondegenerate case, that *all the Segre varieties Q_p of M for $p \notin X$ considered as graphs $w = w_p(z)$ satisfy the system of ODEs $\mathcal{E}(M)$.*

We now turn to property (*) for the ODE system (4.16). For obtaining it, let us recall that defining equations (4.1) of hypersurfaces under consideration satisfy the reality condition:

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$$w \equiv \Theta(z, \bar{z}, \bar{\Theta}(\bar{z}, z, w)) \ \ \forall z, \bar{z}, w \tag{4.22}$$

(see, e.g., [5]). Gathering in (4.22) terms with $z^k \bar{z}^1$ and using (4.1), we obtain

$$0 = \Theta_{k1}(w) + \bar{\Theta}_{1k}(w).$$

Hence we have, in view of (4.15):

$$\operatorname{ord}_0 \Theta_{1k} = m. \tag{4.23}$$

It immediately follows then from the above elimination procedure that the term with $z^0 w^m \zeta^k$ in the expansion of the function Φ_1 in (4.16) is non-zero, and without loss of generality, we assume its coefficient in what follows to be equal to $\pm i$ (even though its exact value is of no special interest to us) and hence Property (*) holds.

4.3. Proof of the main theorem under the k-nondegeneracy assumption

The main technical difficulty of this subsection is to provide an analogue of (3.27) for mappings between hypersurfaces which satisfy the m-nonminimal k-nondegeneracy assumptions. We will expand such a map $H = (F, G)$ in this section as

$$\begin{aligned}
&F = z + S(z, w) + \sum_{j=0}^{k} f_j(w) z^j + f(z, w), \\
&G = T(w) + w^m R(z, w) + w g_0(w) + w^m \sum_{j=1}^{k} g_j(w) z^j + w^m g(z, w), \\
&S_z(0,0) = 0, \quad f(z, w) = O(z^{k+1}), \quad g(z, w) = O(z^{k+1}),
\end{aligned} \tag{4.24}$$

where f_j, g_j, f, g are formal power series, f_j, g_j all vanish to order $k+1$, and $T(w), S(z, w), R(z, w)$ are certain *fixed polynomials* in their variables, exact form of which is of no interest to us (the desired representation of g is possible in view of (4.5)). We think about f_j, g_j and their derivatives as "additional parameters", and for this purpose, we write

$$\alpha_{ij} := f_i^{(j)}(w), \quad \beta_{ij} := g_i^{(j)}(w), \quad \alpha = \{\alpha_{ij}\}, \quad \beta = \{\beta_{ij}\}, \quad 0 \le i \le k, \quad 0 \le j \le k+1.$$

We will see (through a careful analysis of the transformation rules for the associated systems) that we can find holomorphic functions U, V, such that

$$\begin{aligned}
f_{z^{k+1}} &= U\Big(z, w, j^k(f, g), \{f_{z^{k+1-j} w^j}\}_{j=1}^{k+1}, \{g_{z^{k+1-j} w^j}\}_{j=1}^{k+1}, \alpha, \beta\Big), \\
g_{z^{k+1}} &= V\Big(z, w, j^k(f, g), \{f_{z^{k+1-j} w^j}\}_{j=1}^{k+1}, \{g_{z^{k+1-j} w^j}\}_{j=1}^{k+1}, \alpha, \beta\Big).
\end{aligned} \tag{4.25}$$

Let us first show how Theorem 1 follows (in the k-nondegenerate case) from (4.25).

Proof of Theorem 1 under the k-nondegeneracy assumption. Solving (4.25) by the Ovcyannikov theorem (see Section 3) as a Cauchy problem with the initial data

$$f_{z^j}(0,w) = g_{z^j}(0,w) = 0, \quad 0 \le j \le k$$

and the additional parameters α, β, we obtain:

$$f(z,w) = \varphi\big(z,w,\alpha,\beta\big), \quad g(z,w) = \psi\big(z,w,\alpha,\beta\big) \tag{4.26}$$

for two functions φ, ψ, holomorphic in all their arguments. We recall now that, by definition, α and β stand for formal power series *without constant term*, so that substituting back $f_i^{(j)}(w)$ for α_{ij} and $g_i^{(j)}(w)$ for β_{ij} is well defined.

Let us note finally that, combining Proposition 4.2 and the expansion (4.24), we may apply the assertion of Proposition 4.2 to the functions $f_0, ..., f_k, g_0, ..., g_k$. Substituting the arising sectorial functions $f_j^{\pm}, g_j^{\pm}$ into (4.26), we obtain sectorial holomorphic transformations $(F^{\pm}, G^{\pm})$. Then, arguing identically to the proof of Theorem 1 in the generic case (end of Section 3), we obtain the assertion of Theorem 1 in the k-nondegenerate case. □

In what follows, we have to take into consideration the space $J^{k+1}(\mathbb{C}, \mathbb{C})$ of $(k+1)$-jets of holomorphic maps from $\mathbb{C}$ into itself. We use the notations

$$(z, w, w_1, ..., w_{k+1})$$

for the coordinates in the jet space (here w_j corresponds to the derivative $w^{(j)}(z)$). A system (4.16) shall be regarded then as a submanifold in $J^{k+1}(\mathbb{C}, \mathbb{C})$ of dimension 3 (with the local coordinates z, w, w_k):

$$w_1 = \Phi_1\left(z, w, \frac{w_k}{w^m}\right), \cdots, w_{k-1} = \Phi_{k-1}\left(z, w, \frac{w_k}{w^m}\right), \;\; w_{k+1} = \Phi\left(z, w, \frac{w_k}{w^m}\right). \tag{4.27}$$

Next, we consider the $(k+1)$-jet prolongation

$$\begin{aligned} &H^{(k+1)}(z, w, w_1, ..., w_{k+1}) = \\ &\quad = \big(F(z,w), G(z,w), G^{(1)}(z,w,w_1), G^{(2)}(z,w,w_1,w_2), ..., G^{(k+1)}(z,w,w_1,...,w_{k+1})\big) \end{aligned} \tag{4.28}$$

of the map $H = (F, G)$. Introducing the total derivation operator

$$D := \partial_z + w_1 \partial_w + \sum_{j \ge 1} w_{j+1} \partial_{w_j}, \tag{4.29}$$

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we can inductively compute the components of the prolonged map (see [8, (3.96d) of section 2.3.1]) as

$$G^{(j)} = \frac{DG^{(j-1)}}{DF},\ j \geq 1, \quad \text{where} \quad G^{(0)} := G. \tag{4.30}$$

Note that, in fact,

$$DF = F_z + w_1 G_w.$$

As follows from (4.30), each $G^{(j)}(z, w, w_1, ..., w_j)$ is an expression, *rational* in the first jet variable w_1 and *polynomial* in the remaining jet variables $w_2, ..., w_j$; its coefficients are *universal polynomials* in the j-jet of (F, G). For certain precise values of j (e.g. $j = 1, 2, 3$), the j-jet prolongation formulas can be written explicitly. For example, we have:

$$\begin{aligned} G^{(1)}(z, w, w_1) &= \frac{G_z + w_1 G_w}{F_z + w_1 F_w}, \\ G^{(2)}(z, w, w_1, w_2) &= \frac{1}{(F_z + w_1 F_w)^3}\Big[(F_z + w_1 F_w)(G_{zz} + 2w_1 G_{zw} + (w_1)^2 G_{ww} + w_2 G_w) - \\ &\quad - (G_z + w_1 G_w)(F_{zz} + 2w_1 F_{zw} + (w_1)^2 F_{ww} + w_2 F_w)\Big]. \end{aligned} \tag{4.31}$$

For some higher orders see, e.g., [8]. However, for a general j, only certain summation formulas exist, which can not always be worked out. That is why we will use only a few properties of the prolonged maps, which are useful for our consideration. For example, we can claim that, for maps of the kind (4.5), the denominator of it is non-vanishing at $z = w = w_1 = ... = w_j = 0$. This can be easily proved by induction, by using (4.30) and the fact that $F_z(0, 0) = 1$.

According to the outcome of the previous section and the discussion in Section 2, the prolonged map $H^{(k+1)}$ transforms the submanifolds $\mathcal{E}(M), \mathcal{E}(M^*) \subset J^{k+1}(\mathbb{C}, \mathbb{C})$ into each other. That is, we have the following basic identity (we set $W_k := (w_1, ..., w_k)$):

$$\begin{aligned} G^{(1)}(z, w, w_1) &= \Phi_1^*\left(F(z, w), G(z, w), \frac{G^{(k)}(z, w, W_k)}{G^m(z, w)}\right), \\ &\cdots \\ G^{(k-1)}(z, w, w_1, \dots, w_{k-1}) &= \Phi_{k-1}^*\left(F(z, w), G(z, w), \frac{G^{(k)}(z, w, W_k)}{G^m(z, w)}\right), \\ G^{(k+1)}(z, w, w_1, \dots, w_{k+1}) &= \Phi^*\left(F(z, w), G(z, w), \frac{G^{(k)}(z, w, W_k)}{G^m(z, w)}\right), \end{aligned} \tag{4.32}$$

subject to the restriction

$$w_1 = \Phi_1\left(z, w, \frac{w_k}{w^m}\right), \cdots, w_{k-1} = \Phi_{k-1}\left(z, w, \frac{w_k}{w^m}\right), \;\; w_{k+1} = \Phi\left(z, w, \frac{w_k}{w^m}\right) \tag{4.33}$$

(here we used the star notation for the target ODE system). We claim that, by setting

$$\zeta := \frac{w_k}{w^m}, \tag{4.34}$$

we can understand (4.32) as an identity of formal power series in the *independent* variables z, w, ζ.

Indeed, we first note that the substitution $w_k = w^m\zeta$ makes all expressions in (4.33) power series in z, w, ζ (divisible by ζ, in view of (4.17)). Further, we consider the singular expression $\frac{G^{(k)}(z,w,w_1,..,w_k)}{G^m(z,w)}$ in (4.32) as a ratio of two formal power series $P(z, w, w_1, ..., w_k), Q(z, w, w_1, ..., w_k)$, each of which is polynomial in $w_1, ..., w_k$. The denominator Q can be factorized as $w^m \cdot \widetilde{Q}(z, w)$ with $\widetilde{Q}(0, 0) \neq 0$ (as follows from (3.2)). Next, the "constant" term of the polynomial P obtained by setting $w_j = 0$ for all j, can be inductively computed using the scheme

$$c_1 = \frac{G_z}{F_z}, \quad c_j = \frac{\partial_z(c_{j-1})}{F_z}, \quad 2 \le j \le k+1 \tag{4.35}$$

(as follows from (4.30)), and it follows then from (4.5) that the desired constant term $c_k(z, w)$ is divisible by w^m. All the other terms in P are (i) either divisible by w_k, hence the substitution $w_k = w^m\zeta$ makes them divisible by w^m, or (ii) divisible by some w_j, $j = 1, ..., k-1$, and hence the substitution $w_j = \Phi_j(z, w, w_1, ..., w_j)$ makes them divisible by w^m (in view of (4.17)). We conclude that P subject to the restriction (4.33) is divisible by w^m (after the substitution $w_k = w^m\zeta$), and this proves the claim.

We use the following

Convention. In what follows,

$$h_{z^k w^l}$$

denotes the partial derivative $\frac{\partial^{k+l}}{\partial z^k \partial z^l}$ for a function $h(z, w)$.

We then consider the last equation in (4.32) subject to (4.33) as an identity in z, w, ζ and collect within it all terms with ζ^0. Then:

(i) in the left hand side, we obtain the expression $c_{k+1}(z, w)$ from (4.35); it is easy to see that this expression can be written as

$$\frac{1}{(F_z)^{k+1}}\left(G_{z^{k+1}} \cdot F_z - F_{z^{k+1}} \cdot G_z + \cdots\right),$$

where dots stand for a polynomial in $F_z, F_{zz}, ..., F_{z^k}, G_z, g_{zz}, ..., G_{z^k}$; substituting (4.24), we obtain

$$w^m\big(g_{z^{k+1}}\cdot(1+A)+f_{z^{k+1}}\cdot B+C\big),$$

where A,B,C are *holomorphic* expressions in $j^kf,j^kg,z,w,\alpha,\beta$, and A,B vanish at the origin. (In fact, A,B,C have more specific form, but we do not need these further details);

(ii) for the right hand side, we argue as above and conclude that, for the singular argument $\frac{G^{(k)}(z,w,w_1,..,w_k)}{G^m(z,w)}$, evaluating $\zeta=0$ and substituting (4.24) makes the numerator divisible by w^m. Taking further (4.17) into account, we conclude that the right hand side in the identity under consideration as well has the form

$$w^m\widetilde{C},$$

where $\widetilde{C}$ is an expression, *holomorphic* in $j^kf,j^kg,z,w,\alpha,\beta$.

We summarize that, gathering in the last identity in (4.32) terms with ζ^0 gives:

$$g_{z^{k+1}}\cdot(1+A)+f_{z^{k+1}}\cdot B=\widehat{C}, \tag{4.36}$$

where $A,B,\widehat{C}$ are holomorphic expressions as above, and A,B vanish at the origin.

It remains for us to obtain one more identity of the kind (4.36), solvable already in $f_{z^{k+1}}$. For doing so, we consider in the last identity in (4.32) (subject to restriction (4.33)) terms with ζ^k.

Claim. *The result of collecting terms with ζ^k in* (4.32) *(subject to restriction* (4.33)*) can be written in the form*

$$-f_{z^{k+1}}\cdot\big(\pm i+L_0\big)+\sum_{j=1}^{k+1}L_j\cdot f_{z^{k+1-j}w^j}+\sum_{j=0}^{k+1}M_j\cdot g_{z^{k+1-j}w^j}+N=\widetilde{N}, \tag{4.37}$$

where the expressions $L_j,M_j,N,\widetilde{N}$ are described identically to the expressions $A,B,\widehat{C}$ in (4.36) *and, moreover, L_0 vanishes at the origin.*

To prove the claim, we have to analyze the jet prolonged component $G^{(k+1)}$ with more details. Recall that $G^{(k+1)}$ is a rational in w_1 and polynomial in $w_2,...,w_{k+1}$ expression, coefficients of which are certain universal polynomials in $j^{k+1}(F,G)$. Its denominator is nonvanishing for $z=w=w_1=...=w_{k+1}$, as discussed above. Hence, we may expand

$$G^{(k+1)}=\sum_{l_1,\dots,l_{k+1}\geq 0}E_{l_1,\dots,l_{k+1}}(w_1)^{l_1}\cdots(w_{k+1})^{l_{k+1}}, \tag{4.38}$$

where $E_{l_1,\dots,l_{k+1}}$ are all certain universal polynomials in $j^{k+1}(F,G)$ and the ratio $\frac{1}{F_z}$ (the latter fact can be seen from (4.30), induction in k and the chain rule). Recall that the constant term $E_{0,\dots,0}$ can be computed via (4.35). For all the other terms, we have to distinguish $E_{l_1,\dots,l_{k+1}}$ depending on the *highest order derivatives* $F_{z^pw^q},G_{z^pw^q}$, $p+q=k+1$. In this regard, we have

Lemma 4.5. *The only coefficients $E_{l_1,\dots,l_{k+1}}$ in (4.38) depending on the highest order derivatives $F_{z^pw^q}, G_{z^pw^q}$, $p+q=k+1$ are $E_{l,0,\dots,0}$, $l\geq 0$. Moreover, $E_{1,0,\dots,0}$ has the form identical to the left hand side of (4.37), where the expressions L_j, M_j, N are certain universal polynomials in $j^k(F,G)$ and $\frac{1}{F_z}$, and, in addition, L_0 vanishes when $j^k(F,G)$ is evaluated at $z=w=0$.*

Proof. For $k=0$, the assertion follows from the formula (4.31) and the chain rule. For $k>0$, we apply the iterative formula (4.30), induction in k and the chain rule. Then the assertion of the lemma follows by a straightforward inspection. □

We immediately conclude that, when collecting terms with ζ^k in the left hand side of the last identity in (4.32), highest order derivatives may arise only from terms with $(w_1)^l$, $l\geq 1$. Next, we note that the term $E_{1,0,\dots,0}\cdot w_1$, subject to constraint (4.33), contributes

$$w^m(\pm i+o(1))\cdot E_{1,0,\dots,0}$$

(as follows from the Property $(*)$ of Φ_1). Substituting the expansions (4.24) for F,G, we obtain an expression of the kind (4.37) multiplied by w^m. Further, the constant term $E_{0,\dots,0}$ does not contribute to ζ^k (as it doesn't depend on the w_j's). All terms with $(w_1)^l$, $l\geq 2$ may contribute to ζ^k, however, in view of the factorization property (4.17) their contribution gives at least the factor $O(w^{2m})$ in front of the highest order derivatives. All other terms do *not* contribute to ζ^k with the highest order derivatives, as follows from Lemma 4.5. They, however, necessarily give the factor w^m, in view of (4.17), (4.34).

We finally conclude that the left hand side of the identity under discussion has the form identical to the left hand side in (4.37) multiplied by w^m.

To study the right hand side of the last identity in (4.32) subject to (4.33), we recall that

(i) the denominator of the singular argument $\frac{G^{(k)}(z,w,w_1,..,w_k)}{G^m(z,w)}$ has the form $w^m\cdot\widetilde{G}(z,w)$, $\widetilde{G}(0,0)\neq 0$;

(ii) the constant term in the numerator of the same expression is divisible by w^m, after substituting (4.24) (as discussed above);

(iii) the substitutions (4.34), (4.33) together with the factorization (4.17) make the rest of the numerator divisible by w^m;

(iv) the factorization (4.17) applied to Φ^* makes the right hand side under consideration in addition divisible by w^m (after substituting (4.24)).

In this way, we conclude that the right hand side of the identity under discussion has the form identical to the right hand side in (4.37) multiplied by w^m. Dividing the latter identity by w^m finally proves the claim. □

We can now consider the identities (4.36), (4.37) as a linear system in $f_{z^{k+1}}, g_{z^{k+1}}$. Solving it by the Cramer rule, we finally obtain (4.25). □

4.4. Pure order of a nonminimal hypersurface

It might still happen that an m-nonminimal hypersurface (4.1) does *not* satisfy the k-nondegeneracy assumption. To deal with this case, we do (in appropriate coordinates) a blow up in the space of parameters of the Segre family. Related to this procedure is an important invariant of real hypersurface which we call the *pure order*. From the point of view of our method, the pure order replaces, in a certain sense, the nonminimality order.

Definition 4.6. Let $M \subset \mathbb{C}^2$ be a real-analytic Levi-nonflat hypersurface given by (4.1). The *pure order* of M at 0 is the integer p such that

$$p + 1 = \min_{k,l \geq 1} \{l + \operatorname{ord}_0 \Theta_{kl}(\bar{w})\}. \tag{4.39}$$

In other words, $p + 1$ is the minimal possible $l + s$ such that for some $k > 0$ the term with $z^k \bar{z}^l \bar{w}^s$ in the expansion of Θ does not vanish.

Note that:
(i) for a Levi-nonflat hypersurface p is well defined and nonnegative;
(ii) for a Levi-nondegenerate hypersurface we have $p = 0$;
(iii) for an m-nonminimal hypersurface we have $p \geq m$;
(iv) for an m-nonminimal hypersurface with $M \setminus X$ Levi-nondegenerate (the generic case from Section 3) we have $p = m$.

We start with showing that the integer p is a (formal) invariant of a real-analytic hypersurface.

Proposition 4.7. *The pure order of a Levi-nonflat hypersurface is invariant under (formal) invertible transformations of hypersurfaces* (4.1).

Proof. We note that the pure type introduced above actually comes from an *invariant pair* as introduced in [17]; the invariance of those is proved in that paper. □

We now apply the notion of the pure type to prove the following factorization property for CR-maps.

Proposition 4.8. *Let $M, M^* \subset \mathbb{C}^2$ be two real-analytic nonminimal at the origin hypersurfaces, and $H = (F, G) : (M, 0) \mapsto (M^*, 0)$ a formal map. Then*

$$F_z(0,0) \neq 0, \quad G_w(0,0) = G_0'(0) \neq 0, \quad G(z,w) = O(w), \quad G_z(z,w) = O(w^{p+1}), \tag{4.40}$$

where p is the pure order of M, M^* at 0.

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Proof. The proof of all the assertions except the last one goes identically to the proof of (3.2). For the property $G_z(z,w) = O(w^{p+1})$, we consider the basic identity

$$G(z,w) = \Theta^*(F(z,w), \bar{F}(\bar{z},\bar{w}), \bar{G}(\bar{z},\bar{w}))|_{w=\Theta(z,\bar{z},\bar{w})}. \tag{4.41}$$

Putting $\bar{z} = 0$, we get $w = \bar{w}$, and further differentiating in z gives:

$$G_z(z,\bar{w}) = \frac{\partial}{\partial z}\Big[\Theta^*(F(z,\bar{w}), \bar{F}(0,\bar{w}), \bar{G}(0,\bar{w}))\Big]. \tag{4.42}$$

We note now that every non-zero term in the expansion of Θ^* in $z, \bar{z}, \bar{w}$ has total degree at least $p+1$ in $\bar{z}, \bar{w}$ (by the definition of p). At the same time, since (F,G) preserves the origin, we have $g(0,\bar{w}) = O(\bar{w})$, $F(0,\bar{w}) = O(\bar{w})$, so that the whole expression in the square brackets in (4.42) becomes divisible by $\bar{w}^{p+1}$. This property persists after differentiating in z, and this proves the proposition. □

Next, we prove

Proposition 4.9. *Let $M \subset \mathbb{C}^2$ be a real-analytic Levi-nonflat hypersurface, and p is its pure type at 0. Then there exist local holomorphic coordinates (4.1) for M at the origin with (4.2), such that for certain $k \geq 1$ we have:*

$$ord_0\, \Theta_{k1}(\bar{w}) = p. \tag{4.43}$$

Proof. Let us choose any coordinates (4.1) for M at 0 with (4.2). As was discussed above, change of coordinates (4.1) corresponds to choosing a curve $\gamma \subset M$ being transformed to the canonical curve (2.2). Let us choose $\gamma \subset M$ of the kind

$$z = \alpha u, \quad w = u + iq(u), \quad u \in \mathbb{R}$$

for an appropriate real-valued $q(u)$ and a generic $\alpha \in \mathbb{C}$. Then there exists a biholomorphic transformation of the form,

$$z \mapsto z - \alpha w, \quad w \mapsto g(z,w), \quad g(0,0) = 0, \tag{4.44}$$

mapping M into another hypersurface M^* of the form (4.1) and γ into the curve (2.2) (e.g., [13][25]). If we now fix in the expansion (4.1) of M the non-zero term $z^k\bar{z}^j\bar{w}^l$, $j+l = p+1$ with the minimal $k \geq 1$, then it is easy to verify from the basic identity that the substitution (4.44) creates, for a generic α, a non-zero term $z^k\bar{z}\bar{w}^p$ in the expansion (4.1) for M^*. In view of the invariance of the pure order this means the validity of (4.43) for M^*. Moreover, for a generic α in (4.44) the condition (4.2) persists as well, and this proves the proposition. □

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4.5. Proof of the main theorem

Having Theorem 1 proved in the k-nondegenerate case (subsection 4.3) and having the relations (4.40), (4.43), we are now in the position to prove Theorem 1 in its full generality.

Proof of Theorem 1. According to the outcome of subsection 4.3, it remains to prove the theorem in the case when M is m-nonminimal at 0 but is not k-nondegenerate for any $k \geq 1$. Let us choose for M, M^* local holomorphic coordinates according to Proposition 4.9. Then we have the identity (4.43), for both the source and the target. Let us then consider the Segre family $\mathcal{S} = \{Q_p\}_{p=(\bar{a},\bar{b})}$ of M as a 2-parameter holomorphic family in a, b. Next, let us perform the following blow up in the space of parameters:

$$a = \tilde{a}\tilde{b}, \quad b = \tilde{b}. \tag{4.45}$$

Let us denote the new parameterized family by $\widetilde{\mathcal{S}}$, and keep denoting for simplicity the new parameters by a, b. Then, considering an element of the family $\mathcal{S}$ as a graph $w = w(z)$ and expanding in z, a, b, we see that terms $\lambda z^k a^j b^l$ get transformed (after the blow up (4.45)) into $\lambda z^k a^j b^{j+l}$. We obtain from here the crucial corollary that *all terms in the expansion of $w(z, a, b)$ except the very first term $z^0 a^0 b^1$ are divisible by b^{p+1}*. Furthermore, the non-zero (in view of (4.43)) term $\lambda z^k ab^p$, $k \geq 1$ gets transformed into $z^k ab^{p+1}$. We conclude that the transformed family $\widetilde{\mathcal{S}}$ has the form identical to (4.18) with the nondegeneracy property (4.19), with the only difference that m is replaced by $p + 1$. Hence, arguing identically to subsection 4.2, we conclude that the family $\widetilde{\mathcal{S}}$ (and hence the family $\mathcal{S}$!) satisfy a system of ODEs, identical to (4.16) with the only difference that, again, m is replaced by $p + 1$. The same statement applies for the target M^*, and we conclude that the given formal map (F, G) between $(M, 0)$ and $(M^*, 0)$ satisfies an identity similar to (4.32) with m replaced by $p + 1$.

We finally recall that (F, G) satisfies the factorization (4.40), which is identical to (4.5) with, again, m replaced by $p+1$. This allows to repeat the proof in the k-nondegenerate case word-by-word (we recall that the crucial Proposition 4.2 is valid without any further assumptions and thus is applicable to the map (F, G)). Theorem 1 is proved. □

In the end of the paper, we are able to finally formulate and prove the following expanded version of Theorem 3, which is also a much stronger version of the main result.

Theorem 5. *Let $M, M^* \subset \mathbb{C}^2$ be two real-analytic Levi-nonflat hypersurfaces, both of infinite type at 0, and let $\widehat{H} : (M, 0) \mapsto (M^*, 0)$ be a formal CR equivalence. Then there exist a constant $s > 0$, local holomorphic coordinates (z, w) for M, M^* at 0 at which the complex locus of both M and M^* is $\{w = 0\}$, a disc $\Delta \subset \mathbb{C}$, sectors $S^\pm \subset (\mathbb{C}, 0)$ with vertex at 0 containing the directions $\mathbb{R}^\pm$, respectively, and holomorphic maps*

$H_\pm\colon \Delta \times S^\pm \to \mathbb{C}^2$ such that $\widehat{H}$ is the $(0,s)$ multi Gevrey asymptotic expansion of $H_\pm$ and $H_\pm(M \cap (\Delta \times S^\pm)) \subset M^*$; in particular, $\widehat{H}(z,w)$ belongs to the $(0,s)$ multi Gevrey class, and $H_\pm|_M$ defines a CR diffeomorphism H of M onto M^*. Furthermore, the formal power series $\widehat{H}$ has the multisummability property.

Proof of Theorem 5. The assertion of the theorem immediately follows from the crucial Corollary 3.3 and Proposition 4.2, and the representations (3.9), (4.26). □

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