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Geometry of hyperbolic Cauchy–Riemann singularities and KAM-like theory for holomorphic involutions

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Abstract
This article is concerned with the geometry of germs of real analytic surfaces in $(\mathbb{C}^2, 0)$ having an isolated Cauchy–Riemann (CR) singularity at the origin. These are perturbations of *Bishop quadrics*. There are two kinds of CR singularities stable under perturbation: *elliptic* and *hyperbolic*. Elliptic case was studied by Moser–Webster (Acta Math 150(3–4), 255–296, 1983) who showed that such a surface is locally, near the CR singularity, holomorphically equivalent to *normal form* from which lots of geometric features can be read off. In this article we focus on perturbations of *hyperbolic* quadrics. As was shown by Moser and Webster (1983), such a surface can be transformed to a formal *normal form* by a formal change of coordinates that may not be holomorphic in any neighborhood of the origin. Given a *non-degenerate* real analytic surface M in $(\mathbb{C}^2, 0)$ having a *hyperbolic* CR singularity at the origin, we prove the existence of a non-constant Whitney smooth family of connected holomorphic curves intersecting M along holomorphic hyperbolas. This is the very first result concerning hyperbolic CR singularity not equivalent to quadrics. This is a consequence of a non-standard KAM-like theorem for pair of germs of holomorphic involutions $\{\tau_1, \tau_2\}$ at the origin, a common fixed point. We show that such a pair has large amount of

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invariant analytic sets biholomorphic to $\{z_1z_2 = const\}$ (which is not a torus) in a neighborhood of the origin, and that they are conjugate to restrictions of linear maps on such invariant sets.

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1 Introduction

In this article, we are concerned with the local holomorphic invariants of a real analytic submanifold M in $\mathbb{C}^n$. If the tangent space of M at a point p_0 contains a maximal *complex* subspace, the dimension d of which does not depend on p_0, then we say that M is a *Cauchy–Riemann (CR) submanifold*. Since the work of Cartan [11] in the 1930s, lots of studies were devoted to this geometry (see for instance [4, 12, 35, 36]). As "baby" example, one can consider an open neighborhood U of a point p_0 in $\mathbb{R}^n$ in $\mathbb{C}^n$. The local hull of holomorphy of U is the largest open set in $\mathbb{C}^n$ containing U and over which all holomorphic functions defined on U can be holomorphically extended to. It can be shown that, in that case, the hull of holomorphy of U is nothing but U. This situation is quite different when considering a neighborhood of a CR singularity, that is a point p_0 in the real submanifold M in $\mathbb{C}^n$ such that the maximal complex tangent spaces do not have a constant dimension in any neighborhood of p_0. A real submanifold with a CR singularity must have codimension at least 2.

The study of real submanifolds with CR singularities was initiated by Bishop [6] in his pioneering work, and followed by Moser–Webster [40]. They considered higher-order analytic perturbations of the elementary models called Bishop quadrics $\mathcal{Q}_\gamma \subset \mathbb{C}^2$, depending on the Bishop invariant $0 \leq \gamma \leq \infty$:

- for $0 \leq \gamma < \infty$, $\mathcal{Q}_\gamma : z_2 = Q_\gamma(z_1, \bar{z}_1) := |z_1|^2 + \gamma(z_1^2 + \bar{z}_1^2)$,
- for $\gamma = \infty$, $\mathcal{Q}_\infty$: $z_2 = z_1^2 + \bar{z}_1^2$.

When $\gamma \neq \frac{1}{2}$, such a surface has an isolated CR singularity at the origin as it is totally real (i.e., $d = 0$) everywhere but at the origin at which the tangent space is the complex line $\{z_2 = 0\}$ (i.e., $d = 1$). When $0 < \gamma < \frac{1}{2}$, one says that this singularity is *elliptic*. In their seminal work, Moser–Webster [40] considered higher-order analytic perturbations of elliptic quadrics $\mathcal{Q}_\gamma$. They proved that such a submanifold is holomorphically equivalent to a *normal form*, $z_2 = |z_1|^2 + (\gamma + \epsilon\,\mathrm{Re}(z_2)^s)(z_1^2 + \bar{z}_1^2)$, for some $\epsilon \in \{-1, 0, 1\}$ and $s \in \mathbb{N}^* \cup \{\infty\}$. Lots of geometric features can be read off from such a normal form. They also considered n-dimensional submanifolds in $\mathbb{C}^n$ which have a complex tangent at the origin of minimal (positive) dimension. This has been recently extended to CR singularity with *maximal complex tangent* by Gong and the first author [23, 24]. When $\gamma = 0$ (degenerate elliptic case), Moser [39] constructed a formal power series normal form. Although it is still not known whether such a normal form can be obtained through a convergent transformation, Huang-Yin [28] did the achievement of obtaining the holomorphic classification of analytic perturbations of $\mathcal{Q}_0$. Relatively recently, related problems such as flattening [29–31] or quadric rigidity [27] have been successfully considered by Huang and co-authors. Some results on

CR singularities of k-dimensional submanifolds in $\mathbb{C}^n$, $k \neq n$ have been obtained by Coffman [14, 15].

In the so-called *hyperbolic* case, i.e., higher-order analytic perturbation of $\mathcal{Q}_\gamma$ with $\gamma > \frac{1}{2}$, not much is known. Moser–Webster [40] showed that some analytic perturbations of $\mathcal{Q}_\gamma$ may not be holomorphically equivalent to a normal form as in the elliptic case. Forstnerič–Stout [19] proved that such a perturbation is always polynomially convex near such a hyperbolic CR singularity. Gong [20] showed that if the higher-order analytic perturbation of $\mathcal{Q}_\gamma$ is *formally* equivalent to $\mathcal{Q}_\gamma$ (i.e., by the mean of formal power series transformation) and if a *Diophantine condition* associated to γ is satisfied, then the perturbation is actually holomorphically equivalent to the quadric. He also proved the existence of higher-order analytic perturbations of a hyperbolic quadric which are formally equivalent to the hyperbolic quadric but not holomorphically equivalent to it [22]. On the other hand, Klingenberg [33] showed that under a similar Diophantine condition, for a given higher-order analytic perturbation M of the quadric, there always exists a holomorphic curve that intersects M along two transverse totally real curves. Both results have been extended in higher dimension in the case of maximal complex tangent [23].

In both elliptic and hyperbolic cases, the CR singularity is stable under perturbation and is not removable.

The aim of this work is to prove that *non-degenerate analytic perturbations* of hyperbolic quadrics, i.e., perturbations which are not formally equivalent to quadrics, contain a large number of analytic hyperbolas. By this, we mean that there exists a compact set $\mathcal{K} \subset \mathbb{R}$ of positive measure such that for all $\omega \in \mathcal{K}$, there exists a connected holomorphic curve $\mathcal{S}_\omega$ that intersects the non-degenerate analytic perturbation M along two distinguished real analytic curves that are simultaneously holomorphically mapped to the two branches of the real hyperbola $\{\xi\eta = \omega\}$ (in a neighborhood of the origin). We remark that it is elementary that a real analytic curve in the real analytic surface is contained in a holomorphic curve. Having a connected holomorphic curve that intersects M in two distinct real analytic curves is, however, one of main conclusions of this paper.

To do so, we shall develop a new KAM theory (named after Kolmogorov–Arnold–Moser [1, 34, 38]) for a pair of (germs of) holomorphic involutions in a neighborhood of a fixed point (say 0) in $\mathbb{C}^2$, which is swapped by conjugacy with some anti-holomorphic involution. Initially, KAM theory was conceived as an answer to the fundamental problem arising in Dynamical Systems and in particular in Celestial Mechanics [13, 18]. It can be formulated as follows: Given a *completely integrable* Hamiltonian dynamical system written in *action-angle coordinates* $(\theta, I) \in \mathbb{T}^n \times \mathbb{R}^n$ of the form $\dot\theta = \omega(I)$, $\dot I = 0$, where ω denotes an analytic function. For each I_0, the manifold $\mathbb{T}^n \times \{I_0\}$ is invariant and the motion on it is a constant rotation of angle $\omega(I_0)$. In the nature, these systems are rather rare but one encounters small perturbations of them under the form $(*)$ $\dot\theta = \omega(I) + \epsilon f(I, \theta)$, $\dot I = \epsilon g(I, \theta)$ with f, g analytic functions and ϵ a small number. Essentially, KAM theorem states that if the system is *non-degenerate* in some sense then there exists a large (in measure) compact set $\mathcal{K}$ such that for all $I \in \mathcal{K}$, the system $(*)$ has an invariant manifold which is diffeomorphic to a torus the dynamical system on which is conjugated to the rotation of angle $\tilde{\omega}(I)$ on that torus. In some sense, a lot of the invariant tori $\mathbb{T}^n \times \{I\}$ "survive" under a small non-degenerate perturbation.

As mentioned earlier, to obtain a connected holomorphic curve intersecting the real surface is a main result. In our context, we shall go one step further by proving, for a sufficiently small $R > 0$, the existence of a compact set $\mathcal{O}_\infty(R) \subset]-R^2, R^2[$ of positive measure such that for each $\omega \in \mathcal{O}_\infty(R)$, there exists an invariant connected complex submanifold $\tilde{\mathcal{S}}_\omega$ in $\Delta_2(0, R^{\frac{1}{2}}) := \{(\xi, \eta) \in \mathbb{C}^2 : |\xi|, |\eta| < R^{\frac{1}{2}}\}$, which is the image of the connected holomorphic manifold $\mathcal{C}^R_\omega := \{(\xi, \eta) \in \mathbb{C}^2 : \xi\eta = \omega,\ |\xi|, |\eta| < R\}$ by a biholomorphism $\Psi_\omega : \mathcal{C}^R_\omega \to \tilde{\mathcal{S}}_\omega$. The inverse of the latter, Ψ_ω^{-1}, conjugates the restrictions of nonlinear involutions to $\tilde{\mathcal{S}}_\omega$ to the restrictions to $\mathcal{C}^R_\omega$ of linear ones. We emphasize that $\mathcal{C}^R_\omega \cap \Delta_2(0, a\sqrt{|\omega|})$ contains the graph $\zeta \mapsto (\zeta, \frac{\omega}{\zeta})$ over the annulus $\frac{\sqrt{|\omega|}}{a} < |\zeta| < \sqrt{|\omega|}a$. This KAM-like result is non-standard as one does not expect to obtain *invariant tori* as in [5, 7, 17, 42] but different kind of invariant manifolds of the form $\{z_1 z_2 = \omega\}$ (in a neighborhood of the origin; when $\omega \neq 0$). The role played by the rotation is played by linear involutions. In a similar spirit, but in a different context, a KAM-like theory was obtained by the first author for germs of holomorphic vector fields at a fixed point [48]. We emphasize that the KAM-like statement in this paper is different from an apparently similar *real* problem for which one obtains a lot of invariant tori. The main achievements in this direction are due to Sevryuk [43, 45, 46] near an elliptic fixed point. The non-standard hyperbola character of our KAM-like result near an elliptic fixed point of reversible holomorphic mappings unveils new unexpected difficulties.

Hard implicit function theorem, Nash–Moser theorem, Newton Scheme or KAM process are various names in the literature that stand for "rapid iteration scheme" usually needed to solve functional equations in Fréchet spaces [25]. This appears in particular in conjugacy problem to normal forms of vector fields at a fixed point [9, 10, 47], of interval exchange maps [37] or in reducibility problems of quasi-periodic cocycles [3, 16, 26], the latter being related to spectral theory as well.

2 Main results

We shall here summarize some statements of [40]. Let us consider Bishop's *hyperbolic* quadric, a real quadratic surface in $\mathbb{C}^2$ given by

$$Q_\gamma : z_2 = Q_\gamma(z_1, \bar{z}_1) = |z_1|^2 + \gamma(z_1^2 + \bar{z}_1^2), \quad \gamma > \frac{1}{2}.$$

Let M be a higher-order analytic perturbation of Q_γ given by

$$M : z_2 = Q_\gamma(z_1, \bar{z}_1) + f(z_1, \bar{z}_1), \quad f(z_1, \bar{z}_1) = O^3(z_1, \bar{z}_1). \tag{1}$$

To such a real surface, one associates a local dynamical system in $(\mathbb{C}^2, 0)$, $\{\tau_1^o, \tau_2^o, \rho\}$, where τ_1^o, τ_2^o are local holomorphic involutions fixing 0, ρ is an anti-holomorphic involution. They satisfy $\tau_j^o \circ \tau_j^o = \mathrm{Id}$ and $\tau_2^o = \rho \circ \tau_1^o \circ \rho$. Moser–Webster's construction goes as follow. We first "complexify" M as a complex surface $\mathcal{M}$ in $\mathbb{C}^4$ by considering two new complex independent variables w_1, w_2, playing the role of $\bar{z}_1, \bar{z}_2$ respectively. In these coordinates,

$$\mathcal{M} : \begin{cases} z_2 = Q_\gamma(z_1, w_1) + f(z_1, w_1) \\ w_2 = Q_\gamma(z_1, w_1) + \bar{f}(w_1, z_1). \end{cases}$$

There are two natural holomorphic mappings $\pi_i : (\mathbb{C}^4, 0) \cap \mathcal{M} \to (\mathbb{C}^2, 0)$, $i = 1, 2$, defined as $\pi_1(z, w) = z$ and $\pi_2(z, w) = w$. It happens that these are $2 - 1$ branched coverings. Each mapping τ_i^o is defined to be the deck transformation (different from identity) of π_i, that is

$$\pi_1(\tau_1^o(z, w)) = z, \quad \pi_2(\tau_2^o(z, w)) = w.$$

For instance, τ_1^o can be regarded as a mapping defined as $\tau_1^o(z_1, w_1) = (z_1, \phi_1(z_1, w_1))$ such that

$$Q_\gamma(z_1, \phi_1(z_1, w_1)) + f(z_1, \phi_1(z_1, w_1)) = Q_\gamma(z_1, w_1) + f(z_1, w_1).$$

The linear part T_1 of the mapping τ_1^o at the fixed point 0, is obtained by solving the equation $Q_\gamma(z_1, T_1(z_1, w_1)) = Q_\gamma(z_1, w_1)$. An immediate computation shows that $T_1(z_1, w_1) = (z_1, -\gamma^{-1} z_1 - w_1)$. In good local holomorphic coordinates (ξ, η), T_1 is rewritten as $T_1(\xi, \eta) = (\delta\eta, \delta^{-1}\xi)$ for some complex number δ.

Such a triple $\{\tau_1^o, \tau_2^o, \rho\}$ completely characterizes the holomorphic equivalent class of the real surface M (cf. [40, Proposition 1.1] or [23, Proposition 2.8]). It is also useful to consider the germ of biholomorphism $\sigma_o := \tau_1^o \circ \tau_2^o$. In good local holomorphic coordinates (ξ, η), we have $\rho(\xi, \eta) = (\bar{\xi}, \bar{\eta})$,

$$\tau_1^o(\xi, \eta) = \begin{pmatrix} e^{\frac{i}{2}\lambda}\eta + p^o(\xi, \eta) \\ e^{-\frac{i}{2}\lambda}\xi + q^o(\xi, \eta) \end{pmatrix}, \tag{2}$$

$$\tau_2^o(\xi, \eta) = \left(\rho \circ \tau_1^o \circ \rho\right)(\xi, \eta) = \begin{pmatrix} e^{-\frac{i}{2}\lambda}\eta + \bar{p}^o(\xi, \eta) \\ e^{\frac{i}{2}\lambda}\xi + \bar{q}^o(\xi, \eta) \end{pmatrix}, \tag{3}$$

where $\bar{h}$ denotes $\bar{h}(\xi, \eta) := \sum_{k,l \geq 0} \bar{\check{h}}_{k,l}\xi^k\eta^l$ if $h(\xi, \eta) := \sum_{k,l \geq 0} \check{h}_{k,l}\xi^k\eta^l$. We also have

$$\sigma_o(\xi, \eta) = \begin{pmatrix} \mu\xi + f^o(\xi, \eta) \\ \mu^{-1}\eta + g^o(\xi, \eta) \end{pmatrix}, \quad \mu = e^{i\lambda}, \quad |e^{i\lambda}| = 1. \tag{4}$$

Here, $e^{\frac{i}{2}\lambda}, e^{-\frac{i}{2}\lambda}$ are the roots of the quadratic equation $\gamma X^2 - X + \gamma = 0$ and p^o, q^o, f^o, g^o are germs of holomorphic functions of order ≥ 2 at the origin (i.e., the functions and their first-order derivatives vanish at 0). In the case $M = \mathcal{Q}_\gamma$, τ_1^o, τ_2^o are the linear involutions

$$\tau_1^o(\xi, \eta) = \begin{pmatrix} e^{\frac{i}{2}\lambda}\eta \\ e^{-\frac{i}{2}\lambda}\xi \end{pmatrix}, \quad \tau_2^o(\xi, \eta) = \begin{pmatrix} e^{-\frac{i}{2}\lambda}\eta \\ e^{\frac{i}{2}\lambda}\xi \end{pmatrix}.$$

In the sequel, we shall assume that the submanifold M (or their associated involutions τ_1^o, τ_2^o) is *non-exceptional*, meaning that $e^{\frac{i}{2}\lambda}$ is not a root of unity. In this case, Moser–Webster showed (cf. [40, Lemma 3.2, Theorem 3.4]) that there exists a formal transformation $\hat{\Psi}$ satisfying $\hat{\Psi} \circ \rho = \rho \circ \hat{\Psi}$ such that

$$\hat{\tau}_1 := (\hat{\Psi}^{-1} \circ \tau_1^o \circ \hat{\Psi})(\xi, \eta) = \begin{pmatrix} \Lambda(\xi\eta)\eta \\ \Lambda^{-1}(\xi\eta)\xi \end{pmatrix}, \tag{5}$$

$$\hat{\tau}_2 := (\hat{\Psi}^{-1} \circ \tau_2^o \circ \hat{\Psi})(\xi, \eta) = \begin{pmatrix} \Lambda(\xi\eta)^{-1}\eta \\ \Lambda(\xi\eta)\xi \end{pmatrix}, \tag{6}$$

$$\hat{\sigma} := (\hat{\Psi}^{-1} \circ \sigma_o \circ \hat{\Psi})(\xi, \eta) = \begin{pmatrix} \hat{M}(\xi\eta)\xi \\ \hat{M}(\xi\eta)^{-1}\eta \end{pmatrix}. \tag{7}$$

Here, $\Lambda(z)$ and $\hat{M}(z)$ are formal power series of the one-dimensional variable z and satisfy:

$$\Lambda(z)\bar{\Lambda}(z) = 1, \quad \hat{M}(z) = \Lambda(z)^2, \quad \Lambda(0) = e^{\frac{i}{2}\lambda}, \quad \hat{M}(0) = \mu.$$

The maps $\hat{\tau}_j$ and $\hat{\sigma}$ are called formal *normal form*. Furthermore, the pair $\{\hat{\tau}_1, \hat{\tau}_2\}$ is said to be *formally integrable*. It would have been called *integrable* over a domain in $\mathbb{C}^2$ if Λ was holomorphic in that domain. The map $\hat{\Psi}$ is called the *normalizing transformation*. Contrary to the elliptic case, one cannot expect the normalizing transformation to converge in a neighborhood of the origin. This is due to the presence of *small divisors* (we recall that $\{|\mu^k - 1|\}_{k\in\mathbb{N}^*}$ accumulate at the origin when $|\mu| = 1$) as emphasized in [40, Section 6 (b)].

If $\Lambda(z) = \Lambda(0)$, then $\{\tau_1^o, \tau_2^o\}$ is formally linearizable by a formal transformation that commutes with ρ. Hence, the submanifold is formally equivalent to the quadric $\mathcal{Q}_\gamma$. Gong's theorem [20] asserts that, if a *Diophantine condition* is satisfied, i.e., there exist $r, c > 0$, such that for $k \in \mathbb{N}^*$, $|\mu^k - 1| \geq \frac{c}{k^r}$, then the submanifold is actually holomorphically equivalent to the quadric $\mathcal{Q}_\gamma$ near the origin.

In what follows, we shall focus on the *non-degenerate* case, i.e., we assume that $\Lambda(z) \neq \Lambda(0)$ and assume that s is the smallest positive integer l such that $\Lambda^{(l)}(0) \neq 0$. We can normalize $\frac{\Lambda^{(s)}(0)}{s!} = 1$.

2.1 KAM-like theorem for reversible holomorphic maps

We assume that $\tau_1^o, \tau_2^o, \sigma_o$ are defined in $\{|\xi|, |\eta| < r\}$ for some $0 < r < \frac{1}{4}$ as in (2)–(4), and

(A) $\lambda \in [0, 4\pi[$ with $\frac{\lambda}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$,
(B) p^o and q^o are convergent power series on $\{|\xi|, |\eta| < r\}$ of order ≥ 2, i.e.,

$$p^o(\xi, \eta) = \sum_{\substack{l+j\geq 2\\ l,j\geq 0}} \breve{p}^o_{l,j}\xi^l\eta^j, \quad q^o(\xi, \eta) = \sum_{\substack{l+j\geq 2\\ l,j\geq 0}} \breve{q}^o_{l,j}\xi^l\eta^j,$$

with coefficients $\breve{p}^o_{l,j},\ \breve{q}^o_{l,j} \in \mathbb{C}$.

It is easy to verify that σ_o is *reversible* w.r.t. the involution ρ, i.e., $\sigma_o^{-1} = \rho \circ \sigma_o \circ \rho$.

As above, let $\hat{\Psi}$ be the unique normalized formal transformation together with the formal power series $\Lambda = \Lambda(z)$. We assume that $\Lambda(z)$ is not constant. Let $s \in \mathbb{N}^*$ be the smallest positive integer such that $\Lambda^{(s)}(0) \neq 0$. More precisely, we assume that

$$\Lambda(z) = e^{\frac{\mathrm{i}}{2}\lambda} + \sum_{j \geq s} \tilde{C}_j z^s, \quad \tilde{C}_s \neq 0. \tag{8}$$

For $r > 0$, let $\Delta_2(0, r) := \{(\xi, \eta) \in \mathbb{C}^2 : |\xi|, |\eta| < r\}$ and for $\omega \in \mathbb{R}$, let $\mathcal{C}^r_\omega := \left\{(\xi, \eta) \in \mathbb{C}^2 : \xi\eta = \omega,\ |\xi|, |\eta| < r\right\}$. Obviously, $\mathcal{C}^r_\omega$ is empty if $|\omega| \geq r^2$. The following theorem shows that there is a family of invariant closed curves for the involutions τ^o_j and the reversible map σ_o in any neighborhood of the origin.

Theorem 2.1 *With the notations above and under assumption (8), there exists a small enough $R = R(\lambda, r, s) > 0$ such that there is a compact set $\mathcal{O}_\infty(R) \subset]-R^2, R^2[$ satisfying*[1]

$$\frac{|\mathcal{O}_\infty(R)|}{2R^2} \to 1, \quad R \to 0, \tag{9}$$

such that for any $\omega \in \mathcal{O}_\infty(R)$, one can find $\mu_\omega \in \mathbb{R}$ and a holomorphic transformation $\Psi_\omega : \mathcal{C}^R_\omega \to \Delta_2(0, R^{\frac{1}{2}})$ with $\Psi_\omega \circ \rho = \rho \circ \Psi_\omega$, such that, on $\mathcal{C}^R_\omega$:

$$(\Psi_\omega^{-1} \circ \tau^o_1 \circ \Psi_\omega)(\xi, \eta) = \begin{pmatrix} e^{\frac{\mathrm{i}}{2}\mu_\omega}\eta \\ e^{-\frac{\mathrm{i}}{2}\mu_\omega}\xi \end{pmatrix}, \quad (\Psi_\omega^{-1} \circ \tau^o_2 \circ \Psi_\omega)(\xi, \eta) = \begin{pmatrix} e^{-\frac{\mathrm{i}}{2}\mu_\omega}\eta \\ e^{\frac{\mathrm{i}}{2}\mu_\omega}\xi \end{pmatrix},$$

$$(\Psi_\omega^{-1} \circ \sigma_o \circ \Psi_\omega)(\xi, \eta) = \begin{pmatrix} e^{\mathrm{i}\mu_\omega}\xi \\ e^{-\mathrm{i}\mu_\omega}\eta \end{pmatrix}, \quad (\xi, \eta) \in \mathcal{C}^R_\omega.$$

In other words, τ^o_1, τ^o_2 and σ_o have $\Psi_\omega(\mathcal{C}^R_\omega)$ as holomorphic invariant set and their restrictions to it are conjugate to the restrictions to $\mathcal{C}^R_\omega$ of linear maps defined above. Moreover, $\mu_\omega \in]\lambda - \frac{\pi}{4}, \lambda + \frac{\pi}{4}[$ depends on ω smoothly in the sense of Whitney, and $\Psi_\omega = \check{\Psi} \circ (\mathrm{Id} + \phi_\omega)$, with $\check{\Psi}$ biholomorphic on the neighborhood $\Delta_2(0, R)$, fixing the origin, and ϕ_ω is smooth with respect to ω and sufficiently small in the sense of Whitney.

Remark 2.2 If the surface M can be holomorphically flattened, that is, if it can be holomorphically mapped into $\mathrm{Im}(z_2) = 0$, then the situation is much simpler. Indeed, in that case, the associated dynamical system has an extra holomorphic first integral [21]. It implies that automatically, in good holomorphic coordinates near the origin, all curves $\{\xi\eta = constant\}$ are left invariant by the original dynamics. One thus needs to prove that for suitable values of these constants (i.e. ω's), one has a conjugacy to linear maps on the associated $\{\xi\eta = constant\}$ as mentioned by Sevryuk [44] in his Mathematical review of Gong's article [21].

[1] Through the paper, for any $\mathcal{S} \subset \mathbb{R}$, $|\mathcal{S}|$ denotes its Lebesgue measure.

Remark 2.3 μ_ω obtained in Theorem 2.1 is such that $\frac{\mu_\omega}{\pi}$ is irrational and $\{|e^{\mathrm{i}n\mu_\omega}-1|\}_{n\in\mathbb{N}^*}$ does not accumulate at the origin too quickly (see (67)), which guarantees that the restriction $\sigma|_{\mathcal{S}_\omega}$ is an irrational rotation on $\mathcal{S}_\omega$. This contrasts with the example given Section 6(b) of [40]. Indeed, the divergence of the formal normalizing transformation $\hat{\Psi}$ in (5)–(7) cannot be avoided because of the periodic orbits of σ. Such periodic orbits, as well as its invariant curves, do not have any immediate geometrical significance since they do not lie on M but only on its complexification $\mathcal{M}\subset\mathbb{C}^4$.

Sketch of proof of Theorem 2.1. For τ_1^o and τ_2^o given as in (2) and (3), our aim is to eliminate the perturbation p^o and q^o (hence $\bar{p}^o$ and $\bar{q}^o$) by a sequence of holomorphic transformations which commute with ρ.

After finitely many steps of normalization in the sense of Poincaré–Dulac in the neighborhood of origin, we obtain a pair of involutions of the form

$$\check{\tau}_1(\xi,\eta)=\begin{pmatrix} e^{\frac{\mathrm{i}}{2}\check{\alpha}(\xi\eta)}\eta+\check{p}(\xi,\eta)\\ e^{-\frac{\mathrm{i}}{2}\check{\alpha}(\xi\eta)}\xi+\check{q}(\xi,\eta)\end{pmatrix},\quad \check{\tau}_2=\rho\circ\check{\tau}_1\circ\rho,$$

with a non-degenerate $\check{\alpha}=\check{\alpha}(z)$ (as in (69)) and higher-order perturbations $\check{p},\check{q}$ (as in (70)). Hence, we can make the norm of the perturbation small enough by choosing a small enough neighborhood of origin $\{|\xi|,|\eta|<r_*\}$. By a possible normalization on the "crown", $\{|\xi\eta-\omega|<\beta\}$, around $\{\xi\eta=\omega\}$ with ω well chosen from a compact positive-measure subset of $]-r_*^2,r_*^2[$, the system enters into a general iteration scheme (a KAM-like process, see Proposition 4.4), under an additional assumption on the perturbation.

By the iteration process, we build a sequence of involutions $\tau_\nu^{(1)}$, $\nu\in\mathbb{N}$, (hence $\tau_\nu^{(2)}=\rho\circ\tau_\nu^{(1)}\circ\rho$ and $\sigma_\nu=\tau_\nu^{(1)}\circ\tau_\nu^{(2)}$) of the form

$$\tau_\nu^{(1)}(\xi,\eta)=\begin{pmatrix} e^{\frac{\mathrm{i}}{2}\alpha_\nu(\xi\eta)}\eta+p_\nu(\xi,\eta)\\ e^{-\frac{\mathrm{i}}{2}\alpha_\nu(\xi\eta)}\xi+q_\nu(\xi,\eta)\end{pmatrix}$$

on crowns around $\{\xi\eta=\omega\}$, that shrink to the connected holomorphic curve $\{\xi\eta=\omega\}$ when ν tends to infinite. On the other hand, when restricted to $\{\xi\eta=\omega\}$, α_ν tends to a real number $\alpha_\infty(\omega)$ and the perturbation (p_ν,q_ν) tends to zero, as ν tends to infinite. In order to control this process, one has to exclude some parameters ω from the previous set and to show that, the set of admissible parameters ω for full process is non-void.

The required supplementary condition mentioned above is that the "crossing term", called *skew term* below, $e^{\frac{\mathrm{i}}{2}\alpha_\nu(\xi\eta)}\eta q_\nu+e^{-\frac{\mathrm{i}}{2}\alpha_\nu(\xi\eta)}\xi p_\nu$ of $\tau_\nu^{(1)}$ is much smaller than p_ν and q_ν (see (52)) on the crown. With this condition, we are able to construct a suitable holomorphic transformation of the form

$$\psi_\nu(\xi,\eta)=\begin{pmatrix}\xi+u_\nu(\xi,\eta)\\ \eta+v_\nu(\xi,\eta)\end{pmatrix},$$

which conjugates $\tau_\nu^{(1)}$ into $\tau_{\nu+1}^{(1)}$ with perturbation of much smaller size on a smaller crown around $\{\xi\eta = \omega\}$. Here ω is chosen from a suitable real parameter set, which is related to the small-divisor conditions and guarantees the convergence for the product of sequence of transformations $\{\psi_\nu\}$ on $\{\xi\eta = \omega\}$.

Indeed, the supplementary condition on the skew term of $\tau_\nu^{(1)}$ implies the skew term $\eta u_\nu + \xi v_\nu$ of transformation ψ_ν is much smaller (similar to (145) in Lemma 7.5). As a consequence, the error term coming from the *non-degeneracy* of the "eigenvalues" $e^{\frac{i}{2}\alpha_\nu(\xi\eta)}$:

$$e^{\pm\frac{i}{2}\alpha_\nu(\xi\eta+\eta u_\nu+\xi v_\nu+u_\nu v_\nu)} - e^{\pm\frac{i}{2}\alpha_\nu(\xi\eta)}$$

is so small that it can be directly put into the new perturbation. We emphasize that this supplementary hypothesis on the skew terms of $\tau_\nu^{(1)}$ has to be assumed only at the initial KAM step (i.e., for $\nu = 0$), as after each $(\nu + 1) -$ th KAM step, the new skew term of $\tau_{\nu+1}^{(1)}$ is automatically much smaller than $p_{\nu+1}$ and $q_{\nu+1}$. This is due to a subtle cancellation of main parts (see (65) and (66) in Theorem 4.7 and its proof). As mentioned above, by an initial preparation of the involutions, we can make them to satisfy this supplementary condition required in the iteration process.

2.2 Geometry of hyperbolic CR singularity

We recall that M is *non-exceptional*, since $\frac{\lambda}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$ in the associated involutions τ_1^o and τ_2^o given in (2) and (3). Hence, (5) and (6) hold. Let us show that Theorem 2.1 enables us to obtain the result on the geometry of real analytic surfaces with a hyperbolic CR singularity.

As mentioned above, the triple $\{\tau_1^o, \tau_2^o, \rho\}$ given in (2), (3) completely characterizes the holomorphic equivalent class of the submanifold M given in (1). Indeed, following Moser–Webster [40], we can reconstruct a submanifold from a pair of involutions. Let us define two holomorphic mappings φ_1, Φ fixing the origin of $\mathbb{C}^2$ as follows:

$$\varphi_1(\xi, \eta) := \xi + \xi \circ \tau_1^o, \quad \varphi_2 := \overline{\varphi_1 \circ \rho}, \quad \rho(\xi, \eta) = (\bar{\xi}, \bar{\eta}). \tag{10}$$

The latter implies that the biholomorphic mapping, fixing the origin of $\mathbb{C}^2$,

$$\varphi(\xi, \eta) = (\varphi_1(\xi, \eta), \varphi_2(\xi, \eta)) =: (z', w') \tag{11}$$

transforms ρ into the standard complex conjugation $(z', w') \to (\overline{w}', \overline{z}')$. Define

$$\Phi(\xi, \eta) := (\xi \circ \tau_1^o(\xi, \eta)) \cdot \xi, \tag{12}$$

where $\xi \circ \tau_1^o(\xi, \eta)$ denotes the ξ-coordinate of $\tau_1^o(\xi, \eta)$. We verify that φ_1 and Φ are invariant by τ_1^o. Then the local analytic submanifold defined by the local equation

$$z_2 = (\Phi \circ \varphi^{-1})(z_1, \bar{z}_1), \tag{13}$$

has $\{\tau_1^o, \tau_2^o, \rho\}$ as associated Moser–Webster involutions. Assume that the pair of involutions τ_1^o and τ_2^o associated to the real surface M satisfies the assumption of Theorem 2.1. Then, for each $\omega \in \mathcal{O}_\infty(R)$, there is a holomorphic map Ψ_ω defined on $\mathcal{C}_\omega^R = \{(\xi, \eta) \in \mathbb{C}^2 : \xi\eta = \omega,\ |\xi|, |\eta| < R\}$. It is a small perturbation of the the identity. Let H_ω^R be the real hyperbola of $\mathcal{C}_\omega^R$, i.e., $H_\omega^R := \{(\xi, \eta) \in \mathbb{R}^2 : \xi\eta = \omega,\ |\xi|, |\eta| < R\}$. Let us consider the connected holomorphic curve defined as the image of $\mathcal{C}_\omega^R$ (which contains a graph over an annulus):

$$\mathcal{S}_\omega : \begin{cases} z_1 = \varphi_1 \circ \Psi_\omega(\xi, \eta) \\ z_2 = \Phi \circ \Psi_\omega(\xi, \eta) \end{cases}, \quad (\xi, \eta) \in \mathcal{C}_\omega^R.$$

as well as $\mathcal{H}_\omega^R$, the image of the restriction, $\tilde{\Psi}_\omega := (\varphi_1, \Phi) \circ \Psi_\omega|_{H_\omega^R}$, to H_ω^R. Hence, $\mathcal{H}_\omega^R \subset \mathcal{S}_\omega$ shrinks to zero with R.

Theorem 2.4 *Under the assumption of Theorem 2.1 and the notation above, the family $\{\mathcal{S}_\omega\}_{\omega \in \mathcal{O}_\infty(R)}$ is a non-constant Whitney smooth family of connected holomorphic curves. Each of them intersects M, in a neighborhood of the origin, along the holomorphic hyperbola $\mathcal{H}_\omega^R$.*

Remark 2.5 Assumptions of the previous theorem, through (8), implies that the real analytic surface M given in (1) is not formally equivalent to $\mathcal{Q}_\gamma$.

Remark 2.6 The conclusion of the previous theorem contrasts with that of the elliptic case treated by Moser–Webster. Indeed, in the holomorphic normalizing coordinates, there is a real analytic family of holomorphic curves $\mathcal{S}_c : z_2 = c$ for c in a real neighborhood of the origin, and for every c, $\mathcal{S}_c$ intersects M along the ellipse $c = |z_1|^2 + (\gamma + \epsilon c^s)(z_1^2 + \bar{z}_1^2)$ (Fig. 1).

Proof According to Theorem 2.1, for any good parameter $\omega \in \mathcal{O}_\infty(R)$, there exists a connected holomorphic curve $\mathcal{S}_\omega$ invariant by the dynamics and ρ, such that $\tau_j^o|_{\mathcal{S}_\omega}$ is conjugated to the restriction to $\mathcal{C}_\omega^R$ of the linear involutions:

$$T_j : (\xi, \eta) \mapsto \left(e^{\frac{i}{2}(-1)^{j-1}\mu_\omega}\eta,\ e^{-\frac{i}{2}(-1)^{j-1}\mu_\omega}\xi\right).$$

Indeed, with the definitions given in (10) and (12), we have, for all $(\xi, \eta) \in \mathcal{C}_\omega^R$,

$$\begin{aligned}(\varphi_1 \circ \Psi_\omega)(\xi, \eta) &= (\xi \circ \Psi_\omega)(\xi, \eta) + \left(\xi \circ \Psi_\omega \circ \left(\Psi_\omega^{-1} \circ \tau_1^o \circ \Psi_\omega\right)\right)(\xi, \eta) \\ &= (\xi \circ \Psi_\omega)(\xi, \eta) + (\xi \circ \Psi_\omega)\left(e^{\frac{i}{2}\mu_\omega}\eta, e^{-\frac{i}{2}\mu_\omega}\xi\right), \\ (\Phi \circ \Psi_\omega)(\xi, \eta) &= (\xi \circ \Psi_\omega)(\xi, \eta) \cdot \left((\xi \circ \Psi_\omega)\left(e^{\frac{i}{2}\mu_\omega}\eta, e^{-\frac{i}{2}\mu_\omega}\xi\right)\right).\end{aligned}$$

We define the connected holomorphic curve $\mathcal{S}_\omega$ as the image of the holomorphic curve $\mathcal{C}_\omega^R$ by the holomorphic map:

$$\mathcal{S}_\omega : \begin{cases} z_1 = (\varphi_1 \circ \Psi_\omega)(\xi, \eta) \\ z_2 = (\Phi \circ \Psi_\omega)(\xi, \eta) \end{cases}, \quad (\xi, \eta) \in \mathcal{C}_\omega^R.$$

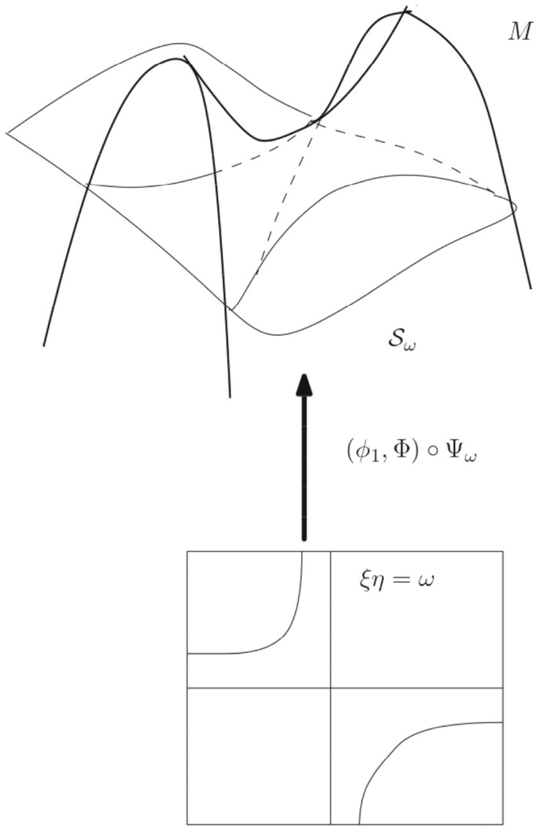

Fig. 1 Holomorphic hyperbola: intersection of M by a holomorphic curve

On the one hand, since Ψ_ω and ρ commutes, we have

$$\varphi \circ \Psi_\omega = \left(\varphi_1 \circ \Psi_\omega, \overline{\varphi_1 \circ \Psi_\omega \circ \rho}\right).$$

On the other hand, $H_\omega^R := \mathcal{C}_\omega^R \cap \mathrm{Fix}(\rho)$ is the union of two branches of real hyperbola $\{\xi\eta = \omega,\ \xi = \bar{\xi},\ \eta = \bar{\eta},\ |\xi|, |\eta| < R\}$. Hence, we have

$$\varphi \circ \Psi_\omega|_{H_\omega^R} = (z_1, \bar{z}_1)|_{H_\omega^R}.$$

As a consequence, the complex curve $\mathcal{S}_\omega$ intersects M given in (13) along the image of H_ω^R by $(\varphi_1, \Phi) \circ \Psi_\omega$,

$$z_2 = (\Phi \circ \Psi_\omega) \circ \left(\Psi_\omega^{-1} \circ \varphi^{-1}\right)(z_1, \bar{z}_1)|_{H_\omega^R} = (\Phi \circ \Psi_\omega) \circ (\varphi \circ \Psi_\omega)^{-1}(z_1, \bar{z}_1)|_{H_\omega^R}.$$

We recall that $\Psi_\omega = \check{\Psi} \circ (\mathrm{Id} + \phi_\omega)$. We can assume that $\check{\Psi} = \mathrm{Id}$ for convenience. Then, with $\Lambda_\omega := e^{\frac{\mathrm{i}}{2}\mu_\omega}$, $\mathcal{S}_\omega$ is defined as

$$\begin{cases} z_1 = \xi + \Lambda_\omega \eta + (\xi \circ \phi_\omega)(\xi, \eta) + (\xi \circ \phi_\omega)(\Lambda_\omega \eta, \Lambda_\omega^{-1}\xi) \\ z_2 = \Lambda_\omega \omega + \xi \cdot ((\xi \circ \phi_\omega)(\Lambda_\omega \eta, \Lambda_\omega^{-1}\xi)) + \Lambda_\omega \eta \cdot ((\xi \circ \phi_\omega)(\xi, \eta)) \\ \quad + ((\xi \circ \phi_\omega)(\xi, \eta)) \cdot ((\xi \circ \phi_\omega)(\Lambda_\omega \eta, \Lambda_\omega^{-1}\xi)) \end{cases},$$

where $(\xi, \eta) \in \mathcal{C}_\omega^R$, that is $\eta = \frac{\omega}{\xi}$, $\frac{|\omega|}{R} < |\xi| < R$. Since ϕ_ω is sufficiently small and smooth with respect to ω in the sense of Whitney, $\mathcal{S}_\omega$ is a small perturbation of the main part

$$\begin{cases} z_1 = \xi + \Lambda_\omega \frac{\omega}{\xi} \\ z_2 = \quad \Lambda_\omega \omega \end{cases},$$

which varies with ω as $\Lambda_\omega \omega$ does. Indeed, assume that there exist $\omega, \omega' \in \mathcal{O}_\infty(R) \subset \mathbb{R}$ with $\omega \neq \omega'$ such that $\Lambda_\omega \omega = \Lambda_{\omega'} \omega'$. Since $\Lambda_\omega = e^{\frac{\mathrm{i}}{2}\mu_\omega}$ and $\Lambda_{\omega'} = e^{\frac{\mathrm{i}}{2}\mu_{\omega'}}$, we have

$$e^{\frac{\mathrm{i}}{2}(\mu_{\omega'} - \mu_\omega)} = \frac{\omega}{\omega'} \in \mathbb{R},$$

which implies that $\mu_{\omega'} - \mu_\omega = (4k+2)\pi$ for some $k \in \mathbb{Z}$. This contradicts with the fact that, $\mu_\omega \in]\lambda - \frac{\pi}{4}, \lambda + \frac{\pi}{4}[$ for every $\omega \in \mathcal{O}_\infty(R)$. Hence, $\mathcal{S}_\omega$ varies with ω.

The Whitney smoothness of $\mathcal{S}_\omega$ follows immediately from that of Ψ_ω and μ_ω. □

The rest of paper will be organized as follows. In Sect. 3, the precise definition of crowns around the curve $\{\xi\eta = constant\} \subset \mathbb{C}^2$ and the norm of holomorphic functions on them are introduced, and basic properties associated with reversible map are given. In Sect. 4, we give an abstract KAM-like theorem, which is used to prove Theorem 2.1 in Sect. 5. A preliminary normalization (which is required to start the KAM-like process, also known as Newton method), as well as the Whitney smoothness of the family of invariant curves, is also given in Sect. 5. In Sect. 6, we describe properties of the pair of holomorphic involutions $\{\tau_1, \tau_2 = \rho \circ \tau_1 \circ \rho\}$ and in particular of their non-degenerate principal parts, as well as of their reversible associated composition $\sigma = \tau_1 \circ \tau_2$. In Sect. 7, two types of holomorphic transformations of $\{\tau_1, \tau_2\}$, commuting with ρ, are introduced, which is used to complete the proof of the KAM-like theorem.

3 Preliminaries and notations

3.1 Basic property of reversible map

Let us define the involution $\rho(\xi, \eta) = (\bar{\xi}, \bar{\eta})$. An invertible map $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ is called *reversible* with respect to ρ if $\sigma^{-1} = \rho \circ \sigma \circ \rho$.

Lemma 3.1 *Given $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ with $\psi = (u, v)$.*
Then $\psi \circ \rho = \rho \circ \psi$ if and only if $u = \bar{u}$, $v = \bar{v}$.

Proof Since $\rho \circ \rho = \mathrm{Id}$, we have $\rho^{-1} \circ \psi \circ \rho = \rho \circ \psi \circ \rho = (\bar{u}, \bar{v})$ which equals to ψ if and only if $u = \bar{u}$, $v = \bar{v}$. □

Lemma 3.2 *Let $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ be a reversible map w.r.t. ρ, and let $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ be an invertible map commuting with ρ, Then $\psi^{-1} \circ \sigma \circ \psi$ is also reversible w.r.t. ρ.*

Proof The conclusion follows from

$$\begin{aligned}(\psi^{-1} \circ \sigma \circ \psi)^{-1} &= \psi^{-1} \circ \sigma^{-1} \circ \psi \\ &= (\rho \circ \psi^{-1} \circ \rho) \circ (\rho \circ \sigma \circ \rho) \circ (\rho \circ \psi \circ \rho) \\ &= \rho \circ (\psi^{-1} \circ \sigma \circ \psi) \circ \rho.\end{aligned}$$

□

3.2 Function space and norms

Given $0 < r < \frac{1}{4}$ and $0 \le \beta < r^2$, for $\omega \in] -r^2 + \beta, r^2 - \beta[$, we define

$$\mathcal{C}^r_\omega := \left\{ (\xi, \eta) \in \mathbb{C}^2 : \xi\eta = \omega,\ |\xi|, |\eta| < r \right\}, \quad \mathcal{C}_\omega := \bigcup_{r>0} \mathcal{C}^r_\omega, \tag{14}$$

$$\mathcal{C}^r_{\omega,\beta} := \left\{ (\xi, \eta) \in \mathbb{C}^2 : |\xi\eta - \omega| \le \beta,\ |\xi|, |\eta| < r \right\}, \quad \mathcal{C}_{\omega,\beta} := \bigcup_{r>0} \mathcal{C}^r_{\omega,\beta}. \tag{15}$$

For a power series

$$f(\xi, \eta) = \sum_{l,j \ge 0} \breve{f}_{l,j} \xi^l \eta^j, \quad \breve{f}_{l,j} \in \mathbb{C},$$

we have the unique decomposition

$$f(\xi, \eta) = f_{0,0}(\xi\eta) + \sum_{l \ge 1} f_{l,0}(\xi\eta)\, \xi^l + \sum_{j \ge 1} f_{0,j}(\xi\eta)\, \eta^j = \sum_{\substack{l,j \ge 0 \\ l \cdot j = 0}} f_{l,j}(\xi\eta) \xi^l \eta^j, \tag{16}$$

with the *coefficients* of f, depending on the product $\xi\eta$, given by

$$f_{l,j}(\xi\eta) = \sum_{k \ge 0} \breve{f}_{k+l,k+j} \cdot (\xi\eta)^k, \quad l \cdot j = 0.$$

Sometimes, by defining $f_{l,j} = 0$ for $lj \ne 0$, we rewrite f as

$$f(\xi, \eta) = \sum_{l,j \ge 0} f_{l,j}(\xi\eta)\, \xi^l \eta^j.$$

Let us consider the anti-holomorphic involution $\rho : (\xi, \eta) \mapsto (\bar{\xi}, \bar{\eta})$. We define the conjugate of f to be $\bar{f}$, whose Taylor expansion coefficients at the origin are the complex conjugates of those of f. Obviously, $f = \bar{f}$ if and only if $\breve{f}_{l,j} \in \mathbb{R}$ for all $l, j \geq 0$.

Let $h = h(\xi\eta)$ be a function of the product $\xi\eta$. We define

$$|h|_{\omega,\beta} := \sup_{(\xi,\eta)\in\mathcal{C}_{\omega,\beta}} |h(\xi\eta)| = \sup_{|z-\omega|<\beta} |h(z)|, \quad |h|_{\omega} := \sup_{(\xi,\eta)\in\mathcal{C}_{\omega}} |h(\xi\eta)| = \sup_{z=\omega} |h(z)|.$$

Given a power series f, we define the norms

$$|f|_r := \sum_{l,j\geq 0} |\breve{f}_{l,j}| r^{l+j}, \quad \|f\|_{\omega,\beta,r} := \sum_{\substack{l,j\geq 0\\ lj=0}} |f_{l,j}|_{\omega,\beta}\, r^{l+j}. \tag{17}$$

In particular, for a function f of $\xi\eta$, we have $\|f\|_{\omega,\beta,r} = |f|_{\omega,\beta}$. For $\omega \in]-r^2+\beta, r^2-\beta[$, it is easy to see that

$$\sup_{(\xi,\eta)\in\mathcal{C}^r_{\omega,\beta}} |f(\xi,\eta)| \leq \|f\|_{\omega,\beta,r} \leq |f|_r. \tag{18}$$

The definition of norm $\|\cdot\|_{\omega,\beta,r}$ implies that

$$\begin{aligned} \|f(\xi,\eta)\|_{\omega,\beta,r} = \|f(\eta,\xi)\|_{\omega,\beta,r} = \|\bar{f}(\xi,\eta)\|_{\omega,\beta,r}, \\ |f_{l,j}|_{\omega,\beta} \leq \|f\|_{\omega,\beta,r} r^{-(l+j)}, \quad l, j \geq 0. \end{aligned} \tag{19}$$

For $\mathcal{O} \subset]-r^2+\beta, r^2-\beta[$, we also define the norm

$$\|f\|_{\mathcal{O},\beta,r} := \sup_{\substack{\omega\in\mathcal{O}\\ |\omega|<r^2-\beta}} \|f\|_{\omega,\beta,r}.$$

In particular, for the coefficients of f,

$$\|f_{l,j}\|_{\mathcal{O},\beta,r} = \sup_{\substack{\omega\in\mathcal{O}\\ |\omega|<r^2-\beta}} |f_{l,j}|_{\omega,\beta}, \quad l, j \geq 0, \; lj = 0.$$

Lemma 3.3 *For given power series f and g, if $|\omega| < r^2 - \beta$, then $\|fg\|_{\omega,\beta,r} \leq \|f\|_{\omega,\beta,r}\|g\|_{\omega,\beta,r}$.*

Proof Decompose fg as in (16), we have

$$(fg)(\xi,\eta) = (fg)_{0,0} + \sum_{l\geq 1}(fg)_{l,0}\xi^l + \sum_{j\geq 1}(fg)_{0,j}\eta^j$$

with the coefficients given by

$$(fg)_{0,0} = f_{0,0}g_{0,0} + \sum_{k\geq 1}(f_{k,0}g_{0,k} + f_{0,k}g_{k,0})(\xi\eta)^k, \tag{20}$$

$$(fg)_{l,0} = \sum_{k=0}^{l-1}(f_{l-k,0}g_{k,0} + f_{k,0}g_{l-k,0}) + \sum_{k\geq 1}(f_{l+k,0}g_{0,k} + f_{0,k}g_{l+k,0})(\xi\eta)^k, \quad (21)$$

and similar expression for $(fg)_{0,j}$. If $|\omega| < r^2 - \beta$ and $(\xi,\eta) \in \mathcal{C}_{\omega,\beta}$, then $|\xi\eta| < r^2$. Therefore, we have

$$|(fg)_{0,0}|_{\omega,\beta} \leq |f_{0,0}|_{\omega,\beta}|g_{0,0}|_{\omega,\beta} + \sum_{k\geq 1}\left(|f_{k,0}|_{\omega,\beta}|g_{0,k}|_{\omega,\beta} + |f_{0,k}|_{\omega,\beta}|g_{k,0}|_{\omega,\beta}\right) r^{2k},$$

$$\begin{aligned} |(fg)_{l,0}|_{\omega,\beta}r^l \leq & \sum_{k=0}^{l-1}\left(|f_{l-k,0}|_{\omega,\beta}|g_{k,0}|_{\omega,\beta} + |f_{k,0}|_{\omega,\beta}|g_{l-k,0}|_{\omega,\beta}\right) r^l \\ & + \sum_{k\geq 1}\left(|f_{l+k,0}|_{\omega,\beta}|g_{0,k}|_{\omega,\beta} + |f_{0,k}|_{\omega,\beta}|g_{l+k,0}|_{\omega,\beta}\right) r^{2k+l}, \end{aligned}$$

and a similar estimate for $|(fg)_{0,j}|_{\omega,\beta}r^j$. Since $\|fg\|_{\omega,\beta,r} = \sum_{l,j\geq 0}|(fg)_{l,j}|_{\omega,\beta}r^{l+j}$ with $(fg)_{l,j} = 0$ whenever $lj \neq 0$, we obtain

$$\|fg\|_{\omega,\beta,r} \leq \left(\sum_{l,j\geq 0}|f_{l,j}|_{\omega,\beta}r^{l+j}\right)\left(\sum_{l,j\geq 0}|g_{l,j}|_{\omega,\beta}r^{l+j}\right) = \|f\|_{\omega,\beta,r}\|g\|_{\omega,\beta,r}. \square$$

Let $0 \leq \beta < r^2$ and $\mathcal{O} \subset]-r^2, r^2[$. We define $\mathcal{H}_{\beta,r}(\mathcal{U})$ to be the set of holomorphic functions in a complex neighborhood $\mathcal{U}$ of

$$\mathcal{O}(r,\beta) := \mathcal{O} \cap]-r^2+\beta, r^2-\beta[, \quad (22)$$

and the collections of power series

$$\mathcal{A}_{\beta,r}(\mathcal{O}) := \left\{ f = \sum_{l,j} f_{l,j}(\xi\eta)\xi^l\eta^j : \begin{array}{l} f \text{ is holomorphic on } \bigcup_{\omega\in\mathcal{O}(r,\beta)}\mathcal{C}^r_{\omega,\beta}, \\ f_{l,j} \in \mathcal{H}_{\beta,r}(\mathcal{U}),\ \mathcal{U} \text{ complex neighborhood of } \mathcal{O}(r,\beta) \end{array} \right\},$$
$$\mathcal{A}^{\mathbb{R}}_{\beta,r}(\mathcal{O}) := \left\{ f \in \mathcal{A}_{\beta,r}(\mathcal{O}) : f = \bar{f} \right\}.$$

In the above definition, $\mathcal{U}$ denotes an unprecised neighborhood over which all $f_{l,j}$'s are holomorphic. Finally for any $\tilde{\mathcal{O}} \subset \mathbb{R}$ and any function h defined on $\tilde{\mathcal{O}}$, we define the norm $|h|_{\tilde{\mathcal{O}}} := \sup_{\omega\in\tilde{\mathcal{O}}}|h(\omega)|$. By (19), we see that

$$|f_{l,j}|_{\mathcal{O}(r,\beta)} = \|f_{l,j}\|_{\mathcal{O},\beta,r} \leq \|f\|_{\mathcal{O},\beta,r}r^{-(l+j)}. \quad (23)$$

It is easy to see that:

- *(linear structure)* for $f, g \in \mathcal{A}_{\beta,r}(\mathcal{O})$ (or $\mathcal{A}^{\mathbb{R}}_{\beta,r}(\mathcal{O})$), we have $a_1 f + a_2 g \in \mathcal{A}_{\beta,r}(\mathcal{O})$ (or $\mathcal{A}^{\mathbb{R}}_{\beta,r}(\mathcal{O})$), $a_1, a_2 \in \mathbb{C}$(or $\mathbb{R}$), with

$$\|a_1 f + a_2 g\|_{\mathcal{O},\beta,r} \leq |a_1|\|f\|_{\mathcal{O},\beta,r} + |a_2|\|g\|_{\mathcal{O},\beta,r}. \quad (24)$$

- *(monotonicity)* If $\mathcal{O}' \subset \mathcal{O}$ and $r' \le r$, $\beta' \le \beta$, $r'^2 - \beta' \le r^2 - \beta$, then $\mathcal{A}_{\beta,r}(\mathcal{O}) \subset \mathcal{A}_{\beta',r'}(\mathcal{O}')$ and $\mathcal{A}^{\mathbb{R}}_{\beta,r}(\mathcal{O}) \subset \mathcal{A}^{\mathbb{R}}_{\beta',r'}(\mathcal{O}')$ with

$$\|f\|_{\mathcal{O}',\beta',r'} \le \|f\|_{\mathcal{O},\beta,r}, \quad \forall\, f \in \mathcal{A}_{\beta,r}(\mathcal{O}) \text{ or } \mathcal{A}^{\mathbb{R}}_{\beta,r}(\mathcal{O}). \tag{25}$$

□

Lemma 3.4 *For f, g with $\|f\|_{\mathcal{O},\beta,r}$, $\|g\|_{\mathcal{O},\beta,r} < \infty$, we have that*

$$\|fg\|_{\mathcal{O},\beta,r} \le \|f\|_{\mathcal{O},\beta,r}\|g\|_{\mathcal{O},\beta,r}. \tag{26}$$

Moreover, if $f, g \in \mathcal{A}_{\beta,r}(\mathcal{O})$, then $fg \in \mathcal{A}_{\beta,r}(\mathcal{O})$.

Proof In view of Lemma 3.3, we obtain (26). Provided that $f, g \in \mathcal{A}_{\beta,r}(\mathcal{O})$, let us prove the analyticity of coefficients $(fg)_{l,j}(\cdot)$, $l, j \ge 0$ with $lj = 0$, on a same neighborhood of $\mathcal{O}(r,\beta)$. By (21), we have, for $l \ge 1$,

$$\begin{aligned}(fg)_{l,0}(\omega) = &\sum_{k=0}^{l-1}\big(f_{l-k,0}(\omega)g_{k,0}(\omega) + f_{k,0}(\omega)g_{l-k,0}(\omega)\big)\\ &+ \sum_{k\ge 1}\big(f_{l+k,0}(\omega)g_{0,k}(\omega) + f_{0,k}(\omega)g_{l+k,0}(\omega)\big)\,\omega^k.\end{aligned}$$

Let $\mathcal{U}$ be a complex neighborhood of $\mathcal{O}(r,\beta)$ over which the $f_{k,j}$, $g_{k,j}$'s are holomorphic. Since $|\omega| < r^2 - \beta$ and, in view of (23), we have

$$|f_{l+k,0}|_{\mathcal{O}(r,\beta)}|g_{0,k}|_{\mathcal{O}(r,\beta)} + |f_{0,k}|_{\mathcal{O}(r,\beta)}|g_{l+k,0}|_{\mathcal{O}(r,\beta)} \le \frac{2\|f\|_{\mathcal{O},\beta,r}\|g\|_{\mathcal{O},\beta,r}}{r^{l+2k}}.$$

The analyticity of the $(fg)_{l,0}$'s on a same neighborhood of $\mathcal{O}(r,\beta)$ follows. The proof for $(fg)_{0,j}$, $j \ge 0$, is similar by the proof of Lemma 3.3. □

Lemma 3.5 *Given $f \in \mathcal{A}_{\beta,r}(\mathcal{O})$ with $\|f\|_{\mathcal{O},\beta,r} < \infty$ and $a \in \mathbb{C}$, we have $e^{af} \in \mathcal{A}_{\beta,r}(\mathcal{O})$.*

Proof By Lemma 3.4, we see that $f^k \in \mathcal{A}_{\beta,r}(\mathcal{O})$ with $\|f^k\|_{\mathcal{O},\beta,r} \le \|f\|^k_{\mathcal{O},\beta,r}$ for every $k \in \mathbb{N}$. Then, according to (23), we have that, for $l, j \ge 0$ with $lj = 0$,

$$|(a^k f^k)_{l,j}|_{\mathcal{O}(r,\beta)} \le \frac{a^k\|f^k\|_{\mathcal{O},\beta,r}}{r^{l+j}} \le \frac{a^k}{r^{l+j}}\|f\|^k_{\mathcal{O},\beta,r}.$$

Developing the exponential function around 0, we have, for $\omega \in \mathcal{O}(r,\beta)$,

$$(e^{af})_{l,j}(\omega) = 1 + \sum_{k\ge 1}\frac{a^k(f^k)_{l,j}(\omega)}{k!}.$$

Hence we obtain the analyticity of $(e^{af})_{l,j}$, since

$$\left|\sum_{k\geq 1}\frac{(a^k f^k)_{l,j}(\omega)}{k!}\right| \leq \frac{1}{r^{l+j}}\sum_{k\geq 1}\frac{a^k\|f\|^k_{\mathcal{O},\beta,r}}{k!}.$$

□

Lemma 3.6 *Given* $0 < r'' < r' < \frac{1}{4}$, $\mathcal{O} \subset]-r'^2, r'^2[$ *and* $0 < 2\beta'' \leq \beta'$, *if* β' *is sufficiently small such that*

$$e^{\frac{9}{8}\beta'}\frac{r''}{r'} < 1 - \frac{\beta'^2}{16}, \quad 8\beta'^{\frac{1}{2}} < (r' - r'')r'', \tag{27}$$

then for $h \in \mathcal{A}_{\beta',r'}(\mathcal{O})$ *with* $\|h\|_{\mathcal{O},\beta',r'} < +\infty$, *for* f_1, f_2, g_1, g_2 *satisfying*

$$\|f_m\|_{\mathcal{O},\beta'',r''}, \|g_m\|_{\mathcal{O},\beta'',r''} < \frac{\beta'^2}{16}, \quad m = 1, 2,$$

we have that

$$\begin{aligned}&\|h(\xi + f_1, \eta + g_1) - h(\xi + f_2, \eta + g_2)\|_{\mathcal{O},\beta'',r''}\\ &\quad < \frac{3r'\|h\|_{\mathcal{O},\beta',r'}}{(r' - r'')\beta'}\max\left\{\|f_1 - f_2\|_{\mathcal{O},\beta'',r''}, \|g_1 - g_2\|_{\mathcal{O},\beta'',r''}\right\}.\end{aligned}$$

Moreover, if $f_1, f_2, g_1, g_2 \in \mathcal{A}_{\beta'',r''}(\mathcal{O})$, *then*

$$h(\xi + f_1, \eta + g_1) - h(\xi + f_2, \eta + g_2) \in \mathcal{A}_{\beta'',r''}(\mathcal{O}).$$

Remark 3.7 Note that the second inequality in (27) implies that $r''^2 - \beta'' < r'^2 - \beta'$. Hence, by the monotonicity, $\mathcal{A}_{\beta',r'}(\mathcal{O}) \subset \mathcal{A}_{\beta'',r''}(\mathcal{O})$.

A more general version of Lemma 3.6 will be given in Sect. 6 (see Lemma 6.11), and will be shown in Appendix A.

Given $(f, g) \in (\mathcal{A}_{\beta,r}(\mathcal{O}))^2$, we define, for $\omega \in \mathcal{O}(r, \beta)$,

$$\|(f, g)\|_{\omega,\beta,r} := \|f\|_{\omega,\beta,r} + \|g\|_{\omega,\beta,r}, \quad \|(f, g)\|_{\mathcal{O},\beta,r} := \|f\|_{\mathcal{O},\beta,r} + \|g\|_{\mathcal{O},\beta,r}.$$

Lemma 3.8 *Given* $0 < r'' < r' < \frac{1}{4}$, $\mathcal{O} \subset]-r'^2, r'^2[$ *and* $\beta' > 0$, *consider the transformation* $\phi = \mathrm{Id} + \mathcal{U}$ *on* $\mathcal{C}^{r'}_{\omega,\beta'}$ *with* $\mathcal{U} \in (\mathcal{A}_{\beta',r'}(\mathcal{O}))^2$. *If* β' *is sufficiently small such that* (27) *is satisfied, and*

$$\|\mathcal{U}\|_{\mathcal{O},\beta',r'} < \frac{\beta'(r' - r'')}{30r'}, \tag{28}$$

then ϕ *is invertible on* $\mathcal{C}^{r'}_{\omega,\beta'}$, *with* $\phi^{-1} - \mathrm{Id} \in (\mathcal{A}_{\frac{\beta'}{2},r''}(\mathcal{O}))^2$ *and*

$$\|\phi^{-1} - \mathrm{Id} + \mathcal{U}\|_{\mathcal{O},\frac{\beta'}{2},r''} \leq \frac{8r'\|\mathcal{U}\|^2_{\mathcal{O},\beta',r'}}{(r'-r'')\beta'}.$$

Proof In view of (28), we see that ϕ is close to identity, hence it is biholomorphic on $\mathcal{C}^{r'}_{\omega,\beta'}$. Let us write $\phi^{-1} =: \mathrm{Id} + \mathcal{V}$. The identity $\phi \circ \phi^{-1} = \mathrm{Id}$ means that

$$\mathcal{V} = -\mathcal{U} \circ (\mathrm{Id} + \mathcal{V}) = -\mathcal{U} - (\mathcal{U} \circ (\mathrm{Id} + \mathcal{V}) - \mathcal{U}). \tag{29}$$

By Lemma 3.6 (since (27) is satisfied) and (29), we have

$$\begin{aligned} \|\mathcal{V}\|_{\mathcal{O},\frac{\beta'}{2},r''} &< \|\mathcal{U}\|_{\mathcal{O},\beta',r'} + \frac{6r'}{(r'-r'')\cdot\beta'}\|\mathcal{V}\|_{\mathcal{O},\frac{\beta'}{2},r''}\|\mathcal{U}\|_{\mathcal{O},\beta',r'} \\ &< \|\mathcal{U}\|_{\mathcal{O},\beta',r'} + \frac{\|\mathcal{V}\|_{\mathcal{O},\frac{\beta'}{2},r''}}{5}, \end{aligned}$$

which implies that $\|\mathcal{V}\|_{\mathcal{O},\frac{\beta'}{2},r''} < \frac{5}{4}\|\mathcal{U}\|_{\mathcal{O},\beta',r'}$. Let us set $\mathcal{U}_1 = -\mathcal{U}$, $\mathcal{V}_1 = \mathcal{U} \circ (\mathrm{Id} + \mathcal{V}) - \mathcal{U}$. Hence, $\mathcal{V} = \mathcal{U}_1 - \mathcal{V}_1$, and, by Lemma 3.6, $\mathcal{U}_1 \in \mathcal{A}_{\frac{\beta'}{2},r''}(\mathcal{O})$,

$$\|\mathcal{V}_1\|_{\mathcal{O},\frac{\beta'}{2},r''} < \frac{6r'}{(r'-r'')\cdot\beta'}\|\mathcal{V}\|_{\mathcal{O},\frac{\beta'}{2},r''}\|\mathcal{U}\|_{\mathcal{O},\beta',r'} < \frac{\|\mathcal{V}\|_{\mathcal{O},\frac{\beta'}{2},r''}}{5} < \frac{\|\mathcal{U}\|_{\mathcal{O},\beta',r'}}{4}.$$

Then, we have

$$\begin{aligned} \mathcal{V} &= -\mathcal{U} \circ (\mathrm{Id} + \mathcal{U}_1 - \mathcal{V}_1) \\ &= -\mathcal{U} \circ (\mathrm{Id} + \mathcal{U}_1) - (\mathcal{U} \circ (\mathrm{Id} + \mathcal{U}_1 - \mathcal{V}_1) - \mathcal{U} \circ (\mathrm{Id} + \mathcal{U}_1)) =: \mathcal{U}_2 - \mathcal{V}_2, \end{aligned}$$

and, by Lemma 3.6, $\mathcal{U}_2 \in \mathcal{A}_{\frac{\beta'}{2},r''}(\mathcal{O})$,

$$\begin{aligned} \|\mathcal{V}_2\|_{\mathcal{O},\frac{\beta'}{2},r''} &< \frac{6r'}{(r'-r'')\cdot\beta'}\|\mathcal{V}_1\|_{\mathcal{O},\frac{\beta'}{2},r''}\|\mathcal{U}\|_{\mathcal{O},\beta',r'} \\ &< \frac{\|\mathcal{V}_1\|_{\mathcal{O},\frac{\beta'}{2},r''}}{4} < \frac{\|\mathcal{U}\|_{\mathcal{O},\beta',r'}}{4^2}. \end{aligned}$$

Assume that, for some $n \in \mathbb{N}$, we have $\mathcal{V} = \mathcal{U}_n - \mathcal{V}_n$ with

$$\mathcal{U}_n \in \mathcal{A}_{\frac{\beta}{2},r''}(\mathcal{O}), \quad \|\mathcal{V}_n\|_{\mathcal{O},\frac{\beta'}{2},r''} < 4^{-n}\|\mathcal{U}\|_{\mathcal{O},\beta',r'}.$$

Then we have $\mathcal{V} = \mathcal{U}_{n+1} - \mathcal{V}_{n+1}$ with

$$\mathcal{U}_{n+1} := -\mathcal{U} \circ (\mathrm{Id} + \mathcal{U}_n), \quad \mathcal{V}_{n+1} := \mathcal{U} \circ (\mathrm{Id} + \mathcal{U}_n - \mathcal{V}_n) - \mathcal{U} \circ (\mathrm{Id} + \mathcal{U}_n).$$

By Lemma 3.6, we have $\mathcal{U}_{n+1} \in \mathcal{A}_{\frac{\beta}{2},r''}(\mathcal{O})$ and

$$\begin{aligned}\|\mathcal{V}_{n+1}\|_{\mathcal{O},\frac{\beta'}{2},r''} &< \frac{6r'}{(r'-r'')\cdot\beta'}\|\mathcal{V}_n\|_{\mathcal{O},\frac{\beta'}{2},r''}\|\mathcal{U}\|_{\mathcal{O},\beta',r'} \\ &< \frac{\|\mathcal{V}_n\|_{\mathcal{O},\frac{\beta'}{2},r''}}{4} < \frac{\|\mathcal{U}\|_{\mathcal{O},\beta',r'}}{4^{n+1}}.\end{aligned}$$

As $n \to \infty$, we see that $\mathcal{V} \in \mathcal{A}_{\frac{\beta}{2},r''}(\mathcal{O})$.

Let $\tilde{\mathcal{U}} := \phi^{-1} - \mathrm{Id} + \mathcal{U}$. By (29), we have $\tilde{\mathcal{U}} = \mathcal{U} - \mathcal{U}\circ(\mathrm{Id}+\mathcal{V})$. Hence, by Lemma 3.6,

$$\begin{aligned}\|\tilde{\mathcal{U}}\|_{\mathcal{O},\frac{\beta'}{2},r''} = \|\mathcal{U}-\mathcal{U}\circ(\mathrm{Id}+\mathcal{V})\|_{\mathcal{O},\frac{\beta'}{2},r''} &< \frac{6r'\|\mathcal{U}\|_{\mathcal{O},\beta',r'}}{(r'-r'')\cdot\beta'}\|\mathcal{V}\|_{\mathcal{O},\frac{\beta'}{2},r''} \\ &< \frac{8r'\|\mathcal{U}\|^2_{\mathcal{O},\beta',r'}}{(r'-r'')\cdot\beta'}.\end{aligned}$$

□

4 An abstract KAM-like theorem

In this section, we give an abstract KAM-like theorem for pairs of holomorphic involutions near a fixed point, which are pairwise conjugate by an anti-holomorphic involution. From this, we obtain the existence of a lot of analytic invariant sets in a neighborhood of the fixed point. This is the core of the proof of Theorem 2.1.

4.1 Sequences of quantities

With fixed $s \in \mathbb{N}^*$, $0 < r_0 < \frac{1}{4}$, $0 < \varepsilon_0 < r_0^2$, $\zeta_0 := \varepsilon_0^{\frac{1}{3}}$, define the sequences, with $\nu \in \mathbb{N}$, $\{\varepsilon_\nu\}$, $\{\beta_\nu\}$, $\{\tilde{\beta}_\nu\}$, $\{\zeta_\nu\}$, $\{r_\nu\}$ and $\{K_\nu\}$ by:

$$\begin{aligned}&\varepsilon_{\nu+1} := \varepsilon_\nu^{\frac{5}{4}}, \qquad \beta_\nu := \varepsilon_\nu^{\frac{1}{40s}}, \qquad \tilde{\beta}_\nu := 16\beta_{\nu+1} = 16\varepsilon_\nu^{\frac{1}{32s}}, \\ &\zeta_{\nu+1} := \zeta_\nu + \varepsilon_\nu^{\frac{1}{3}},\ r_{\nu+1} := r_\nu - \frac{r_0}{2^{\nu+2}},\ K_\nu := \frac{|\ln\varepsilon_\nu|}{\left|\ln\left(\frac{7r_\nu+r_{\nu+1}}{8r_\nu}\right)\right|}.\end{aligned} \tag{30}$$

Between r_ν and $r_{\nu+1}$, we define

$$r_\nu^{(m)} := r_{\nu+1} + \frac{m}{8}(r_\nu - r_{\nu+1}), \quad m = 0, 1, \ldots, 8, \qquad \tilde{r}_\nu := r_\nu^{(4)} = \frac{r_\nu + r_{\nu+1}}{2}.$$

We assume that ε_0 is small enough such that

$$\left(\frac{|\ln \varepsilon_0|}{\left|\ln\left(\frac{7}{8}+\frac{r_1}{8r_0}\right)\right|}+2\right)\frac{(16s+1)^{16s}\varepsilon_0^{\frac{1}{2400s^2}}}{(r_0-r_1)r_1}<1. \tag{31}$$

Lemma 4.1 *Under the assumption* (31), *we have*

$$\left(\frac{|\ln \varepsilon_\nu|}{\left|\ln\left(\frac{7}{8}+\frac{r_{\nu+1}}{8r_\nu}\right)\right|}+2\right)\frac{(16s+1)^{16s}\varepsilon_\nu^{\frac{1}{2400s^2}}}{(r_\nu-r_{\nu+1})r_{\nu+1}}<1, \quad \nu\in\mathbb{N}. \tag{32}$$

Proof The above inequality holds for $\nu=0$ under the assumption (31). Now, assume that, for some $\nu_*\in\mathbb{N}$, we have

$$\left(\frac{\left|\ln \varepsilon_{\nu_*}\right|}{\left|\ln\left(\frac{7}{8}+\frac{r_{\nu_*+1}}{8r_{\nu_*}}\right)\right|}+2\right)\frac{(16s+1)^{16s}\varepsilon_{\nu_*}^{\frac{1}{2400s^2}}}{(r_{\nu_*}-r_{\nu_*+1})r_{\nu_*+1}}<1. \tag{33}$$

The definition of sequence $\{\varepsilon_\nu\}$ implies that, for $\nu\in\mathbb{N}$,

$$|\ln \varepsilon_{\nu+1}|=\frac{5}{4}\,|\ln\varepsilon_\nu|\,, \quad \varepsilon_{\nu+1}^{\frac{1}{2400s^2}}=\varepsilon_\nu^{\frac{5}{4}\cdot\frac{1}{2400s^2}}, \tag{34}$$

and the definition of $\{r_\nu\}$ implies that, for $\nu\in\mathbb{N}$,

$$r_{\nu+1}=r_0\left(1-\sum_{j=0}^{\nu}\frac{1}{2^{j+2}}\right), \quad r_{\nu+1}-r_{\nu+2}=\frac{r_\nu-r_{\nu+1}}{2}. \tag{35}$$

Hence, for $\nu\in\mathbb{N}$, we have

$$\frac{r_\nu}{r_{\nu+1}}\le\frac{4}{3}, \quad 1-\frac{r_{\nu+1}}{r_\nu}=\frac{1}{2^{\nu+1}+2}. \tag{36}$$

Indeed, it is true for $\nu=0$, and for $\nu\in\mathbb{N}^*$,

$$\frac{r_\nu}{r_{\nu+1}}=\frac{1-\sum_{j=0}^{\nu-1}\frac{1}{2^{j+2}}}{1-\sum_{j=0}^{\nu}\frac{1}{2^{j+2}}}=1+\frac{1}{2^{\nu+1}+1}\le\frac{4}{3}, \quad 1-\frac{r_{\nu+1}}{r_\nu}=\frac{1}{2^{\nu+1}+2}.$$

Then, we obtain

$$\left|\frac{\ln\left(1-\frac{1}{8}(1-\frac{r_{\nu+1}}{r_\nu})\right)}{\ln\left(1-\frac{1}{8}(1-\frac{r_{\nu+2}}{r_{\nu+1}})\right)}\right|\le\frac{\frac{3}{2}\cdot\frac{1}{8}(1-\frac{r_{\nu+1}}{r_\nu})}{\frac{3}{4}\cdot\frac{1}{8}(1-\frac{r_{\nu+2}}{r_{\nu+1}})}=4-\frac{2}{2^\nu+1}\le 4. \tag{37}$$

By (34)–(37), combining with the assumption (33), we have

$$\left(\frac{\left|\ln \varepsilon_{\nu_*+1}\right|}{\left|\ln\left(\frac{7}{8}+\frac{r_{\nu_*+2}}{8r_{\nu_*+1}}\right)\right|}+2\right)\frac{(16s+1)^{16s}\varepsilon_{\nu_*+1}^{\frac{1}{2400s^2}}}{(r_{\nu_*+1}-r_{\nu_*+2})r_{\nu_*+2}}$$

$$=\left(\frac{5}{4}\frac{\left|\ln \varepsilon_{\nu_*}\right|}{\left|\ln\left(\frac{7}{8}+\frac{r_{\nu_*+1}}{8r_{\nu_*}}\right)\right|}\cdot\left|\frac{\ln\left(1-\frac{1}{8}(1-\frac{r_{\nu_*+1}}{r_{\nu_*}})\right)}{\ln\left(1-\frac{1}{8}(1-\frac{r_{\nu_*+2}}{r_{\nu_*+1}})\right)}\right|+2\right)$$

$$\cdot\frac{2r_{\nu_*+1}}{r_{\nu_*+2}}\frac{(16s+1)^{16s}\varepsilon_{\nu_*}^{\frac{5}{4}\cdot\frac{1}{2400s^2}}}{(r_{\nu_*}-r_{\nu_*+1})r_{\nu_*+1}}$$

$$\leq\frac{5}{4}\cdot 4\cdot 2\cdot\frac{4}{3}\cdot\varepsilon_{\nu_*}^{\frac{1}{4}\cdot\frac{1}{2400s^2}}\left(\frac{\left|\ln \varepsilon_{\nu_*}\right|}{\left|\ln\left(\frac{7}{8}+\frac{r_{\nu_*+1}}{8r_{\nu_*}}\right)\right|}+2\right)\frac{(16s+1)^{16s}\varepsilon_{\nu_*}^{\frac{1}{2400s^2}}}{(r_{\nu_*}-r_{\nu_*+1})r_{\nu_*+1}}$$

$$<\frac{40}{3}\varepsilon_{\nu_*}^{\frac{1}{4}\cdot\frac{1}{2400s^2}}<1.$$

The last inequality follows from (33) since $\varepsilon_{\nu_*}^{\frac{1}{4}\cdot\frac{1}{2400s^2}}<\varepsilon_0^{\frac{1}{4}\cdot\frac{1}{2400s^2}}<(16s+1)^{-4s}<\frac{3}{40}$. □

4.2 Iteration argument

Let the sequences of quantities be given as in (30). For $\omega\in\mathcal{O}_0\subset]-r_0^2,r_0^2[$ with $|\mathcal{O}_0|>r_0^2$, we consider the pair of germs of holomorphic involutions $\tau_0^{(1)},\tau_0^{(2)}:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$, i.e., they satisfy $\tau_0^{(k)}\circ\tau_0^{(k)}=\mathrm{Id}$, $k=1,2$. Recalling the notation in (22), we set $\mathcal{O}_0(r_0,\beta_0):=\mathcal{O}_0\cap]-r_0^2+\beta_0,r_0^2-\beta_0[$. We assume that they are of the form

$$\tau_0^{(1)}(\xi,\eta)=\begin{pmatrix}e^{\frac{\mathrm{i}}{2}\alpha_0(\xi\eta)}\eta+p_0(\xi,\eta)\\ e^{-\frac{\mathrm{i}}{2}\alpha_0(\xi\eta)}\xi+q_0(\xi,\eta)\end{pmatrix},\tag{38}$$

$$\tau_0^{(2)}(\xi,\eta)=\left(\rho\circ\tau_0^{(1)}\circ\rho\right)(\xi,\eta)=\begin{pmatrix}e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta+\bar{p}_0(\xi,\eta)\\ e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi+\bar{q}_0(\xi,\eta)\end{pmatrix},\tag{39}$$

where, for the fixed $s\in\mathbb{N}^*$,

- *(the principal part)* $\alpha_0=\alpha_0(\xi\eta)\in\mathcal{A}^{\mathbb{R}}_{\beta_0,r_0}(\mathcal{O}_0)$ with

$$\alpha_0(\omega)\in\left]-\frac{1}{8},4\pi+\frac{1}{8}\right[,\quad\omega\in\mathcal{O}_0(r_0,\beta_0),\tag{40}$$

$$\|\alpha_0(\xi\eta)\|_{\mathcal{O}_0,\beta_0,r_0}<4\pi+\frac{1}{4},\tag{41}$$

$$\left|\alpha_0^{(s)} - s!\right|_{\mathcal{O}_0(r_0,\beta_0)} < \frac{s!}{16}, \tag{42}$$

$$\left|\alpha_0^{(k)}\right|_{\mathcal{O}_0(r_0,\beta_0)} < \frac{1}{16}, \qquad 1 \le k \le s-1, \text{ if } s \ge 2, \tag{43}$$

$$\left|\alpha_0^{(k)}\right|_{\mathcal{O}_0(r_0,\beta_0)} < \frac{r_0^{-1}}{4}, \qquad s+1 \le k \le 16s, \tag{44}$$

- *(the perturbation)* $p_0 = p_0(\xi,\eta)$, $q_0 = q_0(\xi,\eta) \in \mathcal{A}_{\beta_0,r_0}(\mathcal{O}_0)$ with

$$\|p_0\|_{\mathcal{O}_0,\beta_0,r_0},\ \|q_0\|_{\mathcal{O}_0,\beta_0,r_0} \le \frac{\varepsilon_0}{10}, \quad \|e^{\frac{i}{2}\alpha_0(\xi\eta)}\eta q_0 + e^{-\frac{i}{2}\alpha_0(\xi\eta)}\xi p_0\|_{\mathcal{O}_0,\beta_0,r_0} < \frac{\varepsilon_0^{\frac{3}{2}}}{3}. \tag{45}$$

Remark 4.2 For instance, $\alpha_0(z) = \lambda + z^s + \sum_{j=s+1}^{m} c_j z^j$, with $\lambda \in [0, 4\pi[$, arbitrary $m \ge s+1$, $c_j \in \mathbb{R}$, and r_0 sufficiently small, is an example of such a function satisfying (40)–(44).

Remark 4.3 In (45), besides the smallness of perturbation (p_0, q_0), the smallness of $e^{\frac{i}{2}\alpha_0(\xi\eta)}\eta q_0 + e^{-\frac{i}{2}\alpha_0(\xi\eta)}\xi p_0$, called the "skew term" of $\tau_0^{(1)}$, is crucial in the iteration.

We also consider the germ of map $\sigma_0 = \tau_0^{(1)} \circ \tau_0^{(2)}$. It is reversible with respect to both ρ and $\tau_0^{(1)}$ since

$$\rho \circ \sigma_0 \circ \rho = \rho \circ \tau_0^{(1)} \circ \rho \circ \rho \circ \tau_0^{(2)} \circ \rho = \tau_0^{(2)} \circ \tau_0^{(1)} = \sigma_0^{-1} = \tau_0^{(1)} \circ \sigma_0 \circ \tau_0^{(1)}.$$

We can write $\sigma_0 = \tau_0^{(1)} \circ \tau_0^{(2)}$ as

$$\sigma_0(\xi,\eta) = \begin{pmatrix} e^{i\alpha_0(\xi\eta)}\xi + f_0(\xi,\eta) \\ e^{-i\alpha_0(\xi\eta)}\eta + g_0(\xi,\eta) \end{pmatrix}.$$

It will be shown in Sect. 6 (see Lemma 6.14 and Corollary 6.15) that, if ε_0 satisfies (31), then $f_0, g_0 \in \mathcal{A}_{\tilde\beta_0, r_0^{(7)}}(\mathcal{O}_0)$ with

$$\|f_0\|_{\mathcal{O}_0,\tilde\beta_0,r_0^{(7)}},\ \|g_0\|_{\mathcal{O}_0,\tilde\beta_0,r_0^{(7)}} \le \frac{\varepsilon_0}{4}.$$

Proposition 4.4 (Iteration scheme) *Assume that* $\varepsilon_0 > 0$ *satisfies* (31). *There exist a sequence of sets* $\{\mathcal{O}_\nu\}$ *with* $\mathcal{O}_\nu \subset]-r_\nu^2, r_\nu^2[$ *satisfying that*

$$\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu \cap]-r_{\nu+1}^2, r_{\nu+1}^2[, \quad \left|(\mathcal{O}_\nu \setminus \mathcal{O}_{\nu+1}) \cap \left]-r_{\nu+1}^2 + \beta_{\nu+1}, r_{\nu+1}^2 - \beta_{\nu+1}\right[\right| < \varepsilon_\nu^{\frac{1}{100s^2}}, \tag{46}$$

and a sequence of maps $\{\sigma_\nu\}$ *given by* $\sigma_\nu = \tau_\nu^{(1)} \circ \tau_\nu^{(2)}$ *with*

$$\tau_\nu^{(1)}(\xi,\eta) = \begin{pmatrix} e^{\frac{i}{2}\alpha_\nu(\xi\eta)}\eta + p_\nu(\xi,\eta) \\ e^{-\frac{i}{2}\alpha_\nu(\xi\eta)}\xi + q_\nu(\xi,\eta) \end{pmatrix}, \quad \tau_\nu^{(2)} = \rho \circ \tau_\nu^{(1)} \circ \rho, \tag{47}$$

satisfying $\tau_\nu^{(k)} \circ \tau_\nu^{(k)} = \mathrm{Id}$, $k = 1, 2$, *such that the following holds.*

(1) $\alpha_\nu = \alpha_\nu(\xi\eta) \in \mathcal{A}^{\mathbb{R}}_{\beta_\nu, r_\nu}(\mathcal{O}_\nu)$ *satisfies that*

$$\left|\alpha_\nu^{(s)} - s!\right|_{\mathcal{O}_\nu(r_\nu, \beta_\nu)} < \left(\frac{1}{16} + \zeta_\nu\right) s!, \tag{48}$$

$$\left|\alpha_\nu^{(k)}\right|_{\mathcal{O}_\nu(r_\nu, \beta_\nu)} < \frac{1}{16} + \zeta_\nu, \qquad 1 \le k \le s - 1, \ if \ s \ge 2, \tag{49}$$

$$\left|\alpha_\nu^{(k)}\right|_{\mathcal{O}_\nu(r_\nu, \beta_\nu)} < \left(\frac{1}{4} + \zeta_\nu\right) r_\nu^{-1}, \qquad s + 1 \le k \le 16s, \tag{50}$$

and for $0 \le k \le 16s$,

$$\left\|(\alpha_{\nu+1} - \alpha_\nu)^{(k)}\right\|_{\mathcal{O}_{\nu+1}, \beta_{\nu+1}, r_{\nu+1}} < \varepsilon_\nu^{\frac{1}{3}}. \tag{51}$$

(2) p_ν, $q_\nu \in \mathcal{A}_{\beta_\nu, r_\nu}(\mathcal{O}_\nu)$ *satisfy that*

$$\|p_\nu\|_{\mathcal{O}_\nu, \beta_\nu, r_\nu}, \|q_\nu\|_{\mathcal{O}_\nu, \beta_\nu, r_\nu} < \frac{\varepsilon_\nu}{10}, \quad \|e^{\frac{\mathrm{i}}{2}\alpha_\nu(\xi\eta)}\eta q_\nu + e^{-\frac{\mathrm{i}}{2}\alpha_\nu(\xi\eta)}\xi p_\nu\|_{\mathcal{O}_\nu, \beta_\nu, r_\nu} < \frac{\varepsilon_\nu^{\frac{3}{2}}}{3}. \tag{52}$$

(3) *The reversible map (w.r.t.* ρ*)* $\sigma_\nu = \tau_\nu^{(1)} \circ \tau_\nu^{(2)}$ *has the form*

$$\sigma_\nu(\xi, \eta) = \begin{pmatrix} e^{\mathrm{i}\alpha_\nu(\xi\eta)}\xi + f_\nu(\xi, \eta) \\ e^{-\mathrm{i}\alpha_\nu(\xi\eta)}\eta + g_\nu(\xi, \eta) \end{pmatrix},$$

with $\|f_\nu\|_{\mathcal{O}_\nu, \tilde{\beta}_\nu, r_\nu^{(7)}}$, $\|g_\nu\|_{\mathcal{O}_\nu, \tilde{\beta}_\nu, r_\nu^{(7)}} < \frac{\varepsilon_\nu}{4}$.

(4) *There is a sequence of transformations* $\{\psi_\nu\}$ *of the form*

$$\psi_\nu(\xi, \eta) = (\mathrm{Id} + \mathcal{U}_\nu)(\xi, \eta) = \begin{pmatrix} \xi + u_\nu(\xi, \eta) \\ \eta + v_\nu(\xi, \eta) \end{pmatrix}, \tag{53}$$

with $\mathcal{U}_\nu \in (\mathcal{A}^{\mathbb{R}}_{\beta_{\nu+1}, r_{\nu+1}}(\mathcal{O}_{\nu+1}))^2$ *satisfying*

$$\|u_\nu\|_{\mathcal{O}_{\nu+1}, \beta_{\nu+1}, r_{\nu+1}}, \ \|v_\nu\|_{\mathcal{O}_{\nu+1}, \beta_{\nu+1}, r_{\nu+1}} < \frac{\varepsilon_\nu^{\frac{49}{50}}}{2}$$

such that, for every $\omega \in \mathcal{O}_{\nu+1}(r_{\nu+1}, \beta_{\nu+1})$, $\psi_\nu : \mathcal{C}^{r_{\nu+1}}_{\omega, \beta_{\nu+1}} \to \mathcal{C}^{r_\nu}_{\omega, \beta_\nu}$ *and, on* $\mathcal{C}^{r_{\nu+1}}_{\omega, \beta_{\nu+1}}$,

$$\sigma_{\nu+1} = \psi_\nu^{-1} \circ \sigma_\nu \circ \psi_\nu, \quad \tau_{\nu+1}^{(k)} = \psi_\nu^{-1} \circ \tau_\nu^{(k)} \circ \psi_\nu, \quad k = 1, 2.$$

Remark 4.5 According to the definition of sequence $\varepsilon_\nu = \varepsilon_0^{\left(\frac{5}{4}\right)^\nu}$ and the fact that $(\frac{5}{4})^x - 1 > \frac{x}{8}$, $\forall\, x > 0$, we have

$$\sum_{\nu\geq 0}\varepsilon_\nu^\varsigma = \varepsilon_0^\varsigma \sum_{\nu\geq 0}\varepsilon_0^{\varsigma\left[\left(\frac{5}{4}\right)^\nu - 1\right]} < \varepsilon_0^\varsigma \sum_{\nu\geq 0}\varepsilon_0^{\frac{\varsigma\nu}{8}} = \frac{\varepsilon_0^\varsigma}{1-\varepsilon_0^{\frac{\varsigma}{8}}}, \quad \forall\, 0 < \varsigma < 1. \tag{54}$$

In particular, $\sum_{\nu\geq 0}\varepsilon_\nu^{\frac{1}{3}} < \frac{\varepsilon_0^{\frac{1}{3}}}{1-\varepsilon_0^{\frac{1}{24}}} < \frac{1}{240}$. Since $\left\|(\alpha_{\nu+1}-\alpha_\nu)^{(k)}\right\|_{\mathcal{O}_{\nu+1},\beta_{\nu+1},r_{\nu+1}} < \varepsilon_\nu^{\frac{1}{3}}$, $0 \leq k \leq 16s$, we obtain, according to (40) and (41), for $\forall\, \nu \in \mathbb{N}$,

$$\alpha_\nu(\omega) \in \left]-\frac{1}{4}, 4\pi + \frac{1}{4}\right[, \quad \forall\, \omega \in \mathcal{O}_\nu(r_\nu, \beta_\nu), \qquad \|\alpha_\nu\|_{\mathcal{O}_\nu,\beta_\nu,r_\nu} < 4\pi + \frac{1}{2}. \tag{55}$$

Moreover, in view of the definition of $\{\zeta_\nu\}$, we see that $\zeta_\nu < \frac{1}{240}$ for every $\nu \in \mathbb{N}$, which implies that $\frac{1}{16} + \zeta_\nu < \frac{1}{15}$, $\frac{1}{4} + \zeta_\nu < \frac{1}{2}$. Then, by (48)–(50),

$$\left|\alpha_\nu^{(s)} - s!\right|_{\mathcal{O}_\nu(r_\nu,\beta_\nu)} < \frac{s!}{15}, \tag{56}$$

$$\left|\alpha_\nu^{(k)}\right|_{\mathcal{O}_\nu(r_\nu,\beta_\nu)} < \frac{1}{15}, \qquad 1 \leq k \leq s-1,\ if\ s \geq 2, \tag{57}$$

$$\left|\alpha_\nu^{(k)}\right|_{\mathcal{O}_\nu(r_\nu,\beta_\nu)} < \frac{r_\nu^{-1}}{2}, \qquad s+1 \leq k \leq 16s. \tag{58}$$

4.3 Proof of Proposition 4.4

Suppose that, at the $(\nu+1)^{\text{th}}$ step, $\nu \geq 0$, we have

$$\tau_\nu^{(1)} = \begin{pmatrix} e^{\frac{i}{2}\alpha_\nu(\xi\eta)}\xi + p_\nu(\xi,\eta) \\ e^{-\frac{i}{2}\alpha_\nu(\xi\eta)}\eta + q_\nu(\xi,\eta) \end{pmatrix}, \quad \tau_\nu^{(2)} = \rho\circ\tau_\nu^{(1)}\circ\rho, \quad \sigma_\nu = \tau_\nu^{(1)}\circ\tau_\nu^{(2)},$$

as described in Proposition 4.4. Our aim is to construct the transformation ψ_ν as in (53), such that $\sigma_{\nu+1} := \psi_\nu^{-1}\circ\sigma_\nu\circ\psi_\nu$, $\tau_{\nu+1}^{(k)} := \psi_\nu^{-1}\circ\tau_\nu^{(k)}\circ\psi_\nu$, $k = 1, 2$, possess similar properties as those of σ_ν, $\tau_\nu^{(k)}$. This will describe an iteration step, hence will give the proof of Proposition 4.4 .

Before starting the construction of ψ_ν, we first introduce another type of transformations that conjugate the pairs of involutions (47) to a perturbation of a new integrable pair. The main feature of the new involutions is that the new perturbation part is much smaller than the initial one, provided that the initial skew term smallness condition is satisfied.

For given $0 < r_+ < r < \frac{1}{4}$ and fixed $s \in \mathbb{N}^*$, assume that $\varepsilon > 0$ satisfies

$$\left(\frac{|\ln\varepsilon|}{\left|\ln\left(\frac{7}{8}+\frac{r_+}{8r}\right)\right|}+2\right)\frac{(16s+1)^{16s}\varepsilon^{\frac{1}{2400s^2}}}{(r-r_+)r_+} < 1.$$

Let us set

$$\beta \in [\varepsilon^{\frac{1}{40s}}, \varepsilon^{\frac{1}{60s}}], \quad \beta_+ = \beta^{\frac{5}{4}} \in [\varepsilon^{\frac{1}{32s}}, \varepsilon^{\frac{1}{48s}}], \tag{59}$$

$$r^{(m)} := r_+ + \frac{m}{8}(r-r_+), \quad m = 0, 1, \ldots, 8, \qquad \tilde{r} := r^{(4)} = \frac{r+r_+}{2}. \tag{60}$$

Given $\mathcal{O} \subset]-r^2, r^2[$, we consider the involutions $\tau_1, \tau_2 = \rho \circ \tau_1 \circ \rho$:

$$\tau_1(\xi,\eta) = \begin{pmatrix} e^{\frac{i}{2}\alpha(\xi\eta)}\eta + p(\xi,\eta) \\ e^{-\frac{i}{2}\alpha(\xi\eta)}\xi + q(\xi,\eta)\end{pmatrix}, \quad \tau_2(\xi,\eta) = \begin{pmatrix} e^{-\frac{i}{2}\alpha(\xi\eta)}\eta + \bar{p}(\xi,\eta) \\ e^{\frac{i}{2}\alpha(\xi\eta)}\xi + \bar{q}(\xi,\eta)\end{pmatrix} \tag{61}$$

with $\alpha = \alpha(\xi\eta) \in \mathcal{A}^{\mathbb{R}}_{\beta,r}(\mathcal{O})$ satisfying (55)–(58) as α_ν, together with $p, q \in \mathcal{A}_{\beta,r}(\mathcal{O})$ satisfying

$$\|p\|_{\mathcal{O},\beta,r}, \ \|q\|_{\mathcal{O},\beta,r} < \frac{\varepsilon}{10}. \tag{62}$$

Remark 4.6 Here, all assumptions of Proposition 4.4 are satisfied but the smallness condition of the skew term $e^{\frac{i}{2}\alpha(\xi\eta)}\eta q + e^{-\frac{i}{2}\alpha(\xi\eta)}\xi p$ in (52).

Theorem 4.7 (Main step) *Given* $\delta \in]\, 80\varepsilon^{\frac{1}{60s}}, 1\, [$, *let*

$$\mathcal{O}_\delta := \left\{\omega \in \mathcal{O} : \ |e^{in\alpha(\omega)} - 1| \geq \delta, \quad \forall\, 0 < |n| \leq K+1, \quad K := \frac{|\ln\varepsilon|}{\left|\ln(r^{(7)}/r)\right|}\right\}. \tag{63}$$

There exists a transformation ψ *of the form*

$$\psi(\xi,\eta) = (\mathrm{Id}+\mathcal{U})(\xi,\eta) = \begin{pmatrix}\xi + u(\xi,\eta)\\ \eta + v(\xi,\eta)\end{pmatrix}$$

with $u, v \in \mathcal{A}^{\mathbb{R}}_{\beta_+,r_+}(\mathcal{O}_\delta)$ *satisfying*

$$\|u\|_{\mathcal{O}_\delta,\beta_+,r_+}, \ \|v\|_{\mathcal{O}_\delta,\beta_+,r_+} < \frac{\varepsilon^{\frac{49}{50}}}{2},$$

such that for every $\omega \in \mathcal{O}_\delta(r_+,\beta_+) = \mathcal{O}_\delta \cap]-r_+^2+\beta_+, r_+^2-\beta_+[$, ψ *is biholomorphic on* $\mathcal{C}^{r_+}_{\omega,\beta_+}$ *with* $\psi\left(\mathcal{C}^{r_+}_{\omega,\beta_+}\right) \subset \mathcal{C}^{r}_{\omega,\beta}$, *and, on* $\mathcal{C}^{r_+}_{\omega,\beta_+}$,

$$(\psi^{-1}\circ\tau_1\circ\psi)(\xi,\eta) = \begin{pmatrix} e^{\frac{i}{2}\alpha_+(\xi\eta)}\eta + p_+(\xi,\eta) \\ e^{-\frac{i}{2}\alpha_+(\xi\eta)}\xi + q_+(\xi,\eta)\end{pmatrix}, \quad (\xi,\eta) \in \mathcal{C}^{r_+}_{\omega,\beta_+},$$

where $\alpha_+ = \alpha_+(\xi\eta) \in \mathcal{A}^{\mathbb{R}}_{\beta_+,r_+}(\mathcal{O}_\delta)$, *with*

$$\left\|(\alpha_+-\alpha)^{(k)}\right\|_{\mathcal{O}_\delta,\beta_+,r_+} < \frac{\varepsilon^{\frac{1}{3}}}{10}, \quad 0 \le k \le 16s, \tag{64}$$

and p_+, $q_+ \in \mathcal{A}_{\beta_+,r_+}(\mathcal{O}_\delta)$, *with*

$$\|p_+\|_{\mathcal{O}_\delta,\beta_+,r_+}, \|q_+\|_{\mathcal{O}_\delta,\beta_+,r_+} < \frac{\varepsilon^{\frac{61}{32}}}{2} + 24(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}, \tag{65}$$

$$\|e^{\frac{\mathrm{i}}{2}\alpha_+}\eta q_+ + e^{-\frac{\mathrm{i}}{2}\alpha_+}\xi p_+\|_{\mathcal{O}_\delta,\beta_+,r_+} < \varepsilon^{\frac{61}{32}}. \tag{66}$$

Remark 4.8 Theorem 4.7 can be applied in two ways, which are indeed two cases described in Sect. 5.2.

- If the skew term $e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p$ of τ_1 satisfies

$$\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r} < \varepsilon^{1+\varsigma}, \quad 0 < \varsigma < 1,$$

  then, according to (65), Theorem 4.7 describes an iteration step (as in Proposition 4.4) with $r = r_\nu$, $r_+ = r_{\nu+1}$, $\varepsilon = \varepsilon_\nu$, and we can simply take $\beta = \varepsilon^{\frac{1}{40s}}$.
- If it is not the case, we cannot apply our Iteration scheme. However, noting that (65) implies $\|p_+\|_{\mathcal{O}_\delta,\beta_+,r_+}, \|q_+\|_{\mathcal{O}_\delta,\beta_+,r_+} < \frac{\varepsilon^{\frac{49}{50}}}{10}$, we see that the Iteration scheme is applicable to $\psi^{-1}\circ\tau_1\circ\psi$ in view of (66). Hence, Theorem 4.7 describes a preliminary step of iteration. In this case, we need to take $\beta > \varepsilon^{\frac{1}{40s}}$ since the new perturbation may be of size $\varepsilon^{\frac{49}{50}}$. This is why β is defined in an interval as in (59).

We postpone the proof of Theorem 4.7 to Sect. 7. The rest of the section is devoted to the proof of Proposition 4.4 from Theorem 4.7.

We want to conjugate the involution $\tau_\nu^{(1)}$ to a new one $\tau_{\nu+1}^{(1)}$. To do so, we need to exclude some parameters so we define the new parameter set as follow:

$$\mathcal{O}_{\nu+1} := \left\{\omega \in \mathcal{O}_\nu \cap]-r_{\nu+1}^2, r_{\nu+1}^2[: |e^{\mathrm{i}n\alpha_\nu(\omega)} - 1| > \varepsilon_\nu^{\frac{1}{64s}}, \ \forall\, 0 < |n| \le K_\nu + 1\right\}. \tag{67}$$

In order to measure its size, let us first recall Pyartli's lemma:

Lemma 4.9 [41, 42] *Let* $f : [a,b] \mapsto \mathbb{R}$ *with* $a < b$ *be a* q*-times continuously differentiable function satisfying*

$$|f^{(q)}(t)| \ge \delta, \quad \forall\, t \in [a,b]$$

for some $q \in \mathbb{N}^*$ *and* $\delta > 0$. *Then, for any* $A > 0$,

$$|\{t \in [a,b] : |f(t)| \le A\}| \le 4\left(q!\frac{A}{2\delta}\right)^{\frac{1}{q}}.$$

We then have

Lemma 4.10 $\left|(\mathcal{O}_\nu \setminus \mathcal{O}_{\nu+1}) \cap \left]-r_{\nu+1}^2+\beta_{\nu+1}, r_{\nu+1}^2-\beta_{\nu+1}\right[\right| < \varepsilon_\nu^{\frac{1}{80s^2}}$.

Proof Since $|\alpha_\nu(\omega)| < 4\pi+1$, it is sufficient to bound from above the measure of the parameter set $\bigcup_{0<|n|\le K_\nu+1} \mathcal{R}_{\nu,n}$, where

$$\mathcal{R}_{\nu,n} := \left\{\omega \in \mathcal{O}_\nu \cap \left]-r_{\nu+1}^2, r_{\nu+1}^2\right[: |n\alpha_\nu(\omega) - 2k\pi| \le \frac{3\varepsilon_\nu^{\frac{1}{64s}}}{2},\ \ k \in \mathbb{Z},\ |k| \le 3|n|\right\}.$$

In view of (48)–(50), we see $\inf_{\omega\in\mathcal{O}_\nu} \left|(n\alpha_\nu)^{(s)}(\omega)\right| \ge \frac{3}{4}|n|s!$. Applying Lemma 4.9 with $q = s$, we have

$$|\mathcal{R}_{\nu,n}| \le 3|n| \cdot 4 \left(\frac{\varepsilon_\nu^{\frac{1}{64s}}}{2|n|}\right)^{\frac{1}{s}} \le 20(K_\nu+1)^{1-\frac{1}{s}} \varepsilon_\nu^{\frac{1}{64s^2}}.$$

Therefore, we obtain

$$\bigcup_{0<|n|\le K_\nu+1} \left|\mathcal{R}_{\nu,n}\right| \le 40(K_\nu+1)^{2-\frac{1}{s}} \varepsilon_\nu^{\frac{1}{64s^2}} < \varepsilon_\nu^{\frac{1}{80s^2}},$$

noting that (32) implies that

$$\begin{aligned}(K_\nu+1)^2 \varepsilon_\nu^{\frac{1}{64s^2}-\frac{1}{80s^2}} &= \left(\frac{|\ln \varepsilon_\nu|}{\left|\ln\left(\frac{7r_\nu+r_{\nu+1}}{8r_\nu}\right)\right|}+1\right)^2 \varepsilon_\nu^{\frac{1}{320s^2}} \\ &< (16s+1)^{-16s} < \frac{1}{40}.\end{aligned}$$

□

Applying Theorem 4.7 to $\tau_1 = \tau_\nu^{(1)}$ with $\delta = \varepsilon_\nu^{\frac{1}{64s}} > 80\varepsilon_\nu^{\frac{1}{60s}}$, we obtain a transformation ψ_ν of the form

$$\psi_\nu(\xi,\eta) = (\mathrm{Id} + \mathcal{U}_\nu)(\xi,\eta) = \begin{pmatrix} \xi + u_\nu(\xi,\eta) \\ \eta + v_\nu(\xi,\eta)\end{pmatrix}$$

with $u_\nu, v_\nu \in \mathcal{A}^{\mathbb{R}}_{\beta_{\nu+1},r_{\nu+1}}(\mathcal{O}_{\nu+1})$ satisfying

$$\|u_\nu\|_{\mathcal{O}_{\nu+1},\beta_{\nu+1},r_{\nu+1}},\ \|v\|_{\mathcal{O}_{\nu+1},\beta_{\nu+1},r_{\nu+1}} < \frac{\varepsilon_\nu^{\frac{49}{50}}}{2},$$

such that, on $\mathcal{C}^{r_{\nu+1}}_{\omega,\beta_{\nu+1}}$, $\omega \in \mathcal{O}_{\nu+1}(r_{\nu+1}, \beta_{\nu+1})$, $\psi_\nu : \mathcal{C}^{r_{\nu+1}}_{\omega,\beta_{\nu+1}} \to \mathcal{C}^{r_\nu}_{\omega,\beta_\nu}$ is injective and holomorphic, and

$$(\psi_\nu^{-1} \circ \tau_\nu^{(1)} \circ \psi_\nu)(\xi, \eta) = \begin{pmatrix} e^{\frac{\mathrm{i}}{2}\alpha_{\nu+1}(\xi\eta)}\eta + p_{\nu+1}(\xi, \eta) \\ e^{-\frac{\mathrm{i}}{2}\alpha_{\nu+1}(\xi\eta)}\xi + q_{\nu+1}(\xi, \eta) \end{pmatrix},$$

where $\alpha_{\nu+1} = \alpha_{\nu+1}(\xi\eta) \in \mathcal{A}^{\mathbb{R}}_{\beta_{\nu+1},r_{\nu+1}}(\mathcal{O}_{\nu+1})$ with

$$\|(\alpha_{\nu+1} - \alpha_\nu)^{(k)}\|_{\mathcal{O}_{\nu+1},\beta_{\nu+1},r_{\nu+1}} < \varepsilon_\nu^{\frac{1}{3}}, \quad 0 \le k \le 16s,$$

which, combining with (48)–(50), implies that

$$\begin{aligned} |\alpha^{(s)}_{\nu+1} - s!|_{\mathcal{O}_{\nu+1}(r_{\nu+1},\beta_{\nu+1})} &< \left(\frac{1}{16} + \zeta_{\nu+1}\right) s!, \\ |\alpha^{(k)}_{\nu+1}|_{\mathcal{O}_{\nu+1}(r_{\nu+1},\beta_{\nu+1})} &< \frac{1}{16} + \zeta_{\nu+1}, \qquad 1 \le k \le s-1, \text{ if } s \ge 2, \\ |\alpha^{(k)}_{\nu+1}|_{\mathcal{O}_{\nu+1}(r_{\nu+1},\beta_{\nu+1})} &< \left(\frac{1}{4} + \zeta_{\nu+1}\right) r^{-1}, \qquad s+1 \le k \le 16s. \end{aligned}$$

In view of (52) and (65), (66), we have the new perturbation $p_{\nu+1}$ and $q_{\nu+1}$ satisfy that

$$\begin{aligned} &\|p_{\nu+1}\|_{\mathcal{O}_{\nu+1},\beta_{\nu+1},r_{\nu+1}}, \|q_{\nu+1}\|_{\mathcal{O}_{\nu+1},\beta_{\nu+1},r_{\nu+1}} \\ &\quad < \frac{\varepsilon_\nu^{\frac{61}{32}}}{2} + 24(K_\nu + 1)\varepsilon_\nu^{-\frac{1}{64s}} \|e^{\frac{\mathrm{i}}{2}\alpha_\nu}\eta q_\nu + e^{-\frac{\mathrm{i}}{2}\alpha_\nu}\xi p_\nu\|_{\mathcal{O}_\nu,\beta_\nu,r_\nu}, \\ &\quad < \frac{\varepsilon_\nu^{\frac{5}{4}}}{10} = \frac{\varepsilon_{\nu+1}}{10}, \end{aligned}$$

and the new skew term satisfies

$$\|e^{\frac{\mathrm{i}}{2}\alpha_{\nu+1}}\eta q_{\nu+1} + e^{-\frac{\mathrm{i}}{2}\alpha_{\nu+1}}\xi p_{\nu+1}\|_{\mathcal{O}_{\nu+1},\beta_{\nu+1},r_{\nu+1}} < \varepsilon_\nu^{\frac{61}{32}} < \frac{\varepsilon_\nu^{\frac{15}{8}}}{3} = \frac{\varepsilon_{\nu+1}^{\frac{3}{2}}}{3}.$$

According to Lemma 3.1, $u_\nu, v_\nu \in \mathcal{A}^{\mathbb{R}}_{\beta_{\nu+1},r_{\nu+1}}(\mathcal{O}_{\nu+1})$ implies that $\rho \circ \psi_\nu = \psi_\nu \circ \rho$. Then, for

$$\tau^{(1)}_{\nu+1} := \psi_\nu^{-1} \circ \tau_\nu^{(1)} \circ \psi_\nu, \quad \tau^{(2)}_{\nu+1} := \psi_\nu^{-1} \circ \tau_\nu^{(2)} \circ \psi_\nu = \rho \circ \tau^{(1)}_{\nu+1} \circ \rho,$$

we still have $\tau^{(k)}_{\nu+1} \circ \tau^{(k)}_{\nu+1} = \mathrm{Id}$, $k = 1, 2$. Moreover, in view of Lemma 3.2, for $\sigma_{\nu+1} = \tau^{(1)}_{\nu+1} \circ \tau^{(2)}_{\nu+1}$, it is still reversible w.r.t. ρ.

5 Proof of Theorem 2.1

This section is dedicated to the proof of Theorem 2.1 by applying the Iteration scheme Proposition 4.4.

5.1 Preliminary normalization

Let us consider the pair of involutions τ_1^o and τ_2^o given in (2) and (3) (and the reversible map $\sigma_o = \tau_1^o \circ \tau_2^o$). In order to start the Iteration scheme Proposition 4.4, we need the involutions to be in a well prepared form as in (38) and (39).

First of all, for any $N > s$, there exists a holomorphic transformation $\check{\Psi}$ in the neighborhood of origin, tangent to identity up to order 2, with $\check{\Psi} \circ \rho = \rho \circ \check{\Psi}$, such that

$$\check{\tau}_1(\xi, \eta) := \left(\check{\Psi}^{-1} \circ \tau_1^o \circ \check{\Psi}\right)(\xi, \eta) = \begin{pmatrix} e^{\frac{i}{2}\check{\alpha}(\xi\eta)}\eta + \check{p}(\xi, \eta) \\ e^{-\frac{i}{2}\check{\alpha}(\xi\eta)}\xi + \check{q}(\xi, \eta) \end{pmatrix}, \tag{68}$$

where we have

$$\check{\alpha}(\xi\eta) := \lambda + (\xi\eta)^s + \sum_{n=s+1}^{N} c_n(\xi\eta)^n, \quad c_n \in \mathbb{R}, \tag{69}$$

and convergent power series at the origin

$$\check{p}(\xi, \eta) = \sum_{\substack{l+j \geq 2N+2 \\ l,j \geq 0}} \check{\check{p}}_{l,j}\xi^l\eta^j, \quad \check{q}(\xi, \eta) = \sum_{\substack{l+j \geq 2N+2 \\ l,j \geq 0}} \check{\check{q}}_{l,j}\xi^l\eta^j. \tag{70}$$

Indeed, since $\frac{\lambda}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$, by classical normal form theory [2, 32, 40], combining with the fact that $\tau_1^o \circ \tau_1^o = \mathrm{Id}$, there is a polynomial transformation Ψ_{PD}, tangent to identity up to order 2 at the origin (composed by finitely many steps of normalization in the sense of Poincaré–Dulac) satisfying $\Psi_{PD} \circ \rho = \rho \circ \Psi_{PD}$, such that

$$\left(\Psi_{PD}^{-1} \circ \tau_1^o \circ \Psi_{PD}\right)(\xi, \eta) = \begin{pmatrix} (e^{\frac{i}{2}\lambda} + \tilde{C}(\xi\eta))\eta + \tilde{p}(\xi, \eta) \\ (e^{\frac{i}{2}\lambda} + \tilde{C}(\xi\eta))^{-1}\xi + \tilde{q}(\xi, \eta) \end{pmatrix}, \quad \tilde{C}(z) = \sum_{j=s}^{N} \tilde{c}_j z^j.$$

Here $\tilde{p}$, $\tilde{q}$ are holomorphic at the origin and of order $\geq 2N + 2$ there. We note that $e^{\frac{i}{2}\lambda} + \tilde{C}(z)$ is actually the truncation of $\Lambda(z)$ in (5)–(7). Recalling the non-degeneracy assumption of Theorem 2.1, we see that $\tilde{c}_s \neq 0$. According to the proof of Theorem 3.4 of [40], we can change $e^{\frac{i}{2}\lambda} + \tilde{C}$ to $(e^{\frac{i}{2}\lambda} + \tilde{C})\mu^{-2}$ by applying the transformation

$$(\xi, \eta) \mapsto (\mu(\xi\eta)\xi, \mu^{-1}(\xi\eta)\eta), \tag{71}$$

where $\mu = \mu(\xi\eta)$ is the fourth root $\mu(\xi\eta) := \left((e^{\frac{i}{2}\lambda} + \tilde{C}(\xi\eta))(e^{-\frac{i}{2}\lambda} + \bar{\tilde{C}}(\xi\eta))\right)^{\frac{1}{4}}$. We see that $\mu(\xi\eta)$ is sufficiently close to 1 and hence well-defined since (ξ, η) belongs to a sufficiently small neighborhood of the origin. A direct computation shows that

$$\begin{aligned}(e^{\frac{\mathrm{i}}{2}\lambda}+\tilde{C})\mu^{-2} &= (e^{\frac{\mathrm{i}}{2}\lambda}+\tilde{C})^{\frac{1}{2}}(e^{-\frac{\mathrm{i}}{2}\lambda}+\bar{\tilde{C}})^{-\frac{1}{2}}\\ &= e^{\frac{\mathrm{i}}{2}\lambda}(1+e^{-\frac{\mathrm{i}}{2}\lambda}\tilde{C})^{\frac{1}{2}}(1+e^{\frac{\mathrm{i}}{2}\lambda}\bar{\tilde{C}})^{-\frac{1}{2}}.\end{aligned}$$

We consider the principal determination of the logarithm to be defined on $\mathbb{C}\setminus\mathbb{R}^-$. Let $A(t)$ be holomorphic germ vanishing at the origin. If t is small enough, there are real numbers $\tilde{b}_k$ such that

$$\ln(1+A(t))-\ln(1+\bar{A}(t))=\sum_{k\geq 1}\frac{A^k(t)-\bar{A}^k(t)}{k}=\mathrm{i}\sum_{k\geq 1}\tilde{b}_k t^k$$

and it converges at the origin. Applying this, we have

$$\ln\left((1+e^{-\frac{\mathrm{i}}{2}\lambda}\tilde{C})^{\frac{1}{2}}(1+e^{\frac{\mathrm{i}}{2}\lambda}\bar{\tilde{C}})^{-\frac{1}{2}}\right)=\frac{1}{2}\left(e^{-\frac{\mathrm{i}}{2}\lambda}\tilde{c}_s-e^{\frac{\mathrm{i}}{2}\lambda}\bar{\tilde{c}}_s\right)(\xi\eta)^s+\sum_{n\geq s+1}b_n(\xi\eta)^n,$$

with $\{b_n\}\subset\mathrm{i}\mathbb{R}$ coefficients of a convergent power series. Define

$$\check{\alpha}(\xi\eta):=\lambda-\mathrm{i}\left(e^{-\frac{\mathrm{i}}{2}\lambda}\tilde{c}_s-e^{\frac{\mathrm{i}}{2}\lambda}\bar{\tilde{c}}_s\right)(\xi\eta)^s-2\mathrm{i}\sum_{n=s+1}^{N}b_n(\xi\eta)^n,$$

which is of the form (69) up to a scaling on the neighborhood of origin. By rewriting $(1+e^{-\frac{\mathrm{i}}{2}\lambda}\tilde{C})^{\frac{1}{2}}(1+e^{\frac{\mathrm{i}}{2}\lambda}\bar{\tilde{C}})^{-\frac{1}{2}}$ as

$$(1+e^{-\frac{\mathrm{i}}{2}\lambda}\tilde{C})^{\frac{1}{2}}(1+e^{\frac{\mathrm{i}}{2}\lambda}\bar{\tilde{C}})^{-\frac{1}{2}}=e^{\frac{\mathrm{i}}{2}(\check{\alpha}-\lambda)}\exp\left\{\sum_{n\geq N+1}b_n(\xi\eta)^n\right\},$$

we see that

$$(e^{\frac{\mathrm{i}}{2}\lambda}+\tilde{C})\mu^{-2}-e^{\frac{\mathrm{i}}{2}\check{\alpha}}=e^{\frac{\mathrm{i}}{2}\lambda}\cdot e^{\frac{\mathrm{i}}{2}(\check{\alpha}-\lambda)}\cdot\left(\exp\left\{\sum_{n\geq N+1}b_n(\xi\eta)^n\right\}-1\right),$$

which contains terms of (ξ,η) of order $\geq 2N+2$. It is similar for $(e^{\frac{\mathrm{i}}{2}\lambda}+\tilde{C})^{-1}\mu^2-e^{-\frac{\mathrm{i}}{2}\check{\alpha}}$. Hence, we obtain $\check{\tau}_1$ in (68), up to a scaling on the neighborhood of origin.

The following lemma shows that, for $N\geq 16s$ large enough, there exists $0<r_*<\frac{1}{4}$ sufficiently small, depending on the coefficients c_n, $n=s+1,\ldots,N$, such that (31) can be satisfied in the two cases (where $A:=10\max\{|\check{p}|_{r_*},\ |\check{q}|_{r_*}\}$):

- $\varepsilon_0=A^{\frac{49}{50}}$, $r_0=\frac{3}{4}r_*$ and $r_1=\frac{9}{16}r_*$,
- $\varepsilon_0=A$, $r_0=r_*$ and $r_1=\frac{3}{4}r_*$.

Lemma 5.1 *If N is large enough, there exists $0 < r_* < \frac{1}{4}$ sufficiently small such that*

(1) $\sup_{|z|<r_*^2} |\check{\alpha}(z) - \lambda| < \frac{1}{8}$, $\sup_{|z|<r_*^2} |\check{\alpha}^{(s)}(z) - s!| < \frac{s!}{20}$,
(2) $\sup_{|z|<r_*^2} |\check{\alpha}^{(k)}(z)| < \frac{1}{20}$, $1 \le k \le s-1$, *if* $s \ge 2$,
(3) $\sup_{|z|<r_*^2} |\check{\alpha}^{(k)}(z)| < \frac{r_*^{-1}}{4}$, $s+1 \le k \le 16s$,
(4) $A = 10\max\{|\check{p}|_{r_*},\ |\check{q}|_{r_*}\}$ *satisfies that*

$$\left(\frac{|\ln A|}{\left|\ln\left(\frac{7}{8}+\frac{1}{8}\cdot\frac{3}{4}\right)\right|}+2\right)\frac{(16s+1)^{16s}A^{\frac{49}{50}\cdot\frac{1}{2400s^2}}}{(\frac{3}{4}r_*-\frac{9}{16}r_*)\cdot\frac{9}{16}r_*} < 1. \tag{72}$$

Proof In view of (69), it is easy to see that (1)–(3) are satisfied for $\check{\alpha}$ for any $N \ge 16s$ and r_* sufficiently small. Let us choose $N = 4900s^2$. We recall that all terms of $\check{p}$ and $\check{q}$ are of order $\ge 2N+2$. According to the definitions of norms in (17), if we replace r_* by $r_k' := 2^{-k}r_*$, then $A_k' := 10\max\{|\check{p}|_{r_k'},\ |\check{q}|_{r_k'}\}$ satisfies that $A_k' \le A\cdot\left(\frac{r_k'}{r_*}\right)^{2N+2} = 2^{-(2N+2)k}A$. Since $0 < r_* < \frac{1}{4}$, we have that $\left((\frac{3}{4}r_*-\frac{9}{16}r_*)\cdot\frac{9}{16}r_*\right)^{-1} = \frac{256}{27}r_*^{-2} < 10r_*^{-2}$. To show (4), it is sufficient to show that

$$(16s+1)^{16s}\left(\frac{|\ln A|}{\left|\ln\left(\frac{31}{32}\right)\right|}+2\right)A^{\frac{49}{50}\cdot\frac{1}{4800s^2}} < 1, \tag{73}$$

$$10r_*^{-2}A^{\frac{49}{50}\cdot\frac{1}{4800s^2}} < 1. \tag{74}$$

Replacing r_* by r_k' in (73) and (74), we see that, as $k \to \infty$,

$$\begin{aligned}
&(16s+1)^{16s}\left(\frac{|\ln A_k'|}{\left|\ln\left(\frac{31}{32}\right)\right|}+2\right)(A_k')^{\frac{49}{50}\cdot\frac{1}{4800s^2}}\\
&\quad< (16s+1)^{16s}\left(\frac{\ln(2)\cdot(9800s^2+2)k+|\ln A|}{\left|\ln\left(\frac{31}{32}\right)\right|}+2\right)2^{-\frac{49}{50}\cdot\frac{9800s^2+2}{4800s^2}k}A^{\frac{49}{50}\cdot\frac{1}{4800s^2}} \to 0,\\
&10r_k'^{-2}\cdot(A_k')^{\frac{49}{50}\cdot\frac{1}{4800s^2}} < 10r_*^{-2}2^{2k}\cdot 2^{-\frac{49}{50}\cdot\frac{9800s^2+2}{4800s^2}k}A^{\frac{49}{50}\cdot\frac{1}{4800s^2}}\\
&\qquad= 10r_*^{-2}A^{\frac{1}{4800s^2}}\cdot 2^{-(\frac{49}{50}\cdot\frac{9800s^2+2}{4800s^2}-2)k} \to 0.
\end{aligned}$$

Hence, there exists a $k_* \in \mathbb{N}^*$ such that if we replace r_* by r_{k_*}', then (73) and (74) are both satisfied. □

5.2 Application of KAM-like Theorem

Take r_* as in Lemma 5.1. Since $\check{p}$ and $\check{q}$ are convergent power series, in view of (18), we have that, for any $\beta_* \in [A^{\frac{1}{40s}}, A^{\frac{1}{60s}}]$,

$$\|\check{p}\|_{]-r_*^2+\beta_*,r_*^2-\beta_*[,\,\beta_*,\,r_*},\ \|\check{q}\|_{]-r_*^2+\beta_*,r_*^2-\beta_*[,\,\beta_*,\,r_*} \le \max\{|\check{p}|_{r_*},\ |\check{q}|_{r_*}\} = \frac{A}{10}. \tag{75}$$

- **Case 1. Small skew term**

If we have

$$\|e^{-\frac{\mathrm{i}}{2}\check{\alpha}}\xi\check{p}+e^{\frac{\mathrm{i}}{2}\check{\alpha}}\eta\check{q}\|_{]-r_*^2+\beta_*,r_*^2-\beta_*[,\,\beta_*,\,r_*}<\frac{A^{\frac{3}{2}}}{3}, \tag{76}$$

then, from Lemma 5.1 and (75), with $\alpha_0(\xi\eta):=\check{\alpha}(\xi\eta)$, $p_0:=\check{p}$, $q_0:=\check{q}$, $r_0:=r_*$, $\varepsilon_0:=A$, $\beta_0:=\varepsilon_0^{\frac{1}{40s}}=A^{\frac{1}{40s}}$ and $\mathcal{O}_0:=]-r_0^2,r_0^2[$, we have that $\alpha_0(\xi\eta)\in\mathcal{A}^{\mathbb{R}}_{\beta_0,r_0}(\mathcal{O}_0)$, $p_0,q_0\in\mathcal{A}_{\beta_0,r_0}(\mathcal{O}_0)$, and (40)–(45) are satisfied. Since, for $r_1:=\frac{3}{4}r_0=\frac{3}{4}r_*$, (31) is satisfied from Lemma 5.1, we can apply Proposition 4.4 to $\tau_0^{(1)}:=\check{\tau}_1$ (hence $\tau_0^{(2)}=\rho\circ\tau_0^{(1)}\circ\rho$ and $\sigma=\tau_0^{(1)}\circ\tau_0^{(2)}$) on $\mathcal{C}^{r_0}_{\omega,\beta_0}$, $\omega\in\mathcal{O}_0(r_0,\beta_0)=\mathcal{O}_0\cap]-r_0^2+\beta_0,r_0^2-\beta_0[$.

- **Case 2. Non-small skew term**

Now assume that (76) is not satisfied. In view of (69) and Lemma 5.1, we see that

$$|e^{\pm\frac{\mathrm{i}}{2}\check{\alpha}}|_{\omega,\beta_*}=|e^{\pm\frac{\mathrm{i}}{2}(\check{\alpha}-\lambda)}|_{\omega,\beta_*}\le\sum_{k\ge0}\frac{1}{k!}\left(\frac{|\check{\alpha}-\lambda|_{\omega,\beta_*}}{2}\right)^k\le e^{\frac{1}{16}},\quad \omega\in]-r_*^2+\beta_*,r_*^2-\beta_*[.$$

Then, by (75),

$$\|e^{-\frac{\mathrm{i}}{2}\check{\alpha}}\xi\check{p}+e^{\frac{\mathrm{i}}{2}\check{\alpha}}\eta\check{q}\|_{\omega,\beta_*,r_*}<\frac{e^{\frac{1}{16}}}{2}\cdot\frac{A}{10}<\frac{A}{10}.$$

With $r=r_*$, $r_+=\frac{3}{4}r_*$ and $r_*^{(j)}:=r_++\frac{j}{8}(r_*-r_+)$, $j=0,\ldots,8$, $\delta:=100A^{\frac{1}{60s}}$, $\beta:=A^{\frac{1}{60s}}$, $\beta_+:=A^{\frac{49}{50}\cdot\frac{1}{40s}}\in[A^{\frac{1}{32s}},A^{\frac{1}{48s}}]$, and

$$\mathcal{O}_\delta:=\left\{\omega\in]-r_*^2,r_*^2[:|e^{\mathrm{i}n\check{\alpha}(\omega)}-1|>\delta,\quad\forall\,0<|n|\le K_*+1:=\frac{|\ln A|}{\left|\ln(r_*^{(7)}/r_*)\right|}+1\right\},$$

we apply Theorem 4.7 to $\check{\tau}_1$ for $\omega\in\mathcal{O}_\delta(r_+,\beta_+):=\mathcal{O}_\delta\cap]-r_+^2+\beta_+,r_+^2-\beta_+[$. We obtain, for all $\omega\in\mathcal{O}_\delta(r_+,\beta_+)$, a biholomorphic transformation $\check{\psi}=\mathrm{Id}+\check{\mathcal{U}}:\mathcal{C}^{r_+}_{\omega,\beta_+}\to\mathcal{C}^{r}_{\omega,\beta}$, with $\check{\mathcal{U}}\in(\mathcal{A}^{\mathbb{R}}_{\beta_+,r_+}(\mathcal{O}_\delta))^2$ and $\|\check{\mathcal{U}}\|_{\omega,\beta_+,r_+}<\frac{A^{\frac{49}{50}}}{2}$. Furthermore, there are $\check{\alpha}_+=\check{\alpha}_+(\xi\eta)\in\mathcal{A}^{\mathbb{R}}_{\beta_+,r_+}(\mathcal{O}_\delta)$, $\check{p}_+,\check{q}_+\in\mathcal{A}_{\beta_+,r_+}(\mathcal{O}_\delta)$ such that

$$(\check{\psi}^{-1}\circ\check{\tau}_1\circ\check{\psi})(\xi,\eta)=\begin{pmatrix}e^{\frac{\mathrm{i}}{2}\check{\alpha}_+(\xi\eta)}\eta+\check{p}_+(\xi,\eta)\\ e^{-\frac{\mathrm{i}}{2}\check{\alpha}_+(\xi\eta)}\xi+\check{q}_+(\xi,\eta)\end{pmatrix},\quad(\xi,\eta)\in\mathcal{C}^{r_+}_{\omega,\beta_+}.$$

They satisfy

$$\left\|(\check{\alpha}_+-\check{\alpha})^{(k)}\right\|_{\mathcal{O}_\delta,\beta_+,r_+}<\frac{A^{\frac{1}{3}}}{10},\quad 0\le k\le16s, \tag{77}$$

$$\|\check{p}_+\|_{\mathcal{O}_\delta,\beta_+,r_+},\ \|\check{q}_+\|_{\mathcal{O}_\delta,\beta_+,r_+}$$

$$< \frac{A^{\frac{61}{32}}}{2} + 18(K_* + 1)\delta^{-1}\|e^{\frac{i}{2}\check{\alpha}}\eta\check{q} + e^{-\frac{i}{2}\check{\alpha}}\xi\check{p}\|_{]-r_*^2+\beta_*,r_*^2-\beta_*[,\beta_*,r_*} < \frac{A^{\frac{49}{50}}}{10},$$

$$\|e^{-\frac{i}{2}\check{\alpha}_+}\xi\check{p}_+ + e^{\frac{i}{2}\check{\alpha}_+}\eta\check{q}_+\|_{\mathcal{O}_\delta,\beta_+,r_+} < A^{\frac{61}{32}} < \frac{A^{\frac{49}{50}\cdot\frac{3}{2}}}{3}.$$

Hence, for $\alpha_0 := \check{\alpha}_+$, $p_0 := \check{p}_+$, $q_0 := \check{q}_+$, $r_0 := r_+ = \frac{3}{4}r_*$, $\varepsilon_0 := A^{\frac{49}{50}}$, and

$$\beta_0 := \beta_+ = \varepsilon_0^{\frac{1}{40s}} = A^{\frac{49}{50}\cdot\frac{1}{40s}},$$

we have (45) for $\mathcal{O}_0 := \mathcal{O}_\delta(r_0, \beta_0)$. According to (77), we obtain (40)–(44) from Lemma 5.1. In particular, Lemma 5.1 (1) and (77) imply that

$$|\alpha_0(\cdot) - \lambda|_{\mathcal{O}_0} < \frac{1}{4}.$$

For $r_0 = \frac{3}{4}r_*$, $r_1 := \frac{3}{4}r_0 = \frac{9}{16}r_*$, (31) is verified by Lemma 5.1, then we can apply Proposition 4.4 to $\tau_0^{(1)} := \check{\psi}^{-1}\circ\check{\tau}_1\circ\check{\psi}$ (hence $\tau_0^{(2)} = \rho\circ\tau_0^{(1)}\circ\rho$ and $\sigma = \tau_0^{(1)}\circ\tau_0^{(2)}$) on $\mathcal{C}^{r_0}_{\omega,\beta_0}$, $\omega \in \mathcal{O}_0(r_0, \beta_0) = \mathcal{O}_0$. According to Lemma 5.1, we see that $|\check{\alpha}^{(s)}(\omega)| \geq \frac{19}{20}s!$ for $\omega \in]-r_*^2, r_*^2[$, then we deduce from Pyartli's lemma (Lemma 4.9) that

$$\left|]-r_0^2+\beta_0, r_0^2-\beta_0[\setminus\mathcal{O}_0\right| < A^{\frac{1}{80s^2}}. \tag{78}$$

The proof of (78) is similar to that of Lemma 4.10.

Let us define $\check{\psi} = \mathrm{Id}$ in Case 1. In Case 2, we have, as above, $\check{\psi} = \mathrm{Id} + \check{\mathcal{U}}$ with $\check{\mathcal{U}} \in (\mathcal{A}^{\mathbb{R}}_{\beta_+,r_+}(\mathcal{O}_\delta))^2$. In both cases, we define $\tilde{\Phi} := \check{\Psi}\circ\check{\psi}$. To summarize, for the involutions τ_1^o given in (2), we have

Proposition 5.2 *There exists $r_0 > 0$ and there exists $\varepsilon_0 > 0$ satisfying (31) with $r_1 = \frac{3r_0}{4}$, and there exists a set $\mathcal{O}_0 \subset]-r_0^2, r_0^2[$ with*

$$\left|]-r_0^2+\beta_0, r_0^2-\beta_0[\setminus\mathcal{O}_0\right| < \varepsilon_0^{\frac{49}{50}\cdot\frac{1}{80s^2}}$$

for $\beta_0 = \varepsilon_0^{\frac{1}{40s}}$, such that the following holds for $\omega \in \mathcal{O}_0$.

There exists a transformation $\tilde{\Phi} : \mathcal{C}^{r_0}_{\omega,\beta_0} \to \mathbb{C}^2$, with $\tilde{\Phi}\circ\rho = \rho\circ\tilde{\Phi}$, such that the involution

$$\tau_0^{(1)}(\xi,\eta) = (\tilde{\Phi}^{-1}\circ\tau_1^o\circ\tilde{\Phi})(\xi,\eta) = \begin{pmatrix} e^{\frac{i}{2}\alpha_0(\xi\eta)}\eta + p_0(\xi,\eta) \\ e^{-\frac{i}{2}\alpha_0(\xi\eta)}\xi + q_0(\xi,\eta) \end{pmatrix},$$

with $\alpha_0(\xi\eta) \in \mathcal{A}^{\mathbb{R}}_{\beta_0,r_0}(\mathcal{O}_0)$, $p_0, q_0 \in \mathcal{A}_{\beta_0,r_0}(\mathcal{O}_0)$ and (40)–(45) satisfied. In particular,

$$|\alpha_0(\cdot) - \lambda|_{\mathcal{O}_0} < \frac{1}{4}. \tag{79}$$

As in (3), $\tau_0^{(2)}$ is obtained by $\tau_0^{(2)} = \rho \circ \tau_0^{(1)} \circ \rho$, and $\sigma_0 = \tau_0^{(1)} \circ \tau_0^{(2)}$. Since $\tilde{\Phi}$ commutes with ρ, we see that $\tau_0^{(2)}$ is still an involution and, in view of Lemma 3.2, σ_0 is still reversible with respect to ρ.

By applying Proposition 4.4 to $\tau_0^{(1)}$ (hence $\tau_0^{(2)} = \rho \circ \tau_0^{(1)} \circ \rho$ and $\sigma_0 = \tau_0^{(1)} \circ \tau_0^{(2)}$) on $\mathcal{C}^{r_0}_{\omega,\beta_0}$, $\omega \in \mathcal{O}_0(r_0, \beta_0)$, we get sequences of involutions $\{\tau_\nu^{(1)}\}_{\nu\in\mathbb{N}}$ and $\{\tau_\nu^{(2)}\}_{\nu\in\mathbb{N}}$ (and hence $\{\sigma_\nu\}_{\nu\in\mathbb{N}}$) on $\mathcal{C}^{r_\nu}_{\omega,\beta_\nu}$ and a sequence of holomorphic transformations $\{\psi_\nu\}_{\nu\in\mathbb{N}}$ of the form $\psi_\nu = \mathrm{Id} + \mathcal{U}_\nu$ with $\mathcal{U}_\nu \in (\mathcal{A}^{\mathbb{R}}_{\beta_{\nu+1},r_{\nu+1}}(\mathcal{O}_{\nu+1}))^2$ and $\|\mathcal{U}_\nu\|_{\mathcal{O}_{\nu+1},\beta_{\nu+1},r_{\nu+1}} < \varepsilon_\nu^{\frac{49}{50}}$, such that, for all $\omega \in \mathcal{O}_{\nu+1}(r_{\nu+1}, \beta_{\nu+1})$, $\psi_\nu : \mathcal{C}^{r_{\nu+1}}_{\omega,\beta_{\nu+1}} \to \mathcal{C}^{r_\nu}_{\omega,\beta_\nu}$,

$$\sigma_{\nu+1} = \psi_\nu^{-1} \circ \sigma_\nu \circ \psi_\nu, \quad \tau_{\nu+1}^{(k)} = \psi_\nu^{-1} \circ \tau_\nu^{(k)} \circ \psi_\nu, \quad k = 1, 2.$$

Recall that $r_{\nu+1} = r_\nu - 2^{-(\nu+2)} r_0$, we see that $r_\nu \to \frac{r_0}{2} =: R$ as $\nu \to \infty$. By (46), we can see that $\mathcal{O}_\nu \to \mathcal{O}_\infty(R)$ for some $\mathcal{O}_\infty(R) \subset \left]-\frac{r_0^2}{4}, \frac{r_0^2}{4}\right[=]-R^2, R^2[$. Moreover, by (46) and (54),

$$\left|\left]-R^2, R^2\right[\setminus \mathcal{O}_\infty(R)\right| < \sum_{\nu\geq 0} \varepsilon_\nu^{\frac{1}{100s^2}} < \frac{\varepsilon_0^{\frac{1}{100s^2}}}{1 - \varepsilon_0^{\frac{1}{800s^2}}} < 2\varepsilon_0^{\frac{1}{100s^2}}.$$

Hence, we have (9) by noting that the Lebesgue density of $\mathcal{O}_\infty(R)$ in $]-R^2, R^2[$ satisfies

$$1 - \frac{\varepsilon_0^{\frac{1}{100s^2}}}{R^2} < \frac{|\mathcal{O}_\infty(R)|}{2R^2} < 1,$$

since $r_0 = r_*$ or $\frac{3}{4} r_*$, and by (72), we have $\varepsilon_0^{\frac{1}{100s^2}} \leq A^{\frac{49}{50}\cdot\frac{1}{100s^2}} < \left(\frac{r_0}{2}\right)^{2400s^2\cdot\frac{1}{100s^2}} = R^{24}$.

For any $\nu \in \mathbb{N}$, let $\Psi_\nu := \check{\psi} \circ \psi_0 \circ \cdots \circ \psi_\nu$, which is well defined and injective on $\mathcal{C}^{r_{\nu+1}}_{\omega,\beta_{\nu+1}}$ for every $\omega \in \mathcal{O}_{\nu+1}(r_{\nu+1}, \beta_{\nu+1})$. By Lemma 3.1, since $\check{\mathcal{U}} \in (\mathcal{A}^{\mathbb{R}}_{\beta_+,r_+}(\mathcal{O}_\delta))^2$ and $u_j, v_j \in \mathcal{A}^{\mathbb{R}}_{\beta_{j+1},r_{j+1}}(\mathcal{O}_{j+1})$, $j = 0, 1, \ldots, \nu$, then $\check{\psi} \circ \rho = \rho \circ \check{\psi}$ and $\psi_j \circ \rho = \rho \circ \psi_j$. Hence $\Psi_\nu \circ \rho = \rho \circ \Psi_\nu$.

Lemma 5.3 *For every* $\nu \in \mathbb{N}$, $\|\Psi_{\nu+1} - \Psi_\nu\|_{\mathcal{O}_{\nu+2},\beta_{\nu+2},r_{\nu+2}} < \varepsilon_{\nu+1}^{\frac{4}{5}}$.

Proof With the smallness of A verified in Lemma 5.1 and recalling that $\varepsilon_0 = A$ or $A^{\frac{49}{50}}$, we have, by Lemma 3.6,

$$\begin{aligned}\|\Psi_0\|_{\mathcal{O}_1,\beta_1,r_1} &\leq \|\check{\psi} \circ \psi_0 - \check{\psi}\|_{\mathcal{O}_1,\beta_1,r_1} + \|\check{\psi}\|_{\mathcal{O}_1,\beta_1,r_1} \\ &\leq \frac{3r_0\|\check{\psi}\|_{\mathcal{O}_0,\beta_0,r_0}\varepsilon_0^{\frac{49}{50}}}{2(r_0 - r_1)\beta_0} + \|\check{\psi}\|_{\mathcal{O}_0,\beta_0,r_0}\end{aligned}$$

$$\leq (2r_0 + A^{\frac{49}{50}}) \left(\frac{3r_0 \varepsilon_0^{\frac{49}{50}}}{2(r_0 - r_1)\beta_0} + 1 \right) \leq 3r_0.$$

Let us show the lemma by induction on ν. When $\nu = 0$, $\Psi_0 = \check{\psi} \circ \psi_0$ and $\Psi_1 = \Psi_0 \circ \psi_1$. For any $(\xi, \eta) \in \mathcal{C}^{r_1}_{\omega,\beta_1}$, we have $\psi_1(\xi, \eta) \in \mathcal{C}^{r_0}_{\omega,\beta_0}$. Since (31) implies that (27) holds for $\beta' = \beta_1$, $\beta'' = \beta_2$, $r' = r_1$, $r'' = r_2$, we have, by Lemma 3.6,

$$\begin{aligned} \|\Psi_1 - \Psi_0\|_{\mathcal{O}_{2,\beta_2,r_2}} &= \|\Psi_0(\xi + u_1, \eta + v_1) - \Psi_0(\xi, \eta)\|_{\mathcal{O}_{2,\beta_2,r_2}} \\ &< \frac{3r_1 \|\Psi_0\|_{\mathcal{O}_{1,\beta_1,r_1}}}{2(r_1 - r_2)\beta_1} \varepsilon_1^{\frac{49}{50}} < \frac{3r_0 r_1 \varepsilon_1^{\frac{49}{50}}}{(r_1 - r_2)\beta_1} < \varepsilon_1^{\frac{4}{5}}. \end{aligned}$$

Given $k \in \mathbb{N}^*$, assume that $\|\Psi_{j+1} - \Psi_j\|_{\mathcal{O}_{j+2,\beta_{j+2},r_{j+2}}} < \varepsilon_{j+1}^{\frac{4}{5}}$ for $0 \leq j \leq k$. Then

$$\begin{aligned} \|\Psi_{k+1}\|_{\mathcal{O}_{k+2,\beta_{k+2},r_{k+2}}} &\leq \|\Psi_0\|_{\mathcal{O}_{1,\beta_1,r_1}} + \sum_{j=0}^{k} \|\Psi_{j+1} - \Psi_j\|_{\mathcal{O}_{j+2,\beta_{j+2},r_{j+2}}} \\ &< 3r_0 + \sum_{j=0}^{k} \varepsilon_{j+1}^{\frac{4}{5}} < 4r_0. \end{aligned} \tag{80}$$

Hence, by Lemma 3.6,

$$\begin{aligned} &\|\Psi_{k+2} - \Psi_{k+1}\|_{\mathcal{O}_{k+3,\beta_{k+3},r_{k+3}}} \\ &\quad = \|\Psi_{k+1}(\xi + u_{k+2}, \eta + v_{k+2}) - \Psi_{k+1}(\xi, \eta)\|_{\mathcal{O}_{k+3,\beta_{k+3},r_{k+3}}} \\ &\quad < \frac{3r_{k+2}\|\Psi_{k+1}\|_{\mathcal{O}_{k+2,\beta_{k+2},r_{k+2}}}}{2(r_{k+2} - r_{k+3})\beta_{k+2}} \varepsilon_{k+2}^{\frac{49}{50}} \\ &\quad < \frac{6r_0 r_{k+2} \varepsilon_{k+2}^{\frac{49}{50}}}{(r_{k+2} - r_{k+3})\beta_{k+2}} < \varepsilon_{k+2}^{\frac{4}{5}}, \end{aligned}$$

since (32) implies that (27) holds for $\beta' = \beta_{k+2}$, $\beta'' = \beta_{k+3}$, $r' = r_{k+2}$, $r'' = r_{k+3}$. □

The above lemma shows that, with $\check{\Psi}$ in (68), for every $\omega \in \mathcal{O}_\infty(R)$, the sequence $\{\check{\Psi} \circ \Psi_\nu\}$ converges uniformly to an injective holomorphic mapping $\Psi_\omega : \mathcal{C}^R_\omega \to \mathbb{C}^2$ as it is a Cauchy sequence:

$$\begin{aligned} &\sup_{(\xi,\eta)\in\mathcal{C}^R_\omega} \|\check{\Psi} \circ \Psi_\nu(\xi, \eta) - \check{\Psi} \circ \Psi_{\nu'}(\xi, \eta)\| \\ &\leq \|\check{\Psi}\|_R \|\Psi_\nu - \Psi_{\nu'}\|_{\omega,0,R} \\ &\leq \|\check{\Psi}\|_R \sum_{j=\nu}^{\nu'} \|\Psi_j - \Psi_{j+1}\|_{\omega,\beta_{j+2},r_{j+2}} \end{aligned}$$

$$< \|\check{\Psi}\|_R \sum_{j\geq\nu} \varepsilon_j^{\frac{4}{5}} \to 0, \quad \nu \to \infty.$$

We shall denote $\Psi_{\omega,\nu}$ the restriction of $\check{\Psi} \circ \Psi_\nu$ to $\mathcal{C}_\omega^R$. Moreover, recalling that $\check{\Psi}$ is tangent to identity and combining with (80), we have, for every $\omega \in \mathcal{O}_\infty(R)$,

$$\sup_{(\xi,\eta)\in\mathcal{C}_\omega^R} \|\Psi_\omega(\xi,\eta)\| \leq \|\check{\Psi}\|_R \left(\|\Psi_0\|_{\mathcal{O}_1,\beta_1,r_1} + \sum_{\nu=0}^{\infty} \|\Psi_{\nu+1} - \Psi_\nu\|_{\mathcal{O}_{\nu+2},\beta_{\nu+2},r_{\nu+2}} \right) < R^{\frac{1}{2}}, \tag{81}$$

which implies that $\Psi_\omega(\mathcal{C}_\omega^R) \subset \Delta_2(0, R^{\frac{1}{2}})$.

For $\omega \in \mathcal{O}_{\nu+1}(r_{\nu+1}, \beta_{\nu+1})$, we have

$$\Psi_\nu^{-1} \circ \check{\Psi}^{-1} \circ \tau_0^{(1)} \circ (\check{\Psi} \circ \Psi_\nu) = \tau_{\nu+1}^{(1)} = \begin{pmatrix} e^{\frac{\mathrm{i}}{2}\alpha_{\nu+1}(\xi\eta)}\xi + p_{\nu+1} \\ e^{-\frac{\mathrm{i}}{2}\alpha_{\nu+1}(\xi\eta)}\eta + q_{\nu+1} \end{pmatrix},$$

with $\|p_{\nu+1}\|_{\omega,\beta_{\nu+1},r_{\nu+1}}, \|q_{\nu+1}\|_{\omega,\beta_{\nu+1},r_{\nu+1}} < \frac{\varepsilon_{\nu+1}}{10}$ and $(\check{\Psi} \circ \Psi_\nu) \circ \rho = \rho \circ (\check{\Psi} \circ \Psi_\nu)$ implies that

$$\begin{aligned} \tau_{\nu+1}^{(2)} &= \rho \circ \tau_{\nu+1}^{(1)} \circ \rho = \Psi_\nu^{-1} \circ \check{\Psi}^{-1} \circ \tau_0^{(2)} \circ (\check{\Psi} \circ \Psi_\nu), \\ \sigma_{\nu+1} &= \tau_{\nu+1}^{(1)} \circ \tau_{\nu+1}^{(2)} = \Psi_\nu^{-1} \circ \check{\Psi}^{-1} \circ \sigma_0 \circ (\check{\Psi} \circ \Psi_\nu). \end{aligned}$$

Hence, for every $\omega \in \mathcal{O}_\infty(R) \subset \left]-R^2, R^2\right[$, the sequence $\{\alpha_\nu\}$ restricted to $\mathcal{C}_\omega^R$ converges to a real number $\mu_\omega = \alpha_\infty(\omega)$ with $\alpha_\infty := \alpha_0 + \sum_{\nu\geq 1}(\alpha_\nu - \alpha_{\nu-1})$. Indeed, since for any $\nu', \nu \in \mathbb{N}$ with $\nu' \geq \nu$,

$$|\alpha_\nu - \alpha_{\nu'}|_{\omega,0} \leq \sum_{j=\nu}^{\nu'} |\alpha_j - \alpha_{j+1}|_{\omega,\beta_{j+1}} < \sum_{j\geq\nu} \varepsilon_j^{\frac{1}{3}} \to 0, \quad \nu \to \infty,$$

it is a Cauchy sequence $\subset]-1, 4\pi+1[$. In particular, combining (51) and (79), we obtain

$$|\alpha_\infty - \lambda|_{\mathcal{O}_\infty(R)} < |\alpha_0 - \lambda|_{\mathcal{O}_0} + \sum_{\nu\geq 1} \|\alpha_\nu - \alpha_{\nu-1}\|_{\mathcal{O}_\nu,\beta_\nu,r_\nu} < \frac{1}{4} + \sum_{\nu\geq 1} \varepsilon_{\nu-1}^{\frac{1}{3}} < \frac{\pi}{4}.$$

Furthermore, we have

$$(\Psi_\omega^{-1} \circ \tau_0^{(1)} \circ \Psi_\omega)(\xi,\eta) = \begin{pmatrix} e^{\frac{\mathrm{i}}{2}\mu_\omega}\eta \\ e^{-\frac{\mathrm{i}}{2}\mu_\omega}\xi \end{pmatrix}, \quad (\Psi_\omega^{-1} \circ \sigma_0 \circ \Psi_\omega)(\xi,\eta) = \begin{pmatrix} e^{\mathrm{i}\mu_\omega}\xi \\ e^{-\mathrm{i}\mu_\omega}\eta \end{pmatrix}.$$

5.3 Whitney-smoothness of holomorphic hyperbolas

Now let us show the smoothness (in the sense of Whitney) of μ_ω and Ψ_ω with respect to $\omega \in \mathcal{O}_\infty = \mathcal{O}_\infty(R)$. We refer to [8, Chapter 6.1.4] for Whitney-smoothness notions.

Given $r \in \mathbb{R}_+ \setminus \mathbb{N}$, let $k_r := \lfloor r \rfloor = \max\{k \in \mathbb{Z} : k < r\}$. Given a closed set $\mathcal{O} \subset \mathbb{R}$, given a family $f = (f_l)_{0 \le l \le k_r} \in (C^0(\mathcal{O}))^{k_r+1}$, we define its *Whitney-C^r norm* to be:

$$|f|_{C^r_W(\mathcal{O})} := \sup_{\substack{\omega\in\mathcal{O}\\ 0\le l\le k_r}} |f_l(\omega)| + \sup_{\substack{(\omega,\omega')\in\mathcal{O}^2,\ \omega\neq\omega'\\ 0\le l\le k_r}} \frac{|f_l(\omega) - P_l(\omega,\omega')|}{|\omega-\omega'|^{r-l}},$$

where P_l is an analogue of the $(k_r - l) -$ th Taylor polynomial for f_l, i.e.,

$$P_l(\omega,\omega') := \sum_{j=0}^{k_r-l} \frac{1}{j!} f_{l+j}(y)(\omega-\omega')^j.$$

Such a family $f = (f_l)_{0 \le l \le k_r} \in (C^0(\mathcal{O}))^{k_r+1}$ is said to define a Whitney-C^r function f if its Whitney-C^r norm is finite:

$$|f|_{C^r_W(\mathcal{O})} < +\infty.$$

Given an open set $\mathcal{U}$ satisfying $\mathcal{O} \subset \mathcal{U} \subset \mathbb{R}$, for $g \in C^{k_r}(\mathcal{U})$, we define

$$|g|_{C^r(\mathcal{U})} := \sup_{\substack{\omega\in\mathcal{U}\\ 0\le l\le k_r}} |g^{(l)}(\omega)| + \sup_{\substack{(\omega,\omega')\in\mathcal{U}^2\\ \omega\neq\omega'}} \frac{|D^{k_r}g(\omega) - D^{k_r}g(\omega')|}{|\omega-\omega'|^{r-k_r}},$$

$$|g|_{C^r(\mathcal{O})} := \sup_{\substack{\omega\in\mathcal{O}\\ 0\le l\le k_r}} |g^{(l)}(\omega)| + \sup_{\substack{(\omega,\omega')\in\mathcal{O}^2\\ \omega\neq\omega'}} \frac{|D^{k_r}g(\omega) - D^{k_r}g(\omega')|}{|\omega-\omega'|^{r-k_r}}.$$

We have $|g|_{C^r(\mathcal{O})} \le |g|_{C^r(\mathcal{U})}$. According to Chapter 6.1.4 of [8], the norms $|\cdot|_{C^r_W(\mathcal{O})}$ and $|\cdot|_{C^r(\mathcal{O})}$ are equivalent. Given $A > 0$, this last norm can be extended to functions defined in a (complex) A-neighborhood $\mathcal{O} + A := \{z \in \mathbb{C} : |z-\omega| < A, \ \text{for some } \omega \in \mathcal{O}\}$ of $\mathcal{O}$ as well. Following Zehnder [49, (2.5), p. 109], Cauchy's estimates can be generalized to "derivatives of non-integer orders" of holomorphic functions g in a neighborhood $\mathcal{U}$ of $\mathcal{O}$ such that $|g|_{C^r(\mathcal{U})} < +\infty$: if $r' < r$ (not necessarily integers) and $A' < A$, then there exists some constant $C_{r,r'} > 1$ such that

$$|g|_{C^r(\mathcal{O}+A)} \le \frac{C_{r,r'}}{(A-A')^{r-r'}} |g|_{C^{r'}(\mathcal{O}+A')}. \tag{82}$$

Let us consider the sum $\alpha_\infty = \alpha_0 + \sum_{\nu\ge1}(\alpha_\nu - \alpha_{\nu-1})$, which converges in $C^{\tilde{s}}(\mathcal{O}_\infty)$ if $\tilde{s} \in \mathbb{R}_+ \setminus \mathbb{N}$ with $\tilde{s} < 16s$. Indeed, according to (51), we apply (82) to $\alpha_{\nu+1} - \alpha_\nu$,

$\nu \in \mathbb{N}$, with $r = \tilde{s}$, $r' = k_{\tilde{s}}$. Since $\beta_{\nu+1} < \beta_{\nu+1}^{\tilde{s}-k_{\tilde{s}}}$, there is some constant $C_{\tilde{s}}$ such that

$$|\alpha_{\nu+1} - \alpha_\nu|_{C^{\tilde{s}}(\mathcal{O}_\infty)} \le C_{\tilde{s}} \cdot \frac{\sup_{0\le l\le k_{\tilde{s}}} \|(\alpha_{\nu+1} - \alpha_\nu)^{(l)}\|_{\mathcal{O}_{\nu+1},\beta_{\nu+1},r_{\nu+1}}}{\beta_{\nu+1}} < C_{\tilde{s}} \varepsilon_\nu^{\frac{1}{3}-\frac{1}{32s}}.$$

Since $\frac{1}{3} - \frac{1}{32s} > 0$, according to (54), we have

$$\sum_{l\ge\nu} |\alpha_{l+1} - \alpha_l|_{C^{\tilde{s}}(\mathcal{O}_\infty)} \le C_{\tilde{s}} \sum_{l\ge\nu} \varepsilon_l^{\frac{1}{3}-\frac{1}{32s}} \le \frac{C_{\tilde{s}} \varepsilon_\nu^{\frac{1}{3}-\frac{1}{32s}}}{1 - \varepsilon_\nu^{\frac{32s-3}{8\cdot 96s}}} \to 0, \quad \nu \to \infty.$$

Hence, it is a Cauchy sequence in $C^{\tilde{s}}(\mathcal{O}_\infty)$. Furthermore, we have

$$|\alpha_\infty - \alpha_0|_{C^{\tilde{s}}_W(\mathcal{O}_\infty)} \le \sum_{\nu\ge1} |\alpha_\nu - \alpha_{\nu-1}|_{C^{\tilde{s}}_W(\mathcal{O}_\infty)} \le \frac{C_{\tilde{s}} \varepsilon_0^{\frac{1}{3}-\frac{1}{32s}}}{1 - \varepsilon_0^{\frac{32s-3}{8\cdot 96s}}}.$$

With fixed $\omega_0 \in \mathcal{O}_\infty$, let us consider the sets

$$U_{\omega_0} := \left\{\omega \in \mathcal{O}_\infty : \left|\frac{\omega}{\omega_0}\right| \le 2\right\}, \quad \mathcal{C}^{\frac{R}{2}}_{\omega_0} = \left\{(\xi, \eta) \in (\mathbb{C}^2, 0) : \xi\eta = \omega_0,\ |\xi|,\ |\eta| < \frac{R}{2}\right\}.$$

Let us define, for all $\omega \in U_{\omega_0}$, the map $\kappa_\omega : \mathcal{C}^{\frac{R}{2}}_{\omega_0} \to \mathcal{C}^R_\omega$ to be $\kappa_\omega(\xi, \eta) := (\frac{\omega}{\omega_0}\xi, \eta)$. We then define $\widetilde{\Psi}_{\omega,\nu} := \Psi_{\omega,\nu} \circ \kappa_\omega$ and $\widetilde{\Psi}_\omega := \Psi_\omega \circ \kappa_\omega$ on $\mathcal{C}^{\frac{R}{2}}_{\omega_0}$. With the same argument, using Lemma 5.3, we have

$$\begin{aligned}
&\sup_{(\xi,\eta)\in\mathcal{C}^{\frac{R}{2}}_{\omega_0}} |\widetilde{\Psi}_\omega(\xi, \eta) - \widetilde{\Psi}_{\omega,0}(\xi, \eta)|_{C^{\tilde{s}}_W(U_{\omega_0})} \\
&= \sup_{(\xi,\eta)\in\mathcal{C}^{\frac{R}{2}}_{\omega_0}} \left|\sum_{\nu\ge1} (\widetilde{\Psi}_{\omega,\nu}(\xi, \eta) - \widetilde{\Psi}_{\omega,\nu-1}(\xi, \eta))\right|_{C^{\tilde{s}}_W(U_{\omega_0})} \\
&\le C(\tilde{s}, \omega_0) \sum_{\nu\ge1} \frac{\|\Psi_\nu - \Psi_{\nu-1}\|_{\mathcal{O}_{\nu+1},\beta_{\nu+1},r_{\nu+1}}}{\beta_{\nu+1}} \\
&\le C(\tilde{s}, \omega_0) \sum_{\nu\ge1} \varepsilon_\nu^{\frac{4}{5}-\frac{1}{32s}} \le \frac{C(\tilde{s}, \omega_0)\varepsilon_0^{\frac{4}{5}-\frac{1}{32s}}}{1 - \varepsilon_0^{\frac{128s-5}{8\cdot 160s}}}.
\end{aligned}$$

Here, $C(\tilde{s}, \omega_0) > 0$ is some constant that depends only on $\tilde{s}$ and ω_0. Hence, $\tilde{\Psi}_\omega$ is Whitney smooth in ω and holomorphic on $\mathcal{C}^{\frac{R}{2}}_{\omega_0}$.

As a consequence, there is a $C^{\tilde{s}}$-Whitney smooth family of holomorphic invariant curves of τ_1^o, τ_2^o and σ_o, for $16s - 1 < \tilde{s} < 16s$.

6 Involutions and reversible maps

In this section, we describe some properties of a pair of germs of involutions τ_1, τ_2 as in (61) with $\alpha = \alpha(\xi\eta) \in \mathcal{A}^{\mathbb{R}}_{\beta,r}(\mathcal{O})$ satisfying (55)–(58) with $\mathcal{O}_\nu = \mathcal{O}$, together with $p, q \in \mathcal{A}_{\beta,r}(\mathcal{O})$ with $\|p\|_{\mathcal{O},\beta,r},\ \|q\|_{\mathcal{O},\beta,r} < \frac{\varepsilon}{10}$. Hence, according to Remark 4.6, if we take the involutions $(\tau_1, \tau_2) = (\tau_\nu^{(1)}, \tau_\nu^{(2)})$ for some $\nu \in \mathbb{N}$, described in Proposition 4.4, then the above assumptions are satisfied. We also consider the germ of map $\sigma = \tau_1 \circ \tau_2$.

Given $0 < r_+ < r$, recall that we have defined $r^{(m)}$ and $\tilde{r}$ in (60) between r and r_+:

$$r^{(m)} = r_+ + \frac{m}{8}(r - r_+), \quad m = 0, 1, \dots, 8, \qquad \tilde{r} = r^{(4)} = \frac{r + r_+}{2}. \tag{83}$$

We assume, from now on, that ε is sufficiently small such that

$$\left(\frac{|\ln \varepsilon|}{\left|\ln\left(\frac{7}{8} + \frac{r_+}{8r}\right)\right|} + 2 \right) \frac{(16s+1)^{16s} \varepsilon^{\frac{1}{2400s^2}}}{(r - r_+) r_+} < 1. \tag{84}$$

It is easy to see that (32) holds if we have

$$\varepsilon_\nu = \varepsilon, \quad r_\nu = r, \quad r_{\nu+1} = r_+.$$

6.1 Properties of $\alpha(\cdot)$

Recalling (22) and (59), we define

$$\beta \in [\varepsilon^{\frac{1}{40s}}, \varepsilon^{\frac{1}{60s}}], \quad \beta_+ = \beta^{\frac{5}{4}} \in [\varepsilon^{\frac{1}{32s}}, \varepsilon^{\frac{1}{48s}}], \quad \tilde{\beta} = 16\beta^{\frac{5}{4}}, \quad \mathcal{O}(r, \beta) = \mathcal{O} \cap] - r^2 + \beta, r^2 - \beta[.$$

The definitions are compatible with (30) if we take

$$\varepsilon = \varepsilon_\nu, \quad \beta = \beta_\nu, \quad \beta_+ = \beta_{\nu+1}, \quad \tilde{\beta} = \tilde{\beta}_\nu, \qquad \nu \in \mathbb{N}.$$

The smallness of ε in (84) implies that of β, β_+ and $\tilde{\beta}$, and we have

$$\beta_+ < \tilde{\beta} < \beta < \beta^{\frac{1}{32}} < 2^{-16}. \tag{85}$$

As it is needed below, we also have $r^{-1} < \beta^{-\frac{1}{32}}$ and

$$e^{\frac{9\tilde{\beta}}{8}} < 1 + \frac{9\tilde{\beta}}{8} \sum_{k \geq 0} \left(\frac{9\tilde{\beta}}{8} \right)^k = 1 + \frac{9\tilde{\beta}}{8} \frac{1}{1 - \frac{9\tilde{\beta}}{8}} < 1 + \frac{7\tilde{\beta}}{6}. \tag{86}$$

Then, according to (55)–(58), we have

Lemma 6.1 $|\alpha'|_{\mathcal{O}(r,\beta)} < \frac{11}{10}$ *and* $|\alpha^{(k)}|_{\mathcal{O}(r,\beta)} < \beta^{-\frac{1}{32}}$ *for* $2 \le k \le 16s$.

Remark 6.2 The non-degeneracy condition (56) is essential through this paper. Nevertheless, we only need the estimates of Lemma 6.1 in the rest of this section.

Lemma 6.3 *For every* $\omega \in \mathcal{O}(r, \beta)$, $|\alpha(\xi\eta) - \alpha(\omega)|_{\omega,\tilde{\beta}} < \frac{9}{8}\tilde{\beta}$ *and*

$$|\alpha^{(j)}(\xi\eta) - \alpha^{(j)}(\omega)|_{\omega,\tilde{\beta}} < \frac{\beta^{\frac{17}{16}}}{2}, \quad 1 \le j \le s.$$

Proof For $(\xi, \eta) \in \mathcal{C}_{\omega,\tilde{\beta}}$, we have $|\xi\eta - \omega| < \tilde{\beta}$. Developing $\alpha(\cdot)$ around ω,

$$|\alpha(\xi\eta) - \alpha(\omega)| \le \sum_{k\ge1} \frac{|\alpha^{(k)}(\omega)|}{k!}|\xi\eta - \omega|^k < \sum_{k\ge1} \frac{|\alpha^{(k)}(\omega)| \cdot \tilde{\beta}^k}{k!}.$$

According to (85), we have

$$\frac{\beta^{-\frac{1}{32}}\tilde{\beta}}{2(1-\tilde{\beta})} = \frac{16\beta^{\frac{5}{4}-\frac{1}{32}}}{2(1-\tilde{\beta})} < 10\beta^{\frac{5}{4}-\frac{1}{32}} < \frac{1}{90}.$$

Then, in view of Lemma 6.1, we have

$$\sum_{k=1}^{16s} \frac{|\alpha^{(k)}(\omega)| \cdot \tilde{\beta}^k}{k!} < \frac{11}{10}\tilde{\beta} + \frac{\beta^{-\frac{1}{32}}}{2}\sum_{k=2}^{16s}\tilde{\beta}^k < \frac{11}{10}\tilde{\beta} + \frac{\beta^{-\frac{1}{32}}\tilde{\beta}^2}{2(1-\tilde{\beta})} < \frac{10}{9}\tilde{\beta}. \tag{87}$$

Since $4\pi + 1 < 2^4$, Cauchy's inequality and (55) lead to

$$|\alpha^{(k)}(\omega)| \le k! \sup_{|z-w|=\frac{\beta}{2}} |\alpha(z)| \cdot \frac{2^k}{\beta^k} < \frac{k! \cdot 2^{k+4}}{\beta^k}, \quad k \ge 16s + 1. \tag{88}$$

Then, we obtain

$$\begin{aligned}\sum_{k\ge16s+1} \frac{|\alpha^{(k)}(\omega)| \cdot \tilde{\beta}^k}{k!} < \sum_{k\ge16s+1} \frac{2^{k+4}\tilde{\beta}^k}{\beta^k} &= 2^4 \sum_{k\ge16s+1} \left(2^5\beta^{\frac{1}{4}}\right)^k \\ &= \frac{2^4\left(2^5\beta^{\frac{1}{4}}\right)^{16s+1}}{1-2^5\beta^{\frac{1}{4}}}.\end{aligned}$$

According to (85), we have $\frac{1}{1-\beta^{\frac{1}{8}}} < \frac{4}{3}$ and

$$2^4\beta^{\frac{16s+1}{8}} = 2^4 \cdot \beta^{\frac{1}{8}} \cdot \beta^{2s} < 2^{-12}\beta^2 < \beta^2.$$

Therefore, we have

$$\sum_{k\geq 16s+1}\frac{|\alpha^{(k)}(\omega)|\cdot\tilde{\beta}^k}{k!}<\frac{2^4\left(2^5\beta^{\frac{1}{4}}\right)^{16s+1}}{1-2^5\beta^{\frac{1}{4}}}<\frac{2^4\beta^{\frac{16s+1}{8}}}{1-\beta^{\frac{1}{8}}}<\frac{4\beta^2}{3}<\frac{\tilde{\beta}}{72}. \tag{89}$$

Adding estimates (87) and (89), we obtain $|\alpha(\xi\eta)-\alpha(\omega)|_{\omega,\tilde{\beta}}<\frac{9}{8}\tilde{\beta}$.

For $(\xi,\eta)\in\mathcal{C}_{\omega,\tilde{\beta}}$, and $1\leq j\leq s$, by Lemma 6.1, we have,

$$\begin{aligned}|\alpha^{(j)}(\xi\eta)-\alpha^{(j)}(\omega)|\leq\sum_{k\geq 1}\frac{|\alpha^{(j+k)}(\omega)|}{k!}|\xi\eta-\omega|^k&<\frac{\beta^{-\frac{1}{32}}}{2}\sum_{k=1}^{16s-j}\tilde{\beta}^k\\&+\sum_{k\geq 16s+1-j}\frac{(j+k)!\cdot 2^{j+k+4}\tilde{\beta}^k}{k!\cdot\beta^{j+k}}.\end{aligned}$$

Note that under (84), for $k\geq 16s+1-j\geq 15s$,

$$\begin{aligned}\frac{(j+k)!\cdot 2^{j+k+4}\tilde{\beta}^k}{k!\cdot\beta^{j+k}}&<2^{j+5k+4}\beta^{\frac{k}{6}-j}\beta^{\frac{k}{15}}\\&\leq 2^{s+4+5k}\beta^{\frac{k}{24}}\cdot\beta^{\frac{k}{8}-j}\beta^{\frac{k}{15}}<\frac{1}{2}\beta^{\frac{k}{15}}.\end{aligned}$$

Hence, we have

$$|\alpha^{(j)}(\xi\eta)-\alpha^{(j)}(\omega)|\leq\frac{\beta}{2}\left(\frac{16\beta^{\frac{7}{32}}}{1-\tilde{\beta}}+\frac{\beta^{\frac{16(s-1)+2-j}{15}}}{1-\beta^{\frac{1}{15}}}\right)<\frac{\beta^{\frac{17}{16}}}{2}.$$

□

By Lemmas 6.1, 6.3 and (88), we have

Corollary 6.4 $\sup_{\omega\in\mathcal{O}(r,\beta)}|\alpha'(\xi\eta)|_{\omega,\tilde{\beta}}<\frac{6}{5}$,

$$\sup_{\omega\in\mathcal{O}(r,\beta)}|\alpha^{(k)}(\xi\eta)|_{\omega,\tilde{\beta}}<\begin{cases}2\beta^{-\frac{1}{32}} & 2\leq k\leq s \text{ if } s\geq 2\\ \frac{k!2^{k+5}}{\beta^k}, & k\geq s+1,\end{cases}.$$

Corollary 6.5 *For any $b\in\mathbb{R}$, and any $0\leq\beta'\leq\tilde{\beta}$, we have*

$$\sup_{\omega\in\mathcal{O}(r,\beta)}|e^{ib\alpha(\xi\eta)}|_{\omega,\beta'}<e^{\frac{9}{8}|b|\tilde{\beta}}.$$

Moreover, for $-1\leq b\leq 1$,

$$\sup_{\omega\in\mathcal{O}(r,\beta)}\left||e^{ib\alpha(\xi\eta)}|-1\right|_{\omega,\beta'}<\frac{5}{4}\tilde{\beta}.$$

Remark 6.6 By Corollary 6.5, for $b \in \mathbb{R}$ with $|b| \leq \beta^{-\frac{1}{4}}$, we have $\sup_{\omega\in\mathcal{O}(r,\beta)} |e^{\mathrm{i}b\alpha(\xi\eta)}|_{\omega,\beta'} < e^{\frac{9}{8}\beta^{-\frac{1}{4}}\cdot\tilde{\beta}} = e^{18\beta}$, for all $0 \leq \beta' \leq \tilde{\beta}$. Since $\beta < 2^{-16}$, we usually use the rough estimate in this paper for convenience:

$$\sup_{\omega\in\mathcal{O}(r,\beta)} |e^{ib\alpha(\xi\eta)}|_{\omega,\beta'} < \frac{101}{100}, \quad \forall\, b \in \mathbb{R} \text{ with } |b| \leq \beta^{-\frac{1}{4}}.$$

Proof of Corollary 6.5. Since $\alpha(\omega) \in \mathbb{R}$, $|e^{\mathrm{i}b\alpha(\omega)}| = 1$. Then $|e^{\mathrm{i}b\alpha(\xi\eta)}| = |e^{\mathrm{i}b(\alpha(\xi\eta)-\alpha(\omega))}|$. For $(\xi, \eta) \in \mathcal{C}_{\omega,\beta'}$, we have

$$e^{\mathrm{i}b(\alpha(\xi\eta)-\alpha(\omega))} = 1 + \sum_{k\geq 1} \frac{\mathrm{i}^k b^k}{k!}(\alpha(\xi\eta) - \alpha(\omega))^k.$$

If $|b| \leq 1$, then, under (84), $|b||\alpha(\xi\eta) - \alpha(\omega)| < \frac{9}{8}\tilde{\beta}$ is sufficiently small, and we have

$$\left||e^{\mathrm{i}b\alpha(\xi\eta)}| - 1\right| = \left||e^{\mathrm{i}b(\alpha(\xi\eta)-\alpha(\omega))}| - 1\right| \leq \left|\sum_{k\geq 1} \frac{\mathrm{i}^k b^k}{k!}(\alpha(\xi\eta) - \alpha(\omega))^k\right|$$
$$\leq \frac{\frac{9}{8}\tilde{\beta}}{1 - \frac{9}{8}\tilde{\beta}} < \frac{5}{4}\tilde{\beta}.$$

Moreover, for any $b \in \mathbb{R}$, any $0 \leq \beta' \leq \tilde{\beta}$,

$$\left|e^{\mathrm{i}b\alpha(\xi\eta)}\right|_{\omega,\beta'} \leq \left|\sum_{k\geq 0} \frac{\mathrm{i}^k b^k}{k!}(\alpha(\xi\eta) - \alpha(\omega))^k\right|_{\omega,\tilde{\beta}}$$
$$< \sum_{k\geq 0} \frac{|b|^k}{k!}\left(\frac{9}{8}\tilde{\beta}\right)^k = e^{\frac{9}{8}|b|\tilde{\beta}}.$$

□

Lemma 6.7 *Given* $0 < r' < r$*, assume* β *is sufficiently small such that*

$$\beta < r^2 - r'^2. \tag{90}$$

Given $0 \leq \beta' \leq \tilde{\beta}$*, and* $f \in \mathcal{A}_{\beta',r'}(\mathcal{O})$ *with*

$$\|f\|_{\mathcal{O},\beta',r'} < \beta^{24s}, \tag{91}$$

then $\alpha(\xi\eta + f) \in \mathcal{A}_{\beta',r'}(\mathcal{O})$ *with*

$$\|\alpha(\xi\eta + f) - \alpha(\xi\eta)\|_{\mathcal{O},\beta',r'} < \frac{5}{4}\|f\|_{\mathcal{O},\beta',r'}, \tag{92}$$

and for $-1 \le b \le 1$, $e^{\mathrm{i}b\alpha(\xi\eta+f)} \in \mathcal{A}_{\beta',r'}(\mathcal{O})$ *with*

$$\|e^{\mathrm{i}b\alpha(\xi\eta+f)} - e^{\mathrm{i}b\alpha(\xi\eta)}\|_{\mathcal{O},\beta',r'} < \frac{4}{3}\|f\|_{\mathcal{O},\beta',r'}. \tag{93}$$

Remark 6.8 Recalling $r^{(m)}$ given in (83), we see that, under the assumption (84), (90) is satisfied for

$$r = r^{(m+1)}, \quad r' = r^{(m)}, \qquad m = 0, 1, \ldots, 7.$$

Indeed, according to (84), we have

$$\begin{aligned}\beta \le \varepsilon^{\frac{1}{60s}} < \frac{(r-r_+)r_+}{(16s+1)^{16s}} < \frac{(r-r_+)(15r_+ + r)}{8} \\ = (r^{(1)})^2 - (r^{(0)})^2 \le (r^{(m+1)})^2 - (r^{(m)})^2.\end{aligned}$$

Proof Let $\varsigma := \|f\|_{\mathcal{O},\beta',r'}$. (91) implies that, for $(\xi,\eta) \in \mathcal{C}_{\omega,\beta'}$, $\omega \in \mathcal{O}(r',\beta')$,

$$|\xi\eta + f(\xi,\eta) - \omega| \le |\xi\eta - \omega| + \|f\|_{\mathcal{O},\beta',r'} \le \tilde{\beta} + \varsigma < \beta,$$

and (90) implies that $r'^2 - \beta' < r'^2 < r^2 - \beta$.

Developing $\alpha(\cdot)$ around $\xi\eta$, we obtain

$$\alpha(\xi\eta + f) - \alpha(\xi\eta) = \sum_{k\ge 1} \frac{\alpha^{(k)}(\xi\eta)}{k!} f^k. \tag{94}$$

By Lemma 3.3, for every $\omega \in \mathcal{O}(r',\beta')$, we have $\|f^k\|_{\omega,\beta',r'} \le \varsigma^k$ for $k \in \mathbb{N}^*$. Then, in view of Corollary 6.4, we have

$$\begin{aligned}&\|\alpha(\xi\eta+f) - \alpha(\xi\eta)\|_{\omega,\beta',r'} \\ &< \begin{cases} \frac{6}{5}\varsigma + 2\beta^{-\frac{1}{32}} \sum_{k=2}^{s} \frac{\varsigma^k}{k!} + 2^5 \sum_{k\ge s+1} \frac{2^k\varsigma^k}{\beta^k}, & s \ge 2 \\ \frac{6}{5}\varsigma + 2^5 \sum_{k\ge 2} \frac{2^k\varsigma^k}{\beta^k}, & s = 1 \end{cases},\end{aligned}$$

which can be bounded by $\frac{5}{4}\varsigma$ under assumption $\varsigma < \beta^{24s}$. Indeed,

- if $s = 1$, we have $\varsigma < \beta^{24s} = \beta^{24}$, then

$$2^5 \sum_{k\ge 2} \frac{2^k\varsigma^k}{\beta^k} < 2^5 \sum_{k\ge 2} \frac{2^k\varsigma^k}{\varsigma^{\frac{k}{24}}} = 2^5 \sum_{k\ge 2} \left(2\varsigma^{\frac{23}{24}}\right)^k = \frac{2^7\varsigma^{\frac{23}{12}}}{1 - 2\varsigma^{\frac{23}{24}}}.$$

- if $s \ge 2$, then, $\varsigma < \beta^{24s}$ implies that

$$2^5 \sum_{k\ge s+1} \frac{2^k\varsigma^k}{\beta^k} < 2^5 \sum_{k\ge s+1} \frac{2^k\varsigma^k}{\varsigma^{\frac{k}{24s}}} < 2^5 \sum_{k\ge 2} \left(2\varsigma^{1-\frac{1}{24s}}\right)^k = \frac{2^7\varsigma^{2-\frac{1}{12s}}}{1 - 2\varsigma^{1-\frac{1}{24s}}},$$

$$2\beta^{-\frac{1}{32}}\sum_{k=2}^{s}\frac{\varsigma^k}{k!}<\beta^{-\frac{1}{32}}\sum_{k=2}^{s}\varsigma^k<\frac{\varsigma^{-\frac{1}{24s\cdot 32}}\cdot\varsigma^2}{1-\varsigma}=\varsigma\cdot\frac{\varsigma^{1-\frac{1}{24s\cdot 32}}}{1-\varsigma}.$$

According to (85), we have

$$\varsigma<\varsigma^{1-\frac{1}{32\cdot 24s}}<\varsigma^{1-\frac{1}{24s}}<\varsigma^{1-\frac{1}{12s}}<\beta^{22}<2^{-16\cdot 32\cdot 22}.$$

Hence, we obtain these rough estimates

$$\frac{2^7\varsigma^{1-\frac{1}{12s}}}{1-2\varsigma^{1-\frac{1}{24s}}}<\frac{1}{200},\quad \frac{\varsigma^{1-\frac{1}{24s\cdot 32}}}{1-\varsigma}<\frac{1}{200},\quad \forall\, s\geq 1.$$

As a consequence, we have, for every $\omega\in\mathcal{O}(r',\beta')$, $\|\alpha(\xi\eta+f)-\alpha(\xi\eta)\|_{\omega,\beta',r'}\leq \frac{6}{5}\varsigma+\frac{\varsigma}{100}\leq\frac{5}{4}\varsigma$. By Lemmas 3.3 and 3.4, we see that $f^k\in\mathcal{A}_{\beta',r'}(\mathcal{O})$ for every $k\in\mathbb{N}$, and, in view of (23),

$$|(f^k)_{l,j}|_{\mathcal{O}(r',\beta')}\leq\|f\|^k_{\mathcal{O},\beta',r'}r^{-(l+j)}=\varsigma^k r^{-(l+j)}.$$

Hence, according to (94), $\alpha(\xi\eta+f)-\alpha(\xi\eta)\in\mathcal{A}_{\beta',r'}(\mathcal{O})$ with

$$\left(\alpha(\xi\eta+f)-\alpha(\xi\eta)\right)_{l,j}(\omega)=\sum_{k\geq 1}\frac{\alpha^{(k)}(\omega)}{k!}(f^k)_{l,j}(\omega),\quad \forall\, l,j\geq 0.$$

For $-1\leq b\leq 1$, we have

$$\begin{aligned}e^{\mathrm{i}b\alpha(\xi\eta+f)}-e^{\mathrm{i}b\alpha(\xi\eta)}&=e^{\mathrm{i}b\alpha(\xi\eta)}\left(e^{\mathrm{i}b(\alpha(\xi\eta+f)-\alpha(\xi\eta))}-1\right)\\&=e^{\mathrm{i}b\alpha(\xi\eta)}\sum_{k\geq 1}\frac{\mathrm{i}^k b^k}{k!}\left(\alpha(\xi\eta+f)-\alpha(\xi\eta)\right)^k.\end{aligned}$$

Then, by Lemma 3.3 and Remark 6.6, we obtain, for every $\omega\in\mathcal{O}(r',\beta')$,

$$\begin{aligned}\|e^{\mathrm{i}b\alpha(\xi\eta+f)}-e^{\mathrm{i}b\alpha(\xi\eta)}\|_{\omega,\beta',r'}&\leq|e^{\mathrm{i}b\alpha(\xi\eta)}|_{\omega,\beta'}\sum_{k\geq 1}\frac{1}{k!}\|\alpha(\xi\eta+f)-\alpha(\xi\eta)\|^k_{\omega,\beta',r'}\\&<\frac{101}{100}\sum_{k\geq 1}\frac{5^k\varsigma^k}{k!4^k}<\frac{4}{3}\varsigma,\end{aligned}$$

which implies (93). By Lemmas 3.4 and 3.5, we see that

$$e^{\mathrm{i}b\alpha(\xi\eta+f)}-e^{\mathrm{i}b\alpha(\xi\eta)}=\left(e^{\mathrm{i}b(\alpha(\xi\eta+f)-\alpha(\xi\eta))}-1\right)e^{\mathrm{i}b\alpha(\xi\eta)}\in\mathcal{A}_{\beta',r'}(\mathcal{O}).$$

□

Lemma 6.9 *Given* $0 < r'' < r' \le r < \frac{1}{4}$, *if* β *is sufficiently small such that*

$$8\beta^{\frac{1}{2}} < (r' - r'')r'', \quad e^{\frac{9}{8}\beta}\frac{r''}{r'} < 1 - \frac{\beta^2}{16}, \tag{95}$$

then for $-1 \le b \le 1$, *for* $0 < \beta' \le \tilde{\beta}$, $h \in \mathcal{A}_{\beta', r'}(\mathcal{O})$ *with* $\|h\|_{\mathcal{O},\beta',r'} < +\infty$, *we have* $h(e^{\mathrm{i}b\alpha(\xi\eta)}\xi, e^{-\mathrm{i}b\alpha(\xi\eta)}\eta) \in \mathcal{A}_{\beta'',r''}(\mathcal{O})$ *with*

$$\|h(e^{\mathrm{i}b\alpha(\xi\eta)}\xi, e^{-\mathrm{i}b\alpha(\xi\eta)}\eta)\|_{\mathcal{O},\beta'',r''} < \|h\|_{\mathcal{O},\beta',r'}.$$

Remark 6.10 Under the assumption (84), (95) is satisfied for

$$r' = r^{(m+1)}, \quad r'' = r^{(m)}, \qquad m = 0, 1, \ldots, 7. \tag{96}$$

Indeed, (84) implies that

$$\begin{aligned}\frac{1}{7} &> -\ln\left(\frac{7}{8} + \frac{r_+}{8r}\right) = \left|\ln\left(\frac{7}{8} + \frac{r_+}{8r}\right)\right| \\ &> \frac{(16s+1)^{16s}|\ln\varepsilon|\cdot\varepsilon^{\frac{1}{2400s^2}}}{(r-r_+)r_+} > 2\beta^{\frac{1}{32}} > \frac{5}{4}\beta. \end{aligned} \tag{97}$$

Then we have

$$8\beta^{\frac{1}{2}} < \frac{(r-r_+)r_+}{7(16s+1)^{16s}} \le \frac{8(r^{(m+1)} - r^{(m)})r^{(m)}}{7(16s+1)^{16s}} < (r^{(m+1)} - r^{(m)})r^{(m)},$$

$$e^{\frac{9}{8}\beta}\frac{r^{(7)}}{r^{(8)}} = e^{\frac{9}{8}\beta}\left(\frac{7}{8} + \frac{r_+}{8r}\right) < e^{-\frac{\beta}{8}} < 1 - \frac{\beta^2}{16}.$$

Hence we obtain (95) for the case (96) by noting that, for $0 \le m \le 6$,

$$\frac{r^{(m)}}{r^{(m+1)}} = \frac{8r_+ + m(r-r_+)}{8r_+ + (m+1)(r-r_+)} < \frac{8r_+ + (m+1)(r-r_+)}{8r_+ + (m+2)(r-r_+)} = \frac{r^{(m+1)}}{r^{(m+2)}}.$$

Proof In view of Corollary 6.5, we have, for every $\omega \in \mathcal{O}(r'', \beta'')$,

$$\begin{aligned}&\left\|h(e^{\mathrm{i}b\alpha}\xi, e^{-\mathrm{i}b\alpha}\eta)\right\|_{\omega,\beta'',r''} \\ &\le \left|h_{0,0}\right|_{\omega,\beta'} + \sum_{l\ge1}\left|e^{\mathrm{i}bl\alpha}h_{l,0}\right|_{\omega,\beta'} r''^l + \sum_{j\ge1}\left|e^{-\mathrm{i}bj\alpha}h_{0,j}\right|_{\omega,\beta'} r''^j \\ &\le \left|h_{0,0}\right|_{\omega,\beta'} + \sum_{l\ge1} e^{\frac{9}{8}l|b|\tilde{\beta}}\left|h_{l,0}\right|_{\omega,\beta'} r''^l + \sum_{j\ge1} e^{\frac{9}{8}j|b|\tilde{\beta}}\left|h_{0,j}\right|_{\omega,\beta'} r''^j \\ &\le \left|h_{0,0}\right|_{\omega,\beta'} + \sum_{l\ge1}\left(e^{\frac{9}{8}\tilde{\beta}}\frac{r''}{r'}\right)^l\left|h_{l,0}\right|_{\omega,\beta'} r'^l + \sum_{j\ge1}\left(e^{\frac{9}{8}\tilde{\beta}}\frac{r''}{r'}\right)^j\left|h_{0,j}\right|_{\omega,\beta'} r'^j\end{aligned}$$

$$< \|h\|_{\omega,\beta',r'}\,.$$

Noting that, for $l,\,j\geq 0$,

$$\left(h(e^{\mathrm{i}b\alpha(\xi\eta)}\xi,e^{-\mathrm{i}b\alpha(\xi\eta)}\eta)\right)_{l,j}(\omega)=h_{l,j}(\omega)e^{\mathrm{i}b(l-j)\alpha(\omega)},$$

we see that $h\left(e^{\mathrm{i}b\alpha(\xi\eta)}\xi,e^{-\mathrm{i}b\alpha(\xi\eta)}\eta\right)\in\mathcal{A}_{\beta'',r''}(\mathcal{O})$. □

Lemma 6.11 *Let* $0<r''<r'\leq r<\frac{1}{4}$ *and* $0<2\beta''\leq\beta'\leq\tilde{\beta}$. *If* β *is small enough such that* (95) *is satisfied, then for* $h\in\mathcal{A}_{\beta',r'}(\mathcal{O})$ *with* $\|h\|_{\mathcal{O},\beta',r'}<+\infty$, *for* $f_1,\,f_2,\,g_1,\,g_2$ *with*

$$\|f_m\|_{\mathcal{O},\beta'',r''},\,\|g_m\|_{\mathcal{O},\beta'',r''}<\frac{\beta'^2}{16},\quad m=1,2, \tag{98}$$

we have that, for $-1\leq b\leq 1$,

$$\begin{aligned}&\|h(e^{\mathrm{i}b\alpha(\xi\eta)}\xi+f_1,e^{-\mathrm{i}b\alpha(\xi\eta)}\eta+g_1)-h(e^{\mathrm{i}b\alpha(\xi\eta)}\xi+f_2,e^{-\mathrm{i}b\alpha(\xi\eta)}\eta+g_2)\|_{\mathcal{O},\beta'',r''}\\&<\frac{3r'\|h\|_{\mathcal{O},\beta',r'}}{(r'-r'')\beta'}\max\left\{\|f_1-f_2\|_{\mathcal{O},\beta'',r''},\|g_1-g_2\|_{\mathcal{O},\beta'',r''}\right\}.\end{aligned}$$

Moreover, if $f_1,\,f_2,\,g_1,\,g_2\in\mathcal{A}_{\beta'',r''}(\mathcal{O})$, *then*

$$h(e^{\mathrm{i}b\alpha(\xi\eta)}\xi+f_1,e^{-\mathrm{i}b\alpha(\xi\eta)}\eta+g_1)-h(e^{\mathrm{i}b\alpha(\xi\eta)}\xi+f_2,e^{-\mathrm{i}b\alpha(\xi\eta)}\eta+g_2)\in\mathcal{A}_{\beta'',r''}(\mathcal{O})\,.$$

Remark 6.12 We deduce Lemma 3.6 from Lemma 6.11 by taking $b=0$, with (27) verified by (95).

We postpone the detailed proof of Lemma 6.11 in Appendix A.

6.2 Properties of the perturbation

Recall that τ_1 and τ_2 are given by (61) with $p,q\in\mathcal{A}_{\beta,r}(\mathcal{O})$ satisfying (62).

Lemma 6.13 *For* $0\leq k\leq 16s$,

$$|(e^{-\frac{\mathrm{i}}{2}\alpha}p_{0,1})^{(k)}|_{\mathcal{O}(r^{(7)},\tilde{\beta})},\ |(e^{\frac{\mathrm{i}}{2}\alpha}\bar{p}_{0,1})^{(k)}|_{\mathcal{O}(r^{(7)},\tilde{\beta})}<\frac{\varepsilon^{\frac{1}{3}}}{10}. \tag{99}$$

Proof By (19), Corollary 6.5 and the assumption $\|p\|_{\mathcal{O},\beta,r}<\frac{\varepsilon}{10}$, we see that

$$\sup_{|z-\omega|<\beta}|(e^{-\frac{\mathrm{i}}{2}\alpha}p_{0,1})(z)|<\frac{101\varepsilon}{1000r},\quad\omega\in\mathcal{O}(r^{(7)},\tilde{\beta}).$$

Hence (99) is true for $k = 0$. According to (84) and (86), we have

$$\frac{(16s)!e^{\frac{9}{16}\tilde{\beta}}\varepsilon^{\frac{1}{2}}}{10r} < \varepsilon^{\frac{1}{2}-\frac{1}{1920}} \cdot \frac{101(16s+1)^{16s}\varepsilon^{\frac{1}{1920s}}}{1000r_+} < \varepsilon^{\frac{1}{2}-\frac{1}{1920}} < \frac{\varepsilon^{\frac{1}{3}}}{10}.$$

Indeed, according to (84), we have

$$\frac{\varepsilon^{\frac{1}{1920s}}}{r_+} < \left|\ln\left(\frac{7}{8}+\frac{r_+}{8r}\right)\right| (16s+1)^{-16s}(r-r_+) < \frac{r-r_+}{7(16s+1)^{16s}} < 1.$$

Hence, $\varepsilon^{\frac{1}{6}-\frac{1}{1920}} < \varepsilon^{\frac{1}{1920s}} < r_+$. Then, applying Cauchy's inequality and recalling that $\tilde{\beta} \in [16\varepsilon^{\frac{1}{32s}}, 16\varepsilon^{\frac{1}{48s}}]$, we have, for $1 \le k \le 16s$, $\omega \in \mathcal{O}(r^{(7)}, \tilde{\beta})$,

$$|(e^{-\frac{\mathrm{i}}{2}\alpha} p_{0,1})^{(k)}(\omega)| \le k! \cdot \frac{\sup_{|z-\omega|=\tilde{\beta}} |(e^{-\frac{\mathrm{i}}{2}\alpha} p_{0,1})(z)|}{16^k \varepsilon^{\frac{k}{32s}}} < \frac{(16s)!e^{\frac{9}{16}\tilde{\beta}}\varepsilon^{\frac{1}{2}}}{10r} < \frac{\varepsilon^{\frac{1}{3}}}{10}.$$

The proof for $e^{\frac{\mathrm{i}}{2}\alpha}\bar{p}_{0,1}$ is similar. □

For $\sigma = \tau_1 \circ \tau_2$, we have

$$\sigma(\xi,\eta) = \begin{pmatrix} \exp\left\{\frac{\mathrm{i}}{2}\left(\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q}) + \alpha(\xi\eta)\right)\right\}\xi \\ + \exp\left\{\frac{\mathrm{i}}{2}\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q})\right\}\bar{q}(\xi,\eta) \\ + p(e^{-\frac{\mathrm{i}}{2}\alpha}\eta + \bar{p}, e^{\frac{\mathrm{i}}{2}\alpha}\xi + \bar{q}) \\ \exp\left\{-\frac{\mathrm{i}}{2}\left(\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q}) + \alpha(\xi\eta)\right)\right\}\eta \\ + \exp\left\{-\frac{\mathrm{i}}{2}\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q})\right\}\bar{p}(\xi,\eta) \\ + q(e^{-\frac{\mathrm{i}}{2}\alpha}\eta + \bar{p}, e^{\frac{\mathrm{i}}{2}\alpha}\xi + \bar{q}) \end{pmatrix}$$
$$=: \begin{pmatrix} e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi + f(\xi,\eta) \\ e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta + g(\xi,\eta) \end{pmatrix}.$$

Hence, we have

$$f = \left(\exp\left\{\frac{\mathrm{i}}{2}\left(\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q}) - \alpha(\xi\eta)\right)\right\} - 1\right) e^{\mathrm{i}\alpha}\xi$$
$$+ \exp\left\{\frac{\mathrm{i}}{2}\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q})\right\}\bar{q} + p(e^{-\frac{\mathrm{i}}{2}\alpha}\eta + \bar{p}, e^{\frac{\mathrm{i}}{2}\alpha}\xi + \bar{q}), \tag{100}$$

$$g = \left(\exp\left\{-\frac{\mathrm{i}}{2}\left(\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q}) - \alpha(\xi\eta)\right)\right\} - 1\right) e^{-\mathrm{i}\alpha}\eta$$
$$+ \exp\left\{-\frac{\mathrm{i}}{2}\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q})\right\}\bar{p} + q(e^{-\frac{\mathrm{i}}{2}\alpha}\eta + \bar{p}, e^{\frac{\mathrm{i}}{2}\alpha}\xi + \bar{q}). \tag{101}$$

Lemma 6.14 *$f, g \in \mathcal{A}_{\tilde{\beta},r^{(7)}}(\mathcal{O})$ with*

$$\left\| f - \frac{\mathrm{i}\alpha'(\xi\eta)}{2}(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p})e^{\mathrm{i}\alpha}\xi - e^{\frac{\mathrm{i}}{2}\alpha}\bar{q} - p(e^{-\frac{\mathrm{i}}{2}\alpha}\eta, e^{\frac{\mathrm{i}}{2}\alpha}\xi)\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{80}, \tag{102}$$

$$\left\| g + \frac{\mathrm{i}\alpha'(\xi\eta)}{2}(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p})e^{-\mathrm{i}\alpha}\eta - e^{-\frac{\mathrm{i}}{2}\alpha}\bar{p} - q(e^{-\frac{\mathrm{i}}{2}\alpha}\eta, e^{\frac{\mathrm{i}}{2}\alpha}\xi)\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{80}. \tag{103}$$

Proof According to Lemmas 6.7 and 6.11, we see that $p(e^{-\frac{\mathrm{i}}{2}\alpha}\eta + \bar{p}, e^{\frac{\mathrm{i}}{2}\alpha}\xi + \bar{q})$, $q(e^{-\frac{\mathrm{i}}{2}\alpha}\eta+\bar{p}, e^{\frac{\mathrm{i}}{2}\alpha}\xi+\bar{q})$ and $\alpha(\xi\eta+e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q}+e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}+\bar{p}\bar{q})$ are elements of $\mathcal{A}_{\tilde{\beta},r^{(7)}}(\mathcal{O})$. Then, by Lemma 3.5,

$$\exp\left\{\pm\frac{\mathrm{i}}{2}\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q})\right\} \in \mathcal{A}_{\tilde{\beta},r^{(7)}}(\mathcal{O}),$$
$$\exp\left\{\pm\frac{\mathrm{i}}{2}\left(\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q}) - \alpha(\xi\eta)\right)\right\} \in \mathcal{A}_{\tilde{\beta},r^{(7)}}(\mathcal{O}).$$

In view of (100) and (101), combining with Lemma 3.4, we obtain that $f, g \in \mathcal{A}_{\tilde{\beta},r^{(7)}}(\mathcal{O})$.

For $(\xi,\eta) \in \mathcal{C}^{r^{(7)}}_{\omega,\tilde{\beta}}$, $\omega \in \mathcal{O}(r^{(7)}, \tilde{\beta})$, we have

$$\begin{aligned}
&\left(\exp\left\{\frac{\mathrm{i}}{2}(\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q}) - \alpha(\xi\eta))\right\} - 1\right)e^{\mathrm{i}\alpha}\xi \\
&= e^{\mathrm{i}\alpha}\xi\sum_{k\geq 1}\frac{\mathrm{i}^k}{2^k k!}\left(\alpha(\xi\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q}) - \alpha(\xi\eta)\right)^k \\
&= \frac{\mathrm{i}}{2}\alpha'(\xi\eta)\left(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}\right)e^{\mathrm{i}\alpha}\xi \qquad (104)\\
&\quad + \frac{\mathrm{i}}{2}\alpha'(\xi\eta)e^{\mathrm{i}\alpha}\xi\cdot\bar{p}\bar{q} + \frac{\mathrm{i}e^{\mathrm{i}\alpha}\xi}{2}\sum_{j\geq 2}\frac{\alpha^{(j)}(\xi\eta)}{j!}(e^{\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{-\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q})^j \qquad (105)\\
&\quad + e^{\mathrm{i}\alpha}\xi\sum_{k\geq 2}\frac{\mathrm{i}^k}{2^k k!}\left(\alpha(\xi\eta + e^{\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{-\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q}) - \alpha(\xi\eta)\right)^k. \qquad (106)
\end{aligned}$$

Since $\|p\|_{\mathcal{O},\beta,r}, \|q\|_{\mathcal{O},\beta,r} < \frac{\varepsilon}{10}$, by Corollary 6.5 and (86), we obtain

$$\|e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + \bar{p}\bar{q}\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{e^{\frac{9}{8}\tilde{\beta}}r^{(7)}\varepsilon}{10} + \frac{\varepsilon^2}{100} < \frac{\varepsilon}{10}.$$

Hence, applying Corollary 6.4 and Lemma 3.4, we have, if $s \geq 2$, then

$$\begin{aligned}&\left\| \frac{\mathrm{i}e^{\mathrm{i}\alpha}\xi}{2}\sum_{j\geq 2}\frac{\alpha^{(j)}(\xi\eta)}{j!}(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q}+e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}+\bar{p}\bar{q})^j\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\\&\leq r^{(7)}e^{\frac{9}{8}\tilde{\beta}}\cdot\beta^{-\frac{1}{32}}\sum_{j=2}^{s}\frac{1}{j!}\left(\frac{\varepsilon}{10}\right)^j+\frac{r^{(7)}e^{\frac{9}{8}\tilde{\beta}}}{2}\sum_{j\geq s+1}\frac{2^{j+5}\varepsilon^j}{\beta^j 10^j}<\frac{\varepsilon^{\frac{63}{32}}}{100}.\end{aligned}$$

Otherwise, for $s = 1$, we have

$$\begin{aligned}&\left\| \frac{\mathrm{i}e^{\mathrm{i}\alpha}\xi}{2}\sum_{j\geq 2}\frac{\alpha^{(j)}(\xi\eta)}{j!}(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q}+e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}+\bar{p}\bar{q})^j\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\\&\leq\frac{r^{(7)}e^{\frac{9}{8}\tilde{\beta}}}{2}\sum_{j\geq 2}\frac{2^{j+5}\varepsilon^j}{\beta^j 10^j}<\frac{\varepsilon^{\frac{31}{16}}}{200}.\end{aligned}$$

Then, the $\|\cdot\|_{\mathcal{O},\tilde{\beta},r^{(7)}}$-norm of terms in (105) is bounded by

$$\frac{\varepsilon^{\frac{31}{16}}}{200}+\frac{3r^{(7)}e^{\frac{9}{8}\tilde{\beta}}}{5}\left(\frac{\varepsilon}{10}\right)^2<\frac{\varepsilon^{\frac{31}{16}}}{160}. \tag{107}$$

Applying Lemma 6.7, we obtain

$$\|\alpha(\xi\eta+e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q}+e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}+\bar{p}\bar{q})-\alpha(\xi\eta)\|_{\mathcal{O},\tilde{\beta},r^{(7)}}<\frac{\varepsilon}{8}.$$

Hence the $\|\cdot\|_{\mathcal{O},\tilde{\beta},r^{(7)}}$-norm of terms (106) is bounded by

$$r^{(7)}e^{\frac{9}{8}\tilde{\beta}}\sum_{k\geq 2}\frac{\varepsilon^k}{k!2^k8^k}<\frac{\varepsilon^2}{100}. \tag{108}$$

Since $\|p\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\leq\|p\|_{\mathcal{O},\beta,r}<\frac{\varepsilon}{10}$, by Lemma 6.11, we have

$$\begin{aligned}&\|p(e^{-\frac{\mathrm{i}}{2}\alpha}\eta+\bar{p},e^{\frac{\mathrm{i}}{2}\alpha}\xi+\bar{q})-p(e^{-\frac{\mathrm{i}}{2}\alpha}\eta,e^{\frac{\mathrm{i}}{2}\alpha}\xi)\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\\&<\frac{3r}{(r-r^{(7)})\tilde{\beta}}\max\{\|p\|_{\mathcal{O},\tilde{\beta},r^{(7)}},\|q\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\}\cdot\|p\|_{\mathcal{O},\beta,r}\\&\leq\frac{3r\varepsilon^2}{100\cdot 16(r-r^{(7)})\varepsilon^{\frac{1}{32s}}}<\frac{\varepsilon^{\frac{31}{16}}}{200}.\end{aligned} \tag{109}$$

Moreover, by (93) in Lemma 6.7, we have

$$\begin{aligned}
&\left\|\exp\left\{\frac{\mathrm{i}}{2}\alpha(\xi\eta+e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q}+e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}+\bar{p}\bar{q})\right\}\bar{q}-e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\bar{q}\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\\
&\leq\left\|\exp\left\{\frac{\mathrm{i}}{2}\alpha(\xi\eta+e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q}+e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}+\bar{p}\bar{q})\right\}-e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\|q\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\\
&<\frac{4}{3}\cdot\frac{\varepsilon}{10}\cdot\frac{\varepsilon}{10}=\frac{\varepsilon^2}{75}.
\end{aligned}\tag{110}$$

Hence, (102) is shown by combining (107)–(110). The proof for (103) is similar. □

Corollary 6.15 $\|f\|_{\mathcal{O},\tilde{\beta},r^{(7)}},\ \|g\|_{\mathcal{O},\tilde{\beta},r^{(7)}}<\frac{\varepsilon}{4}$.

Proof Lemma 6.9 implies that

$$\|p(e^{-\frac{\mathrm{i}}{2}\alpha}\eta,e^{\frac{\mathrm{i}}{2}\alpha}\xi)\|_{\mathcal{O},\tilde{\beta},r^{(7)}},\ \|q(e^{-\frac{\mathrm{i}}{2}\alpha}\eta,e^{\frac{\mathrm{i}}{2}\alpha}\xi)\|_{\mathcal{O},\tilde{\beta},r^{(7)}}<\frac{\varepsilon}{10}.$$

Moreover, we have

$$\begin{aligned}
&\left\|\frac{\mathrm{i}\alpha'(\xi\eta)}{2}(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q}+e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p})e^{\mathrm{i}\alpha}\xi\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}},\\
&\quad\left\|\frac{\mathrm{i}\alpha'(\xi\eta)}{2}(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q}+e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p})e^{-\mathrm{i}\alpha}\eta\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}}<\frac{3e^{2\tilde{\beta}}r^2\varepsilon}{50},\\
&\|e^{\frac{\mathrm{i}}{2}\alpha}\bar{q}\|_{\mathcal{O},\tilde{\beta},r^{(7)}},\ \|e^{-\frac{\mathrm{i}}{2}\alpha}\bar{p}\|_{\omega,\tilde{\beta},r^{(7)}}<\frac{e^{\frac{9}{16}\tilde{\beta}}\varepsilon}{10}.
\end{aligned}$$

By Lemma 6.14, the corollary is shown. □

Let $C(\xi,\eta):=\frac{\mathrm{i}}{2}\alpha'(\xi\eta)\left(e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta\bar{q}(\xi,\eta)+e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi\bar{p}(\xi,\eta)\right)$. Applying (19), we have

Corollary 6.16 *$C\in\mathcal{A}_{\tilde{\beta},r^{(7)}}(\mathcal{O})$ and we have*

$$\left\|f_{l,0}-e^{\frac{\mathrm{i}}{2}\alpha}\bar{q}_{l,0}-e^{\frac{\mathrm{i}}{2}l\alpha}p_{0,l}-e^{\mathrm{i}\alpha}C_{l-1,0}\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}}<\frac{\varepsilon^{\frac{31}{16}}}{80(r^{(7)})^l},\quad l\geq 2,\tag{111}$$

$$\left\|f_{0,j}-e^{\frac{\mathrm{i}}{2}\alpha}\bar{q}_{0,j}-e^{-\frac{\mathrm{i}}{2}j\alpha}p_{j,0}-(\xi\eta)e^{\mathrm{i}\alpha}C_{0,j+1}\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}}<\frac{\varepsilon^{\frac{31}{16}}}{80(r^{(7)})^j},\quad j\geq 0,\tag{112}$$

$$\left\|g_{l,0}-e^{-\frac{\mathrm{i}}{2}\alpha}\bar{p}_{l,0}-e^{\frac{\mathrm{i}}{2}l\alpha}q_{0,l}+(\xi\eta)e^{-\mathrm{i}\alpha}C_{l+1,0}\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}}<\frac{\varepsilon^{\frac{31}{16}}}{80(r^{(7)})^l},\quad l\geq 0,\tag{113}$$

$$\left\| g_{0,j} - e^{-\frac{i}{2}\alpha} \bar{p}_{0,j} - e^{-\frac{i}{2}j\alpha} q_{j,0} + e^{-i\alpha} C_{0,j-1} \right\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{80(r^{(7)})^j}, \quad j \geq 2. \tag{114}$$

Recall that $\tau_1 \circ \tau_1 = \mathrm{Id}$, which means that

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \exp\left\{\frac{i}{2}\left(\alpha(\xi\eta + e^{\frac{i}{2}\alpha}\eta q + e^{-\frac{i}{2}\alpha}\xi p + pq) - \alpha(\xi\eta)\right)\right\}\xi \\ + \exp\left\{\frac{i}{2}\alpha(\xi\eta + e^{\frac{i}{2}\alpha}\eta q + e^{-\frac{i}{2}\alpha}\xi p + pq)\right\} q \\ + p(e^{\frac{i}{2}\alpha}\eta + p, e^{-\frac{i}{2}\alpha}\xi + q), \\ \exp\left\{-\frac{i}{2}\left(\alpha(\xi\eta + e^{\frac{i}{2}\alpha}\eta q + e^{-\frac{i}{2}\alpha}\xi p + pq) - \alpha(\xi\eta)\right)\right\}\eta \\ + \exp\left\{-\frac{i}{2}\alpha(\xi\eta + e^{\frac{i}{2}\alpha}\eta q + e^{-\frac{i}{2}\alpha}\xi p + pq)\right\} p \\ + q(e^{\frac{i}{2}\alpha}\eta + p, e^{-\frac{i}{2}\alpha}\xi + q) \end{pmatrix}.$$

Then, similarly to Lemma 6.14, we have

Lemma 6.17 *We have*

$$\left\| \frac{i\alpha'(\xi\eta)}{2}(e^{\frac{i}{2}\alpha}\eta q + e^{-\frac{i}{2}\alpha}\xi p)\xi + e^{\frac{i}{2}\alpha} q + p(e^{\frac{i}{2}\alpha}\eta, e^{-\frac{i}{2}\alpha}\xi) \right\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{80}, \tag{115}$$

$$\left\| -\frac{i\alpha'(\xi\eta)}{2}(e^{\frac{i}{2}\alpha}\eta q + e^{-\frac{i}{2}\alpha}\xi p)\eta + e^{-\frac{i}{2}\alpha} p + q(e^{\frac{i}{2}\alpha}\eta, e^{-\frac{i}{2}\alpha}\xi) \right\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{80}. \tag{116}$$

Corollary 6.18 *We have, for $l, j \geq 1$*

$$\|(\xi\eta)(e^{\frac{i}{2}\alpha} q_{1,0} + e^{-\frac{i}{2}\alpha} p_{0,1})\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{r^{(7)}\varepsilon^{\frac{31}{16}}}{80}, \tag{117}$$

$$\left\|(\xi\eta)(e^{\frac{i}{2}\alpha} q_{l+1,0} + e^{-\frac{i}{2}(l+1)\alpha} p_{0,l+1}) + e^{-\frac{i}{2}\alpha} p_{l-1,0} + e^{-\frac{i}{2}(l-1)\alpha} q_{0,l-1}\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{40(r^{(7)})^{l-1}}, \tag{118}$$

$$\left\|(\xi\eta)(e^{-\frac{i}{2}\alpha} p_{0,j+1} + e^{\frac{i}{2}(j+1)\alpha} q_{j+1,0}) + e^{\frac{i}{2}\alpha} q_{0,j-1} + e^{\frac{i}{2}(j-1)\alpha} p_{j-1,0}\right\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{40(r^{(7)})^{j-1}}. \tag{119}$$

Proof Due to cancellation of terms, we have that

$$\eta\left(\frac{i\alpha'(\xi\eta)}{2}(e^{\frac{i}{2}\alpha}\eta q + e^{-\frac{i}{2}\alpha}\xi p)\xi + e^{\frac{i}{2}\alpha} q + p(e^{\frac{i}{2}\alpha}\eta, e^{-\frac{i}{2}\alpha}\xi)\right)$$
$$+\xi\left(-\frac{i\alpha'(\xi\eta)}{2}(e^{\frac{i}{2}\alpha}\eta q + e^{-\frac{i}{2}\alpha}\xi p)\eta + e^{-\frac{i}{2}\alpha} p + q(e^{\frac{i}{2}\alpha}\eta, e^{-\frac{i}{2}\alpha}\xi)\right)$$

$$= e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p + \eta p(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi) + \xi q(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi).$$

Hence, by (115) and (116),

$$\left\| e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p + \eta p(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi) + \xi q(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi) \right\|_{\mathcal{O},\tilde{\beta},r^{(7)}}$$
$$\leq 2r^{(7)} \cdot \frac{\varepsilon^{\frac{31}{16}}}{80} = \frac{r^{(7)}\varepsilon^{\frac{31}{16}}}{40}.$$

The corresponding coefficients under the decomposition (16) satisfy (117)–(119). □

Corollary 6.19 *We have*

$$\|e^{\frac{\mathrm{i}}{2}\alpha}q_{1,0} + e^{-\frac{\mathrm{i}}{2}\alpha}p_{0,1}\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{61}{32}}}{60r^{(7)}}, \tag{120}$$

$$\|e^{\frac{\mathrm{i}}{2}\alpha}\bar{q}_{1,0} + e^{\frac{\mathrm{i}}{2}\alpha}p_{0,1} - f_{1,0}\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{61}{32}}}{60r^{(7)}}, \tag{121}$$

$$\|e^{-\frac{\mathrm{i}}{2}\alpha}\bar{p}_{0,1} + e^{-\frac{\mathrm{i}}{2}\alpha}q_{1,0} - g_{1,0}\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{61}{32}}}{60r^{(7)}}. \tag{122}$$

Proof Note that (117) actually means that

$$\|(e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p)_{0,0}\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{r^{(7)}\varepsilon^{\frac{31}{16}}}{80}.$$

Hence we obtain (120), since in (115), the coefficients of the term $\xi^1\eta^0$ satisfies

$$\left\| \frac{\mathrm{i}\alpha'(\xi\eta)}{2}(e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p)_{0,0} + e^{\frac{\mathrm{i}}{2}\alpha}q_{1,0} + e^{-\frac{\mathrm{i}}{2}\alpha}p_{0,1} \right\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{80r^{(7)}}.$$

In (102), the coefficients of the term ξ satisfies

$$\left\| f_{1,0} - \frac{\mathrm{i}\alpha'(\xi\eta)}{2}(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p})_{0,0}e^{\mathrm{i}\alpha} - e^{\frac{\mathrm{i}}{2}\alpha}\bar{q}_{1,0} - e^{\frac{\mathrm{i}}{2}\alpha}p_{0,1} \right\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{80r^{(7)}},$$

then we have (121). The proof for (122) is similar by applying (103). □

6.3 The skew terms

In the KAM-like (or Newton) scheme stated in Sect. 4, we also consider the skew term $e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p$ of τ_1, and the skew term $e^{-\mathrm{i}\alpha}\eta f + e^{\mathrm{i}\alpha}\xi g$ of σ. By Lemma 3.4 and 6.14, we see that both skew terms belong to $\mathcal{A}_{\tilde{\beta},r^{(7)}}(\mathcal{O})$.

Lemma 6.20 *We have*

$$\|e^{-\mathrm{i}\alpha}\eta f + e^{\mathrm{i}\alpha}\xi g\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < 2\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r} + \frac{\varepsilon^{\frac{31}{16}}}{40}. \tag{123}$$

Proof According to (102) and (103), $e^{-\mathrm{i}\alpha}\eta f + e^{\mathrm{i}\alpha}\xi g$ can be approximated by

$$\begin{aligned}
&e^{-\mathrm{i}\alpha}\eta\left(\frac{\mathrm{i}\alpha'}{2}\left(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}\right)e^{\mathrm{i}\alpha}\xi + e^{\frac{\mathrm{i}}{2}\alpha}\bar{q} + p(e^{-\frac{\mathrm{i}}{2}\alpha}\eta, e^{\frac{\mathrm{i}}{2}\alpha}\xi)\right)\\
&\quad + e^{\mathrm{i}\alpha}\xi\left(-\frac{\mathrm{i}\alpha'}{2}\left(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}\right)e^{-\mathrm{i}\alpha}\eta + e^{-\frac{\mathrm{i}}{2}\alpha}\bar{p} + q(e^{-\frac{\mathrm{i}}{2}\alpha}\eta, e^{\frac{\mathrm{i}}{2}\alpha}\xi)\right)\\
&= e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p} + e^{-\frac{\mathrm{i}}{2}\alpha}\left(e^{-\frac{\mathrm{i}}{2}\alpha}\eta\right)p(e^{-\frac{\mathrm{i}}{2}\alpha}\eta, e^{\frac{\mathrm{i}}{2}\alpha}\xi) + e^{\frac{\mathrm{i}}{2}\alpha}\left(e^{\frac{\mathrm{i}}{2}\alpha}\xi\right)q(e^{-\frac{\mathrm{i}}{2}\alpha}\eta, e^{\frac{\mathrm{i}}{2}\alpha}\xi),
\end{aligned}$$

up to an error smaller than $\frac{\varepsilon^{\frac{31}{16}}}{40}$. In view of (19) and (25), we have

$$\|e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}\|_{\mathcal{O},\tilde{\beta},r^{(7)}} = \|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\tilde{\beta},r^{(7)}} \le \|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r},$$

By (19) and Lemma 6.9, we have

$$\begin{aligned}
&\|e^{-\frac{\mathrm{i}}{2}\alpha}(e^{-\frac{\mathrm{i}}{2}\alpha}\eta)p(e^{-\frac{\mathrm{i}}{2}\alpha}\eta, e^{\frac{\mathrm{i}}{2}\alpha}\xi) + e^{\frac{\mathrm{i}}{2}\alpha}(e^{\frac{\mathrm{i}}{2}\alpha}\xi)q(e^{-\frac{\mathrm{i}}{2}\alpha}\eta, e^{\frac{\mathrm{i}}{2}\alpha}\xi)\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\\
&\le \|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}.
\end{aligned}$$

Then (123) is shown by combining the error smaller than $\frac{\varepsilon^{\frac{31}{16}}}{40}$. □

7 Transformations on crowns

Fix $s \in \mathbb{N}^*$, $0 < r_+ < r < \frac{1}{4}$, $0 < \varepsilon, \beta < r^2$ as in Sect. 6. In this section, we introduce two types of transformations on the "crown" $\mathcal{C}^r_{\omega,\beta}$, which will be used in the KAM-like scheme.

7.1 Product-preserving scaling transformation

Consider the map

$$\tau(\xi,\eta) = \begin{pmatrix} (e^{\frac{\mathrm{i}}{2}\theta(\xi\eta)} + A(\xi\eta))\eta + p(\xi,\eta)\\ (e^{\frac{\mathrm{i}}{2}\theta(\xi\eta)} + A(\xi\eta))^{-1}\xi + q(\xi,\eta)\end{pmatrix}, \quad (\xi,\eta)\in\mathcal{C}^{r'}_{\omega,\beta'}, \tag{124}$$

with $0 \le \beta' < \tilde{\beta}$ and $0 < r' < (r^2-\beta)^{\frac{1}{2}}$, where $A = A(\xi\eta) \in \mathcal{A}_{\beta,r}(\mathcal{O})$ with $\|A\|_{\mathcal{O},\beta,r} < \frac{1}{16}$, $p, q \in \mathcal{A}_{\beta,r}(\mathcal{O})$, $\theta = \theta(\xi\eta) \in \mathcal{A}^{\mathbb{R}}_{\beta,r}(\mathcal{O})$, with $\theta(\omega) \in]-\frac{1}{2}, 4\pi + \frac{1}{2}[$

for $\omega \in \mathcal{O}(r,\beta)$ and $\|\theta\|_{\mathcal{O},\beta,r} < 4\pi+1$. Furthermore, we assume $|\theta^{(s)} - s!|_{\mathcal{O}(r,\beta)} < \frac{s!}{15}$ and

$$|\theta^{(k)}|_{\mathcal{O}(r,\beta)} < \begin{cases} \frac{1}{15}, & 1 \le k \le s-1 \text{ and } s \ge 2, \\ \frac{r^{-\frac{1}{2}}}{2}, & s+1 \le k \le 16s. \end{cases}$$

θ satisfies all the hypothesis of α in Sect. 6 (see (55)–(58) in Sect. 4). Hence all the lemmas and corollaries in Sect. 6.1 are applicable on θ. In particular, by Remark 6.6, we have

$$\|e^{\mathrm{i}b\theta}\|_{\mathcal{O},\beta',r'} < \frac{101}{100}, \quad \forall -\beta^{-\frac{1}{4}} \le b \le \beta^{-\frac{1}{4}}. \tag{125}$$

For $(\xi,\eta) \in \mathcal{C}^{r'}_{\omega,\beta'}$, $\omega \in \mathcal{O}(r',\beta')$, let

$$\Theta(\xi\eta) := \left((e^{\frac{\mathrm{i}}{2}\theta(\xi\eta)} + A(\xi\eta))(e^{-\frac{\mathrm{i}}{2}\theta(\xi\eta)} + \bar{A}(\xi\eta))\right)^{\frac{1}{4}} \tag{126}$$

be the fourth root, and let us set

$$\varphi : (\xi,\eta) \mapsto \left(\Theta(\xi\eta)\xi,\, \Theta^{-1}(\xi\eta)\eta\right). \tag{127}$$

It is easy to show that $\rho \circ \varphi = \varphi \circ \rho$.

Lemma 7.1 *For $k = \pm 1, \pm 2$, $\Theta^k \in \mathcal{A}^{\mathbb{R}}_{\beta',r'}(\mathcal{O})$ satisfies that*

$$\|\Theta^k - 1\|_{\mathcal{O},\beta',r'} < \frac{3|k|}{4}\|A\|_{\mathcal{O},\beta',r'}. \tag{128}$$

Proof Since $A \in \mathcal{A}_{\beta,r}(\mathcal{O})$ with $\|A\|_{\mathcal{O},\beta,r} < \frac{1}{16}$, by (125), we have that $e^{\frac{\mathrm{i}}{2}\theta}\bar{A} + e^{-\frac{\mathrm{i}}{2}\theta}A + A\bar{A} \in \mathcal{A}^{\mathbb{R}}_{\beta',r'}(\mathcal{O})$ with

$$\|e^{\frac{\mathrm{i}}{2}\theta}\bar{A} + e^{-\frac{\mathrm{i}}{2}\theta}A + A\bar{A}\|_{\mathcal{O},\beta',r'} < \frac{101}{50}\|A\|_{\mathcal{O},\beta,r} + \|A\|^2_{\mathcal{O},\beta,r} < \frac{21}{10}\|A\|_{\mathcal{O},\beta,r}.$$

For $k = \pm 1, \ \pm 2$, we have

$$\begin{aligned}
&\|(1 + e^{\frac{\mathrm{i}}{2}\theta}\bar{A} + e^{-\frac{\mathrm{i}}{2}\theta}A + A\bar{A})^{\frac{k}{4}} - 1\|_{\mathcal{O},\beta',r'} \\
&\le \frac{|k|}{4} \frac{\|e^{\frac{\mathrm{i}}{2}\theta}\bar{A} + e^{-\frac{\mathrm{i}}{2}\theta}A + A\bar{A}\|_{\mathcal{O},\beta',r'}}{\left(1 - \|e^{\frac{\mathrm{i}}{2}\theta}\bar{A} + e^{-\frac{\mathrm{i}}{2}\theta}A + A\bar{A}\|_{\mathcal{O},\beta',r'}\right)^{1-\frac{k}{4}}} \\
&\le \frac{|k|}{4} \cdot \frac{21}{10}\|A\|_{\mathcal{O},\beta,r} \cdot \left(1 - \frac{21}{10}\|A\|_{\mathcal{O},\beta,r}\right)^{-\frac{3}{2}} \\
&\le \frac{3|k|}{4}\|A\|_{\mathcal{O},\beta,r}.
\end{aligned}$$

By the expression of Θ in (126), we obtain that $\Theta^k \in \mathcal{A}^{\mathbb{R}}_{\beta',r'}(\mathcal{O})$ and (128) is satisfied. □

Proposition 7.2 *If* $\|A\|_{\mathcal{O},\beta,r}$ *satisfies that*

$$\|A\|_{\mathcal{O},\beta,r} < \frac{4(r-r')}{3r'}, \tag{129}$$

then φ*, as defined by* (127)*, satisfies* $\|\varphi - \mathrm{Id}\|_{\mathcal{O},\beta',r'} < \frac{3}{4}\|A\|_{\mathcal{O},\beta,r}$ *and, for every* $\omega \in \mathcal{O}(r',\beta')$*, we have* $\varphi(\mathcal{C}^{r'}_{\omega,\beta'}) \subset \mathcal{C}^{r}_{\omega,\beta}$*. Furthermore, there are* $\theta_+ \in \mathcal{A}^{\mathbb{R}}_{\beta',r'}(\mathcal{O})$*,* $p_+, q_+ \in \mathcal{A}_{\beta',r'}(\mathcal{O})$ *such that*

$$(\varphi^{-1}\circ\tau\circ\varphi)(\xi,\eta) = \begin{pmatrix} e^{\frac{\mathrm{i}}{2}\theta_+(\xi\eta)}\eta + p_+(\xi,\eta) \\ e^{-\frac{\mathrm{i}}{2}\theta_+(\xi\eta)}\xi + q_+(\xi,\eta)\end{pmatrix},$$

$$\theta_+(\xi\eta) - \theta(\xi\eta) = -\mathrm{i}(e^{-\frac{\mathrm{i}}{2}\theta(\xi\eta)}A(\xi\eta) - e^{\frac{\mathrm{i}}{2}\theta(\xi\eta)}\bar{A}(\xi\eta)), \tag{130}$$

$$\|p_+\|_{\mathcal{O},\beta',r'} < \left(1+\frac{3}{4}\|A\|_{\mathcal{O},\beta,r}\right)\|p\|_{\mathcal{O},\beta,r} + \|A\|^2_{\mathcal{O},\beta,r}, \tag{131}$$

$$\|q_+\|_{\mathcal{O},\beta',r'} < \left(1+\frac{3}{4}\|A\|_{\mathcal{O},\beta,r}\right)\|q\|_{\mathcal{O},\beta,r} + \|A\|^2_{\mathcal{O},\beta,r}. \tag{132}$$

Moreover, we have

$$\begin{aligned}&\|e^{-\frac{\mathrm{i}}{2}\theta_+}\xi p_+ + e^{-\frac{\mathrm{i}}{2}\theta_+}\eta q_+\|_{\mathcal{O},\beta',r'} \\ &\quad < \|e^{-\frac{\mathrm{i}}{2}\theta}\xi p + e^{\frac{\mathrm{i}}{2}\theta}\eta q\|_{\mathcal{O},\beta,r} + \|A\|_{\mathcal{O},\beta,r}(\|p\|_{\mathcal{O},\beta,r} + \|q\|_{\mathcal{O},\beta,r}) + \|A\|^2_{\mathcal{O},\beta,r}.\end{aligned} \tag{133}$$

Remark 7.3 At each KAM step, we always work with the involutions of the form (61) with $p, q \in \mathcal{A}_{\beta,r}(\mathcal{O})$ satisfying (62). After conjugacy by the KAM transformation, the new involution, denoted by $\tilde{\tau}_1$ (resp. $\tilde{\tau}_2 = \rho\circ\tilde{\tau}_1\circ\rho$), has the form

$$\tilde{\tau}_1(\xi,\eta) = \begin{pmatrix} \lambda(\xi\eta)\eta + \tilde{p}(\xi,\eta) \\ \lambda^{-1}(\xi\eta)\xi + \tilde{q}(\xi,\eta)\end{pmatrix},$$

with new perturbations $\tilde{p}$ and $\tilde{q}$ of much smaller size than p, q. Nevertheless, the new principal part usually does not satisfy $|\lambda(\omega)| = 1$ (but close to 1). Hence, an additional transformation φ (as in (127), see also (71)) is needed in order to preserve the same form as τ_1 in (61). This is similar to the role of Theorem 3.4 in [40] for the formal hyperbolic non-exceptional manifold case.

Before the proof of Proposition 7.2, we show the following lemma similar to Lemma 6.9.

Lemma 7.4 *For* $0 < r' < r$ *and* $h \in \mathcal{A}_{\beta,r}(\mathcal{O})$ *with* $\|h\|_{\mathcal{O},\beta,r} < +\infty$, *if* $\|A\|_{\mathcal{O},\beta,r}$ *is sufficiently small such that* (129) *is satisfied, then we have* $h(\Theta\xi, \Theta^{-1}\eta) \in \mathcal{A}_{\beta',r'}(\mathcal{O})$ *with*

$$\|h(\Theta\xi, \Theta^{-1}\eta)\|_{\mathcal{O},\beta',r'} < \|h\|_{\mathcal{O},\beta,r}. \tag{134}$$

Proof According to (129), we have $\left(1 + \frac{3}{4}|A|_{\omega,\beta'}\right)\frac{r'}{r} = 1 + \frac{\frac{3}{4}|A|_{\omega,\beta'}r' - (r-r')}{r} < 1$. Therefore, with the help of (128), we obtain for $\omega \in \mathcal{O}(r', \beta')$:

$$\begin{aligned}
\left\|h(\Theta\xi, \Theta^{-1}\eta)\right\|_{\omega,\beta',r'} &\le |h_{0,0}|_{\omega,\beta'} + \sum_{l\ge1}|h_{l,0}|_{\omega,\beta'}|\Theta|^l_{\omega,\beta'}r'^l + \sum_{j\ge1}|h_{0,j}|_{\omega,\beta'}|\Theta^{-1}|^j_{\omega,\beta'}r'^j \\
&< |h_{0,0}|_{\omega,\beta'} + \sum_{l\ge1}\left(1 + \frac{3}{4}|A|_{\omega,\beta'}\right)^l\left(\frac{r'}{r}\right)^l(|h_{l,0}|_{\omega,\beta'} + |h_{0,l}|_{\omega,\beta'})r^l \\
&< \|h\|_{\omega,\beta,r}.
\end{aligned}$$

Since for $l, j \ge 0$ with $lj = 0$ and $\omega \in \mathcal{O}(r', \beta')$, $\left(h(\Theta\xi, \Theta^{-1}\eta)\right)_{l,j}(\omega) = h_{l,j}(\omega) \cdot \Theta^{l-j}(\omega)$, we see that $h(\Theta\xi, \Theta^{-1}\eta) \in \mathcal{A}_{\beta',r'}(\mathcal{O})$. □

Proof of Proposition 7.2. With φ defined in (127), we have

$$\varphi^{-1}\circ\tau\circ\varphi = \begin{pmatrix}\Theta^{-2}(\xi\eta)(e^{\frac{i}{2}\theta(\xi\eta)} + A(\xi\eta))\eta \\ \Theta^2(\xi\eta)(e^{\frac{i}{2}\theta(\xi\eta)} + A(\xi\eta))^{-1}\xi\end{pmatrix} + \begin{pmatrix}\Theta^{-1}(\xi\eta)p(\Theta\xi, \Theta^{-1}\eta) \\ \Theta(\xi\eta)q(\Theta\xi, \Theta^{-1}\eta)\end{pmatrix}.$$

In view of (134), we obtain

$$\left\|p(\Theta\xi, \Theta^{-1}\eta)\right\|_{\mathcal{O},\beta',r'} < \|p\|_{\mathcal{O},\beta,r}, \quad \left\|q(\Theta\xi, \Theta^{-1}\eta)\right\|_{\mathcal{O},\beta',r'} < \|q\|_{\mathcal{O},\beta,r}.$$

Combining with (128) together with $B := 1 + \frac{3}{4}\|A\|_{\mathcal{O},\beta,r}$, we have

$$\left\|\Theta^{-1}p(\Theta\xi, \Theta^{-1}\eta)\right\|_{\mathcal{O},\beta',r'} < B\|p\|_{\mathcal{O},\beta,r}, \quad \left\|\Theta q(\Theta\xi, \Theta^{-1}\eta)\right\|_{\mathcal{O},\beta',r'} < B\|q\|_{\mathcal{O},\beta,r}.$$

By a direct computation, we have, for $(\xi, \eta) \in \mathcal{C}^{r'}_{\omega,\beta'}$, $\omega \in \mathcal{O}(r', \beta')$,

$$\begin{aligned}
\Theta^{-2}(\xi\eta)(e^{\frac{i}{2}\theta(\xi\eta)} + A(\xi\eta)) &= \frac{e^{\frac{i}{2}\theta(\xi\eta)} + A(\xi\eta)}{(e^{\frac{i}{2}\theta(\xi\eta)} + A(\xi\eta))^{\frac12}(e^{-\frac{i}{2}\theta(\xi\eta)} + \bar{A}(\xi\eta))^{\frac12}} \\
&= (e^{\frac{i}{2}\theta(\xi\eta)} + A(\xi\eta))^{\frac12}(e^{-\frac{i}{2}\theta(\xi\eta)} + \bar{A}(\xi\eta))^{-\frac12} \\
&= e^{\frac{i}{2}\theta(\xi\eta)}(1 + e^{-\frac{i}{2}\theta(\xi\eta)}A(\xi\eta))^{\frac12}(1 + e^{\frac{i}{2}\theta(\xi\eta)}\bar{A}(\xi\eta))^{-\frac12}, \\
\Theta^2(\xi\eta)(e^{\frac{i}{2}\theta(\xi\eta)} + A(\xi\eta))^{-1} &= e^{-\frac{i}{2}\theta(\xi\eta)}(1 + e^{-\frac{i}{2}\theta(\xi\eta)}A(\xi\eta))^{-\frac12}(1 + e^{\frac{i}{2}\theta(\xi\eta)}\bar{A}(\xi\eta))^{\frac12}.
\end{aligned}$$

According to (125) and (129), we see that $e^{-\frac{i}{2}\theta(\xi\eta)}A(\xi\eta)$ and $e^{\frac{i}{2}\theta(\xi\eta)}\bar{A}(\xi\eta)$ are small enough. Then, by the expansion of power series, we have

$$\ln\left((1 + e^{-\frac{i}{2}\theta(\xi\eta)}A(\xi\eta))^{\frac12}(1 + e^{\frac{i}{2}\theta(\xi\eta)}\bar{A}(\xi\eta))^{-\frac12}\right)$$

$$= \ln\left(1+\frac{1}{2}\left(e^{-\frac{\mathrm{i}}{2}\theta(\xi\eta)}A(\xi\eta)-e^{\frac{\mathrm{i}}{2}\theta(\xi\eta)}\bar{A}(\xi\eta)\right)+\tilde{\mathcal{P}}(\xi\eta))\right)$$
$$= \frac{1}{2}\left(e^{-\frac{\mathrm{i}}{2}\theta(\xi\eta)}A(\xi\eta)-e^{\frac{\mathrm{i}}{2}\theta(\xi\eta)}\bar{A}(\xi\eta)\right)+\mathcal{P}(\xi\eta),$$

where $\tilde{\mathcal{P}}, \mathcal{P}$ are sums of powers of $(e^{-\frac{\mathrm{i}}{2}\theta}A, e^{\frac{\mathrm{i}}{2}\theta}\bar{A})$ of order greater than 2, satisfying that

$$\|\tilde{\mathcal{P}}\|_{\mathcal{O},\beta',r'},\ \|\mathcal{P}\|_{\mathcal{O},\beta',r'} < \frac{1}{2}\|A\|^2_{\mathcal{O},\beta',r'}.$$

Now we define $\theta_+(\xi\eta) := \theta(\xi\eta) - \mathrm{i}(e^{-\frac{\mathrm{i}}{2}\theta(\xi\eta)}A(\xi\eta) - e^{\frac{\mathrm{i}}{2}\theta(\xi\eta)}\bar{A}(\xi\eta))$ as in (130), which belongs to $\mathcal{A}^{\mathbb{R}}_{\beta',r'}(\mathcal{O})$. Then we rewrite $(1+e^{-\frac{\mathrm{i}}{2}\theta}A)^{\frac{1}{2}}(1+e^{\frac{\mathrm{i}}{2}\theta}\bar{A})^{-\frac{1}{2}}$ as

$$(1+e^{-\frac{\mathrm{i}}{2}\theta}A)^{\frac{1}{2}}(1+e^{\frac{\mathrm{i}}{2}\theta}\bar{A})^{-\frac{1}{2}} = e^{\frac{\mathrm{i}}{2}(\theta_+-\theta)}e^{\mathcal{P}}.$$

Since (125) implies that

$$\|e^{-\frac{\mathrm{i}}{2}\theta}A - e^{\frac{\mathrm{i}}{2}\theta}\bar{A}\|_{\mathcal{O},\beta',r'} \le 2\|e^{-\frac{\mathrm{i}}{2}\theta}A\|_{\mathcal{O},\beta',r'} < \frac{101}{50}\|A\|_{\mathcal{O},\beta,r},$$

we have

$$\begin{aligned}
\left\|\Theta^{-2}\cdot(e^{\frac{\mathrm{i}}{2}\theta}+A)-e^{\frac{\mathrm{i}}{2}\theta_+}\right\|_{\mathcal{O},\beta',r'} &= \left\|e^{\frac{\mathrm{i}}{2}\theta}(1+e^{-\frac{\mathrm{i}}{2}\theta}A)^{\frac{1}{2}}(1+e^{\frac{\mathrm{i}}{2}\theta}\bar{A})^{-\frac{1}{2}}-e^{\frac{\mathrm{i}}{2}\theta_+}\right\|_{\mathcal{O},\beta',r'} \\
&= \left\|e^{\frac{\mathrm{i}}{2}\theta}e^{\frac{\mathrm{i}}{2}(\theta_+-\theta)}e^{\mathcal{P}}-e^{\frac{\mathrm{i}}{2}\theta_+}\right\|_{\mathcal{O},\beta',r'} \\
&\le \|e^{\frac{\mathrm{i}}{2}\theta}\|_{\mathcal{O},\beta',r'}\|e^{\frac{\mathrm{i}}{2}(\theta_+-\theta)}\|_{\mathcal{O},\beta',r'}\|e^{\mathcal{P}}-1\|_{\mathcal{O},\beta',r'} \\
&< \frac{101}{100}\left(1+\frac{101}{40}\|A\|_{\mathcal{O},\beta,r}\right)\cdot\frac{2}{3}\|A\|^2_{\mathcal{O},\beta',r'} \\
&< \|A\|^2_{\mathcal{O},\beta',r'}.
\end{aligned} \tag{135}$$

Similarly, we have

$$\left\|\Theta^2\cdot(e^{\frac{\mathrm{i}}{2}\theta}+A)^{-1}-e^{-\frac{\mathrm{i}}{2}\theta_+}\right\|_{\mathcal{O},\beta',r'} < \|A\|^2_{\mathcal{O},\beta,r}. \tag{136}$$

Then we obtain (131) and (132) by letting

$$\begin{pmatrix} p_+(\xi,\eta) \\ q_+(\xi,\eta)\end{pmatrix} := \begin{pmatrix} (\Theta^{-2}\cdot(e^{\frac{\mathrm{i}}{2}\theta}+A)-e^{\frac{\mathrm{i}}{2}\theta_+})\eta \\ (\Theta^2\cdot(e^{\frac{\mathrm{i}}{2}\theta}+A)^{-1}-e^{-\frac{\mathrm{i}}{2}\theta_+})\xi \end{pmatrix} + \begin{pmatrix} \Theta^{-1}\cdot p(\Theta\xi,\Theta^{-1}\eta) \\ \Theta\cdot q(\Theta\xi,\Theta^{-1}\eta)\end{pmatrix}.$$

Since $\|A\|_{\mathcal{O},\beta,r} \le \frac{1}{16}$, we see that $(e^{\frac{\mathrm{i}}{2}\theta}+A)^{\pm 1} \in \mathcal{A}_{\beta',r'}(\mathcal{O})$. According to Lemma 3.4, Lemma 7.1 and Lemma 7.4, we have that $p_+,\ q_+ \in \mathcal{A}_{\beta',r'}(\mathcal{O})$.

It remains to prove (133). With the above p_+ and q_+, we have that

$$\begin{aligned} &e^{-\frac{\mathrm{i}}{2}\theta_+}\xi p_+ + e^{\frac{\mathrm{i}}{2}\theta_+}\eta q_+ \\ &\quad = e^{-\frac{\mathrm{i}}{2}\theta_+}\cdot(\xi\eta)\left(\Theta^{-2}\cdot(e^{\frac{\mathrm{i}}{2}\theta}+A) - e^{\frac{\mathrm{i}}{2}\theta_+}\right) && (137)\\ &\qquad + e^{-\frac{\mathrm{i}}{2}\theta_+}\cdot(\xi\eta)\left(\Theta^{2}\cdot(e^{\frac{\mathrm{i}}{2}\theta}+A)^{-1} - e^{-\frac{\mathrm{i}}{2}\theta_+}\right) && (138)\\ &\qquad + e^{-\frac{\mathrm{i}}{2}\theta_+}\xi\cdot\Theta^{-1}\cdot p(\Theta\xi,\Theta^{-1}\eta) + e^{\frac{\mathrm{i}}{2}\theta_+}\eta\cdot\Theta\cdot q(\Theta\xi,\Theta^{-1}\eta). && (139)\end{aligned}$$

Since (130) implies $\|\theta_+ - \theta\|_{\mathcal{O},\beta',r'} < \frac{101}{50}\|A\|_{\mathcal{O},\beta,r}$, so that

$$\|e^{\mathrm{i}b\theta_+} - e^{\mathrm{i}b\theta}\|_{\mathcal{O},\beta',r'} < \frac{7}{2}|b|\|A\|_{\mathcal{O},\beta,r}, \quad -1 \le b \le 1,$$

which implies that $\|e^{\mathrm{i}b\theta_+}\|_{\mathcal{O},\beta',r'} < 2$, we see that, in view of (135) and (136), the sum of terms in (137) and (138) is smaller than $\|A\|^2_{\mathcal{O},\beta,r}$. Let us focus on (139), which equals to

$$\begin{aligned} &e^{-\frac{\mathrm{i}}{2}\theta_+}\cdot\Theta^{-1}\xi\cdot p(\Theta\xi,\Theta^{-1}\eta) + e^{\frac{\mathrm{i}}{2}\theta_+}\cdot\Theta\eta\cdot q(\Theta\xi,\Theta^{-1}\eta)\\ &\quad = e^{-\frac{\mathrm{i}}{2}\theta}\cdot(\Theta\xi)p(\Theta\xi,\Theta^{-1}\eta) + e^{\frac{\mathrm{i}}{2}\theta}\cdot(\Theta^{-1}\eta)q(\Theta\xi,\Theta^{-1}\eta)\\ &\qquad + (e^{-\frac{\mathrm{i}}{2}\theta_+}\Theta^{-1} - e^{-\frac{\mathrm{i}}{2}\theta}\Theta)\xi p(\Theta\xi,\Theta^{-1}\eta) + (e^{\frac{\mathrm{i}}{2}\theta_+}\Theta - e^{\frac{\mathrm{i}}{2}\theta}\Theta^{-1})\eta q(\Theta\xi,\Theta^{-1}\eta)\end{aligned}$$

Since Lemma 7.1 implies that $\|\Theta^{-1}\|_{\mathcal{O},\beta',r'} < 1 + \frac{3}{4}\|A\|_{\mathcal{O},\beta,r} < \frac{19}{16}$ and

$$\|\Theta - \Theta^{-1}\|_{\mathcal{O},\beta',r'} \le \|\Theta - 1\|_{\mathcal{O},\beta',r'} + \|\Theta^{-1} - 1\|_{\mathcal{O},\beta',r'} < \frac{3}{2}\|A\|_{\mathcal{O},\beta,r},$$

we obtain that

$$\begin{aligned} &\|e^{-\frac{\mathrm{i}}{2}\theta_+}\Theta^{-1} - e^{-\frac{\mathrm{i}}{2}\theta}\Theta\|_{\mathcal{O},\beta',r'}\\ &\quad \le \|(e^{-\frac{\mathrm{i}}{2}\theta_+} - e^{-\frac{\mathrm{i}}{2}\theta})\Theta^{-1}\|_{\mathcal{O},\beta',r'} + \|e^{-\frac{\mathrm{i}}{2}\theta}(\Theta^{-1} - \Theta)\|_{\mathcal{O},\beta',r'}\\ &\quad < \frac{19}{16}\cdot\frac{7}{4}\|A\|_{\mathcal{O},\beta,r} + \frac{101}{100}\cdot\frac{3}{2}\|A\|_{\mathcal{O},\beta,r} < 4\|A\|_{\mathcal{O},\beta,r}.\end{aligned}$$

Similarly, we have

$$\|e^{\frac{\mathrm{i}}{2}\theta_+}\Theta - e^{\frac{\mathrm{i}}{2}\theta}\Theta^{-1}\|_{\mathcal{O},\beta',r'} < 4\|A\|_{\mathcal{O},\beta,r}.$$

On the other hand, by Lemma 7.4, we see that

$$\|e^{-\frac{\mathrm{i}}{2}\theta}\cdot(\Theta\xi)p(\Theta\xi,\Theta^{-1}\eta) + e^{\frac{\mathrm{i}}{2}\theta}\cdot(\Theta^{-1}\eta)q(\Theta\xi,\Theta^{-1}\eta)\|_{\mathcal{O},\beta',r'} < \|e^{-\frac{\mathrm{i}}{2}\theta}\xi p + e^{\frac{\mathrm{i}}{2}\theta}\eta q\|_{\mathcal{O},\beta,r}.$$

Hence (139) is bounded by $\|e^{-\frac{\mathrm{i}}{2}\theta}\xi p + e^{\frac{\mathrm{i}}{2}\theta}\eta q\|_{\mathcal{O},\beta',r'} + \|A\|_{\mathcal{O},\beta,r}(\|p\|_{\mathcal{O},\beta,r} + \|q\|_{\mathcal{O},\beta,r})$. Combining with the errors in (137) and (138), (133) is shown. □

7.2 Approximated cohomological equations—Proof of Theorem 4.7

For $0 < r_+ < r < \frac{1}{4}$, let $\varepsilon > 0$ be sufficiently small such that (84) is satisfied, let β, β_+, $\tilde{\beta}$ be defined as in (59), and let $r^{(m)}$, $m = 0, 1, \ldots, 8$, be defined as in (60). Consider the holomorphic involution

$$\tau_1(\xi, \eta) = \begin{pmatrix} e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta + p(\xi, \eta) \\ e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi + q(\xi, \eta) \end{pmatrix}, \quad (\xi, \eta) \in \mathcal{C}^r_{\omega,\beta}, \quad \omega \in \mathcal{O}(r, \beta)$$

given as in Sect. 6 (same with that in Theorem 4.7), with $\|p\|_{\mathcal{O},\beta,r}, \|q\|_{\mathcal{O},\beta,r} < \frac{\varepsilon}{10}$.

The rest of this subsection is devoted to the proof of Theorem 4.7. The core of proof is the resolution of the approximated cohomological equations (see Lemma 7.5).

At first, we see that the definition of $\mathcal{O}_\delta$ in (63) implies that

$$|e^{\mathrm{i}n\alpha(\xi\eta)} - 1| \geq \frac{\delta}{2}, \quad \forall\, 0 < |n| \leq K + 1, \quad \forall\, (\xi, \eta) \in \mathcal{C}_{\omega,\tilde{\beta}}, \ \omega \in \mathcal{O}_\delta. \tag{140}$$

Indeed, recalling that $\tilde{\beta} = 16\beta_+$ and $\delta > 80\varepsilon^{\frac{1}{60s}}$, by (84), we have

$$K + 1 = \frac{|\ln \varepsilon|}{\left|\ln(r^{(7)}/r)\right|} + 1 = \frac{|\ln \varepsilon|}{\left|\ln(\frac{7}{8} + \frac{r_+}{8r})\right|} + 1 < \frac{\varepsilon^{-\frac{1}{2400s^2}}}{2}. \tag{141}$$

Hence, by Lemma 6.3, for $0 < |n| \leq K + 1$, we have

$$|n||\alpha(\xi\eta) - \alpha(\omega)|_{\omega,\tilde{\beta}} < (K + 1) \cdot 18\beta_+ \leq 18(K + 1)\varepsilon^{\frac{1}{48s}} < 20\varepsilon^{\frac{1}{60s}} < \frac{\delta}{4}.$$

In view of Remark 6.6, we have

$$\begin{aligned} |e^{\mathrm{i}n\alpha(\xi\eta)} - e^{\mathrm{i}n\alpha(\omega)}|_{\omega,\tilde{\beta}} &\leq e^{\frac{9}{8}|n|\tilde{\beta}}|e^{\mathrm{i}n(\alpha(\xi\eta)-\alpha(\omega))} - 1|_{\omega,\tilde{\beta}} \\ &\leq \frac{101}{100}\sum_{k\geq 1}\frac{|n|^k|\alpha(\xi\eta) - \alpha(\omega)|^k_{\omega,\tilde{\beta}}}{k!} < \frac{\delta}{2}, \end{aligned}$$

and hence, combining with (63), (140) is obtained.

Define p_K and $q_K \in \mathcal{A}_{\beta,r}(\mathcal{O})$ as

$$\begin{pmatrix} p_K \\ q_K \end{pmatrix} := \begin{pmatrix} p_{0,0}(\xi\eta) + \sum_{1\leq l\leq K} p_{l,0}(\xi\eta)\xi^l + \sum_{1\leq j\leq K} p_{0,j}(\xi\eta)\eta^j \\ q_{0,0}(\xi\eta) + \sum_{1\leq l\leq K} q_{l,0}(\xi\eta)\xi^l + \sum_{1\leq j\leq K} q_{0,j}(\xi\eta)\eta^j \end{pmatrix}. \tag{142}$$

Since $\|p\|_{\mathcal{O},\beta,r}, \|q\|_{\mathcal{O},\beta,r} < \frac{\varepsilon}{10}$, we have

$$\|p - p_K\|_{\mathcal{O},\tilde{\beta},r^{(7)}}, \ \|q - q_K\|_{\mathcal{O},\tilde{\beta},r^{(7)}} < \frac{\varepsilon}{10}\left(\frac{r^{(7)}}{r}\right)^{\frac{|\ln \varepsilon|}{\left|\ln(r^{(7)}/r)\right|}} = \frac{\varepsilon^2}{10}. \tag{143}$$

Lemma 7.5 *There is* $\hat{\mathcal{U}} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \in (\mathcal{A}^{\mathbb{R}}_{\tilde{\beta},r^{(7)}}(\mathcal{O}_\delta))^2$, *with* $\hat{u}_{1,0} = \hat{v}_{0,1} = 0$ *satisfying*

$$\left\|\hat{u}\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}, \left\|\hat{v}\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{49}{50}}}{20}, \tag{144}$$

$$\left\|\eta\hat{u} + \xi\hat{v}\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{61}{32}}}{16} + 5(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r} \tag{145}$$

such that

$$\begin{aligned}
&\|e^{\frac{\mathrm{i}}{2}\alpha}\hat{v} - \hat{u}(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi) + p_K - p_{0,1}\eta\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} \\
&\quad < \frac{\varepsilon^{\frac{61}{32}}}{80} + \frac{6(K+1)}{\delta}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}, \qquad (146)\\
&\|e^{-\frac{\mathrm{i}}{2}\alpha}\hat{u} - \hat{v}(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi) + q_K - q_{1,0}\xi\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} \\
&\quad < \frac{\varepsilon^{\frac{61}{32}}}{80} + \frac{6(K+1)}{\delta}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}, \qquad (147)\\
&\Big\|e^{-\frac{\mathrm{i}}{2}\alpha}\xi\left(e^{\frac{\mathrm{i}}{2}\alpha}\hat{v} - \hat{u}(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi) + p_K - p_{0,1}\eta\right) \\
&\qquad + e^{\frac{\mathrm{i}}{2}\alpha}\eta\left(e^{-\frac{\mathrm{i}}{2}\alpha}\hat{u} - \hat{v}(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi) + q_K - q_{1,0}\xi\right)\Big\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{61}{32}}}{20}. \qquad (148)
\end{aligned}$$

Remark 7.6 As mentioned in Sect. 2.1, the aim of the KAM-like process is to eliminate the main part of the perturbation and get a much smaller new perturbation.

In view of (143), we would like to construct $\hat{\mathcal{U}} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$ so that the change of variables $\mathrm{Id} + \hat{\mathcal{U}}$ eliminates p_K and q_K. This amounts to solve the cohomological equations (arising as the "linearized equation" of the conjugacy equation),

$$\begin{aligned}
e^{\frac{\mathrm{i}}{2}\alpha}\hat{v} - \hat{u}(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi) + p_K - p_{0,1}\eta = 0, \\
e^{-\frac{\mathrm{i}}{2}\alpha}\hat{u} - \hat{v}(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi) + q_K - q_{1,0}\xi = 0,
\end{aligned}$$

with $p_{0,1}\eta$ and $q_{1,0}\xi$ added to the new principal part. Based on the fact that τ_1 and τ_2 are involutions and $\tau_1 \circ \tau_2$ is reversible w.r.t. ρ, we solve the above equations approximately, with the errors estimated as in (146) and (147). Then, with the transformation $\mathrm{Id} + \hat{\mathcal{U}}$, the main parts of perturbations of τ_1, τ_2 and $\tau_1 \circ \tau_2$ are all eliminated approximately.

Proof Let us define $\hat{u}$, $\hat{v}$ by giving the coefficients: $\hat{u}_{1,0} = \hat{v}_{0,1} = 0$,

$$\hat{u}_{l,0}(\xi\eta) := \frac{1}{2} \cdot \frac{f_{l,0}(\xi\eta) - e^{\mathrm{i}(l+1)\alpha(\xi\eta)}\bar{f}_{l,0}(\xi\eta)}{e^{\mathrm{i}l\alpha(\xi\eta)} - e^{\mathrm{i}\alpha(\xi\eta)}}, \quad 2 \le l \le K, \tag{149}$$

$$\hat{u}_{0,j}(\xi\eta) := \frac{1}{2}\cdot\frac{f_{0,j}(\xi\eta) - e^{-\mathrm{i}(j-1)\alpha(\xi\eta)}\bar{f}_{0,j}(\xi\eta)}{e^{-\mathrm{i}j\alpha(\xi\eta)} - e^{\mathrm{i}\alpha(\xi\eta)}}, \quad 0 \le j \le K, \tag{150}$$

$$\hat{v}_{l,0}(\xi\eta) := \frac{1}{2}\cdot\frac{g_{l,0}(\xi\eta) - e^{\mathrm{i}(l-1)\alpha(\xi\eta)}\bar{g}_{l,0}(\xi\eta)}{e^{\mathrm{i}l\alpha(\xi\eta)} - e^{-\mathrm{i}\alpha(\xi\eta)}}, \quad 0 \le l \le K, \tag{151}$$

$$\hat{v}_{0,j}(\xi\eta) := \frac{1}{2}\cdot\frac{g_{0,j}(\xi\eta) - e^{-\mathrm{i}(j+1)\alpha(\xi\eta)}\bar{g}_{0,j}(\xi\eta)}{e^{-\mathrm{i}j\alpha(\xi\eta)} - e^{-\mathrm{i}\alpha(\xi\eta)}}, \quad 2 \le j \le K, \tag{152}$$

with other coefficients being 0. Here, f, g are defined by (100) and (101). In view of (149)–(152), we see that if $\hat{u}_{l,j} \neq 0$, then

$$\overline{\hat{u}}_{l,j}(\xi\eta) = \frac{1}{2}\cdot\frac{\bar{f}_{l,j}(\xi\eta) - e^{-\mathrm{i}(l-j+1)\alpha(\xi\eta)}f_{l,j}(\xi\eta)}{e^{-\mathrm{i}(l-j)\alpha(\xi\eta)} - e^{-\mathrm{i}\alpha(\xi\eta)}} = \hat{u}_{l,j}(\xi\eta),$$

and if $\hat{v}_{l,j} \neq 0$, then

$$\overline{\hat{v}}_{l,j}(\xi\eta) = \frac{1}{2}\cdot\frac{\bar{g}_{l,j}(\xi\eta) - e^{-\mathrm{i}(l-j-1)\alpha(\xi\eta)}g_{l,j}(\xi\eta)}{e^{-\mathrm{i}(l-j)\alpha(\xi\eta)} - e^{\mathrm{i}\alpha(\xi\eta)}} = \hat{v}_{l,j}(\xi\eta).$$

Hence, according to the definition of $\mathcal{O}_\delta$ in (63) and Lemma 6.14, $\hat{u}, \hat{v} \in \mathcal{A}^{\mathbb{R}}_{\tilde{\beta},r^{(7)}}(\mathcal{O}_\delta)$.

With the coefficients in (149)–(152), we have, for $lj = 0$, $l, j \le K$ and $(l, j) \neq (0, 1)$,

$$\begin{aligned} &e^{\frac{\mathrm{i}}{2}\alpha}\hat{v}_{l,j} - e^{\frac{\mathrm{i}}{2}(j-l)\alpha}\hat{u}_{j,l} \\ &\quad = \frac{e^{\frac{\mathrm{i}}{2}\alpha}}{2}\frac{g_{l,j} - e^{\mathrm{i}(l-j-1)\alpha}\bar{g}_{l,j}}{e^{\mathrm{i}(l-j)\alpha} - e^{-\mathrm{i}\alpha}} - \frac{e^{\frac{\mathrm{i}}{2}(j-l)\alpha}}{2}\frac{f_{j,l} - e^{\mathrm{i}(j-l+1)\alpha}\bar{f}_{j,l}}{e^{\mathrm{i}(j-l)\alpha} - e^{\mathrm{i}\alpha}}, \end{aligned} \tag{153}$$

and for $lj = 0$, $l, j \le K$, $(l, j) \neq (1, 0)$,

$$\begin{aligned} &e^{-\frac{\mathrm{i}}{2}\alpha}\hat{u}_{l,j} - e^{\frac{\mathrm{i}}{2}(j-l)\alpha}\hat{v}_{j,l} \\ &\quad = \frac{e^{-\frac{\mathrm{i}}{2}\alpha}}{2}\frac{f_{l,j} - e^{\mathrm{i}(l-j+1)\alpha}\bar{f}_{l,j}}{e^{\mathrm{i}(l-j)\alpha} - e^{\mathrm{i}\alpha}} - \frac{e^{\frac{\mathrm{i}}{2}(j-l)\alpha}}{2}\frac{g_{j,l} - e^{\mathrm{i}(j-l-1)\alpha}\bar{g}_{j,l}}{e^{\mathrm{i}(j-l)\alpha} - e^{-\mathrm{i}\alpha}}. \end{aligned} \tag{154}$$

According to Corollary 6.16, we have

$$C = \frac{\mathrm{i}}{2}\alpha'(\xi\eta)\left(e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi\bar{p}\right) \in \mathcal{A}_{\tilde{\beta},r^{(7)}}(\mathcal{O}_\delta).$$

Replacing the coefficients of f and g in (153) and (154) and according to (111)–(114), let us show that, for $lj = 0$ with $l, j \le K$,

$$\|e^{\frac{\mathrm{i}}{2}\alpha}\hat{v}_{l,j} - e^{\frac{\mathrm{i}}{2}(j-l)\alpha}\hat{u}_{j,l} + p_{l,j} - \hat{p}_{l,j}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} < \frac{\delta^{-1}\varepsilon^{\frac{31}{16}}}{16(r^{(7)})^{l+j}}, \quad (l, j) \neq (0, 1), \tag{155}$$

$$\|e^{-\frac{\mathrm{i}}{2}\alpha}\hat{u}_{l,j}-e^{\frac{\mathrm{i}}{2}(j-l)\alpha}\hat{v}_{j,l}+q_{l,j}-\hat{q}_{l,j}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\frac{\delta^{-1}\varepsilon^{\frac{31}{16}}}{16(r^{(7)})^{l+j}},\quad (l,j)\neq(1,0). \tag{156}$$

Here, $\hat{p},\hat{q}\in\mathcal{A}_{\tilde{\beta},r^{(7)}}(\mathcal{O}_\delta)$ with $\hat{p}_{l,j}$ and $\hat{q}_{l,j}$ defined by

$$\hat{p}_{l,0}:=(\xi\eta)\cdot\frac{-e^{-\frac{\mathrm{i}}{2}\alpha}C_{l+1,0}+e^{\mathrm{i}l\alpha}e^{\frac{\mathrm{i}}{2}\alpha}\bar{C}_{l+1,0}+e^{\frac{\mathrm{i}}{2}l\alpha}C_{0,l+1}-e^{-\frac{\mathrm{i}}{2}l\alpha}e^{-\mathrm{i}\alpha}\bar{C}_{0,l+1}}{2(e^{\mathrm{i}l\alpha}-e^{-\mathrm{i}\alpha})}$$
$$+\frac{1}{2}\left(p_{l,0}+e^{-\frac{\mathrm{i}}{2}(l-1)\alpha}q_{0,l}\right),\quad 0\le l\le K, \tag{157}$$

$$\hat{p}_{0,j}:=\frac{-e^{-\frac{\mathrm{i}}{2}\alpha}C_{0,j-1}+e^{-\mathrm{i}j\alpha}e^{\frac{\mathrm{i}}{2}\alpha}\bar{C}_{0,j-1}+e^{-\frac{\mathrm{i}}{2}j\alpha}C_{j-1,0}-e^{\frac{\mathrm{i}}{2}j\alpha}e^{-\mathrm{i}\alpha}\bar{C}_{j-1,0}}{2(e^{-\mathrm{i}j\alpha}-e^{-\mathrm{i}\alpha})}$$
$$+\frac{1}{2}\left(p_{0,j}+e^{\frac{\mathrm{i}}{2}(j+1)\alpha}q_{j,0}\right),\quad 2\le j\le K, \tag{158}$$

$$\hat{q}_{l,0}:=\frac{e^{\frac{\mathrm{i}}{2}\alpha}C_{l-1,0}-e^{\mathrm{i}l\alpha}e^{-\frac{\mathrm{i}}{2}\alpha}\bar{C}_{l-1,0}-e^{\frac{\mathrm{i}}{2}l\alpha}C_{0,l-1}+e^{-\frac{\mathrm{i}}{2}l\alpha}e^{\mathrm{i}\alpha}\bar{C}_{0,l-1}}{2(e^{\mathrm{i}l\alpha}-e^{\mathrm{i}\alpha})}$$
$$+\frac{1}{2}\left(q_{l,0}+e^{-\frac{\mathrm{i}}{2}(l+1)\alpha}p_{0,l}\right),\quad 2\le l\le K, \tag{159}$$

$$\hat{q}_{0,j}:=(\xi\eta)\cdot\frac{e^{\frac{\mathrm{i}}{2}\alpha}C_{0,j+1}-e^{-\mathrm{i}j\alpha}e^{-\frac{\mathrm{i}}{2}\alpha}\bar{C}_{0,j+1}-e^{-\frac{\mathrm{i}}{2}j\alpha}C_{j+1,0}+e^{\frac{\mathrm{i}}{2}j\alpha}e^{\mathrm{i}\alpha}\bar{C}_{j+1,0}}{2(e^{-\mathrm{i}j\alpha}-e^{\mathrm{i}\alpha})}$$
$$+\frac{1}{2}\left(q_{0,j}+e^{\frac{\mathrm{i}}{2}(j-1)\alpha}p_{j,0}\right),\quad 0\le j\le K, \tag{160}$$

and all other coefficients of $\hat{p}$ and $\hat{q}$ are defined as 0. Indeed, by (153), we have

$$e^{\frac{\mathrm{i}}{2}\alpha}\hat{v}_{l,0}-e^{-\frac{\mathrm{i}}{2}l\alpha}\hat{u}_{0,l}=\frac{e^{\frac{\mathrm{i}}{2}\alpha}}{2}\frac{g_{l,0}-e^{\mathrm{i}(l-1)\alpha}\bar{g}_{l,0}}{e^{\mathrm{i}l\alpha}-e^{-\mathrm{i}\alpha}}$$
$$-\frac{e^{-\frac{\mathrm{i}}{2}l\alpha}}{2}\frac{f_{0,l}-e^{-\mathrm{i}(l-1)\alpha}\bar{f}_{0,l}}{e^{-\mathrm{i}l\alpha}-e^{\mathrm{i}\alpha}}=\sum_{k=1}^{4}(\mathcal{M}_k+\mathcal{N}_k),$$

where, $\mathcal{M}_2:=\bar{\mathcal{M}}_1e^{\mathrm{i}\alpha}$, $\mathcal{M}_4:=\bar{\mathcal{M}}_3e^{-\mathrm{i}l\alpha}$, $\mathcal{N}_2:=\bar{\mathcal{N}}_1e^{\mathrm{i}\alpha}$, $\mathcal{N}_4:=\bar{\mathcal{N}}_3e^{-\mathrm{i}l\alpha}$ together with

$$\mathcal{M}_1:=\frac{e^{\frac{3\mathrm{i}}{2}\alpha}}{2(e^{\mathrm{i}(l+1)\alpha}-1)}\left(e^{-\frac{\mathrm{i}}{2}\alpha}\bar{p}_{l,0}+e^{\frac{\mathrm{i}}{2}l\alpha}q_{0,l}-(\xi\eta)e^{-\mathrm{i}\alpha}C_{l+1,0}\right),$$
$$\mathcal{N}_1:=\frac{e^{\frac{3\mathrm{i}}{2}\alpha}}{2(e^{\mathrm{i}(l+1)\alpha}-1)}\left(g_{l,0}-e^{-\frac{\mathrm{i}}{2}\alpha}\bar{p}_{l,0}-e^{\frac{\mathrm{i}}{2}l\alpha}q_{0,l}+(\xi\eta)e^{-\mathrm{i}\alpha}C_{l+1,0}\right),$$
$$\mathcal{M}_3:=-\frac{e^{-\frac{\mathrm{i}}{2}(l+2)\alpha}}{2(e^{-\mathrm{i}(l+1)\alpha}-1)}\left(e^{\frac{\mathrm{i}}{2}\alpha}\bar{q}_{0,l}+e^{-\frac{\mathrm{i}}{2}l\alpha}p_{l,0}+(\xi\eta)e^{\mathrm{i}\alpha}C_{0,l+1}\right),$$
$$\mathcal{N}_3:=-\frac{e^{-\frac{\mathrm{i}}{2}(l+2)\alpha}}{2(e^{-\mathrm{i}(l+1)\alpha}-1)}\left(f_{0,l}-e^{\frac{\mathrm{i}}{2}\alpha}\bar{q}_{0,l}-e^{-\frac{\mathrm{i}}{2}l\alpha}p_{l,0}-(\xi\eta)e^{\mathrm{i}\alpha}C_{0,l+1}\right).$$

By Corollary 6.5 and Remark 6.6, we obtain, for $0 \le l \le K$, $\omega \in \mathcal{O}(r^{(7)}, \tilde{\beta})$,

$$|e^{\frac{3\mathrm{i}}{2}\alpha}|_{\omega,\tilde{\beta}},\ |e^{\mathrm{i}(l+\frac{1}{2})\alpha}|_{\omega,\tilde{\beta}},\ |e^{-\frac{\mathrm{i}}{2}(l+2)\alpha}|_{\omega,\tilde{\beta}},\ |e^{-\frac{3\mathrm{i}}{2}l\alpha}|_{\omega,\tilde{\beta}} < \frac{101}{100},$$

and, by (140), for $0 \le l \le K$, $\left\|(e^{\pm\mathrm{i}(l+1)\alpha} - 1)^{-1}\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} < 2\delta^{-1}$. Then, by (112), (113), we have

$$\sum_{k=1}^{4} \|\mathcal{N}_k\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} \le 4 \cdot \frac{101}{100}\delta^{-1} \cdot \frac{\varepsilon^{\frac{31}{16}}}{80(r^{(7)})^l} < \frac{\delta^{-1}\varepsilon^{\frac{31}{16}}}{16(r^{(7)})^l}. \tag{161}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{M}_1 &= \frac{e^{\frac{\mathrm{i}}{2}\alpha}}{2(e^{\mathrm{i}l\alpha} - e^{-\mathrm{i}\alpha})}\left(e^{-\frac{\mathrm{i}}{2}\alpha}\bar{p}_{l,0} + e^{\frac{\mathrm{i}}{2}l\alpha}q_{0,l} - (\xi\eta)e^{-\mathrm{i}\alpha}C_{l+1,0}\right)\\
&= \frac{1}{2(e^{\mathrm{i}l\alpha} - e^{-\mathrm{i}\alpha})}\left(\bar{p}_{l,0} + e^{\frac{\mathrm{i}}{2}(l+1)\alpha}q_{0,l} - (\xi\eta)e^{-\frac{\mathrm{i}}{2}\alpha}C_{l+1,0}\right),\\
\mathcal{M}_2 &= -\frac{1}{2(e^{\mathrm{i}l\alpha} - e^{-\mathrm{i}\alpha})}\left(e^{\mathrm{i}l\alpha}p_{l,0} + e^{\frac{\mathrm{i}}{2}(l-1)\alpha}\bar{q}_{0,l} - (\xi\eta)e^{\mathrm{i}(l+\frac{1}{2})\alpha}\bar{C}_{l+1,0}\right),\\
\mathcal{M}_3 &= \frac{1}{2(e^{\mathrm{i}l\alpha} - e^{-\mathrm{i}\alpha})}\left(e^{\frac{\mathrm{i}}{2}(l-1)\alpha}\bar{q}_{0,l} + e^{-\mathrm{i}\alpha}p_{l,0} + (\xi\eta)e^{\frac{\mathrm{i}}{2}l\alpha}C_{0,l+1}\right),\\
\mathcal{M}_4 &= -\frac{1}{2(e^{\mathrm{i}l\alpha} - e^{-\mathrm{i}\alpha})}\left(e^{-\frac{\mathrm{i}}{2}(l+1)\alpha}q_{0,l} + \bar{p}_{l,0} + (\xi\eta)e^{-\frac{\mathrm{i}}{2}(l+2)\alpha}\bar{C}_{0,l+1}\right).
\end{aligned}$$

Hence, adding the above terms, we have

$$\begin{aligned}
p_{l,0} + \sum_{k=1}^{4}\mathcal{M}_k &= p_{l,0} + \frac{1}{2(e^{\mathrm{i}l\alpha} - e^{-\mathrm{i}\alpha})}\left[\left(\bar{p}_{l,0} - \bar{p}_{l,0}\right) + \left(e^{\frac{\mathrm{i}}{2}(l-1)\alpha}\bar{q}_{0,l} - e^{\frac{\mathrm{i}}{2}(l-1)\alpha}\bar{q}_{0,l}\right)\right]\\
&\quad + \frac{(e^{-\mathrm{i}\alpha}p_{l,0} - e^{\mathrm{i}l\alpha}p_{l,0}) + (e^{\frac{\mathrm{i}}{2}(l+1)\alpha}q_{0,l} - e^{-\frac{\mathrm{i}}{2}(l+1)\alpha}q_{0,l})}{2(e^{\mathrm{i}l\alpha} - e^{-\mathrm{i}\alpha})}\\
&\quad + (\xi\eta)\cdot\frac{e^{\frac{\mathrm{i}}{2}l\alpha}C_{0,l+1} - e^{-\frac{\mathrm{i}}{2}l\alpha}e^{-\mathrm{i}\alpha}\bar{C}_{0,l+1} - e^{-\frac{\mathrm{i}}{2}\alpha}C_{l+1,0} + e^{\mathrm{i}l\alpha}e^{\frac{\mathrm{i}}{2}\alpha}\bar{C}_{l+1,0}}{2(e^{\mathrm{i}l\alpha} - e^{-\mathrm{i}\alpha})}\\
&= \frac{p_{l,0} + e^{-\frac{\mathrm{i}}{2}(l-1)\alpha}q_{0,l}}{2}\\
&\quad + (\xi\eta)\cdot\frac{-e^{-\frac{\mathrm{i}}{2}\alpha}C_{l+1,0} + e^{\mathrm{i}l\alpha}e^{\frac{\mathrm{i}}{2}\alpha}\bar{C}_{l+1,0} + e^{\frac{\mathrm{i}}{2}l\alpha}C_{0,l+1} - e^{-\frac{\mathrm{i}}{2}l\alpha}e^{-\mathrm{i}\alpha}\bar{C}_{0,l+1}}{2(e^{\mathrm{i}l\alpha} - e^{-\mathrm{i}\alpha})}\\
&= \hat{p}_{l,0}.
\end{aligned} \tag{162}$$

By (161) and (162), we obtain (155) for the case $(l, j) = (l, 0)$, $l \ge 0$. The proof is similar for the case $(l, j) = (0, j)$, $j \ge 2$. It is also similar for proving (156).

Now we are going to show (146)–(148). Since (155) and (156) imply that

$$\|e^{\frac{\mathrm{i}}{2}\alpha}\hat{v}-\hat{u}(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi)+p_K-p_{0,1}\eta-\hat{p}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\frac{K\delta^{-1}\varepsilon^{\frac{31}{16}}}{8},$$
$$\|e^{-\frac{\mathrm{i}}{2}\alpha}\hat{u}-\hat{v}(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi)+q_K-q_{1,0}\xi-\hat{q}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\frac{K\delta^{-1}\varepsilon^{\frac{31}{16}}}{8},$$

it is sufficient to prove

$$\|\hat{p}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}},\ \|\hat{q}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\frac{\varepsilon^{\frac{61}{32}}}{100}+6(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q+e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}, \tag{163}$$

$$\|e^{-\frac{\mathrm{i}}{2}\alpha}\xi\hat{p}+e^{\frac{\mathrm{i}}{2}\alpha}\eta\hat{q}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\frac{\varepsilon^{\frac{61}{32}}}{40}. \tag{164}$$

By (19), Lemma 3.3 and Corollary 6.4, we have

$$\begin{aligned}\|C\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} &\le \frac{\|\alpha'\|_{\mathcal{O},\beta,r}}{2}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q+e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\\ &<\frac{3}{5}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q+e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r},\end{aligned}$$

which gives the estimates for the coefficients: for $lj=0$,

$$\|C_{l,j}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\frac{3}{5(r^{(7)})^{l+j}}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q+e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}.$$

Then, in view of (115), we have that

$$\begin{aligned}&\|e^{\frac{\mathrm{i}}{2}\alpha}q+p(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi)\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}\\ &\le\|e^{\frac{\mathrm{i}}{2}\alpha}q+p(e^{\frac{\mathrm{i}}{2}\alpha}\eta, e^{-\frac{\mathrm{i}}{2}\alpha}\xi)-\bar{C}\xi\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}+\|C\xi\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}\\ &\le\frac{\varepsilon^{\frac{31}{16}}}{80}+\frac{3r^{(7)}}{5}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q+e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r},\end{aligned}$$

which gives the estimates for the coefficients: for $lj=0$,

$$\begin{aligned}&\|e^{\frac{\mathrm{i}}{2}\alpha}q_{l,j}+e^{\frac{\mathrm{i}}{2}(j-l)\alpha}p_{j,l}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}},\ \|e^{-\frac{\mathrm{i}}{2}\alpha}p_{l,j}+e^{\frac{\mathrm{i}}{2}(j-l)\alpha}q_{j,l}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}\\ &<\left(\frac{\varepsilon^{\frac{31}{16}}}{80}+\frac{3r^{(7)}}{5}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q+e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}\right)(r^{(7)})^{-(l+j)}.\end{aligned}$$

Recalling the coefficients in (157)–(160), and combining with (140), we obtain

$$\|\hat{p}_{l,j}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}},\ \|\hat{q}_{l,j}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}$$

$$
\begin{aligned}
&< \frac{101}{200}\left(\frac{\varepsilon^{\frac{31}{16}}}{80} + \frac{3r^{(7)}}{5}\|e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}\right)(r^{(7)})^{-(l+j)} \\
&\quad + 4\cdot\frac{101}{100}\delta^{-1}\cdot\frac{3}{5}\|e^{-\frac{\mathrm{i}}{2}\alpha}\eta\bar{q} + e^{\frac{\mathrm{i}}{2}\alpha}\xi\bar{p}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}(r^{(7)})^{-(l+j)} \\
&< \left(\frac{\varepsilon^{\frac{31}{16}}}{100} + 3\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}\right)(r^{(7)})^{-(l+j)},
\end{aligned}
$$

which implies (163). By Corollary 6.18, we have, for $l \geq 1$,

$$
\begin{aligned}
&\|e^{-\frac{\mathrm{i}}{2}\alpha}(p_{l-1,0} + e^{-\frac{\mathrm{i}}{2}(l-2)\alpha}q_{0,l-1}) \\
&\quad + (\xi\eta)e^{\frac{\mathrm{i}}{2}\alpha}(q_{l+1,0} + e^{-\frac{\mathrm{i}}{2}(l+2)\alpha}p_{0,l+1})\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{40(r^{(7)})^{l-1}}.
\end{aligned}
\tag{165}
$$

Moreover, we have

$$
e^{-\frac{\mathrm{i}}{2}\alpha}\cdot\frac{-(\xi\eta)e^{-\frac{\mathrm{i}}{2}\alpha}C_{l,0}}{2(e^{\mathrm{i}(l-1)\alpha} - e^{-\mathrm{i}\alpha})} + (\xi\eta)e^{\frac{\mathrm{i}}{2}\alpha}\cdot\frac{e^{\frac{\mathrm{i}}{2}\alpha}C_{l,0}}{2(e^{\mathrm{i}(l+1)\alpha} - e^{\mathrm{i}\alpha})} = 0, \tag{166}
$$

$$
e^{-\frac{\mathrm{i}}{2}\alpha}\cdot\frac{(\xi\eta)e^{\mathrm{i}(l-1)\alpha}e^{\frac{\mathrm{i}}{2}\alpha}\bar{C}_{l,0}}{2(e^{\mathrm{i}(l-1)\alpha} - e^{-\mathrm{i}\alpha})} + (\xi\eta)e^{\frac{\mathrm{i}}{2}\alpha}\cdot\frac{-e^{\mathrm{i}(l+1)\alpha}e^{-\frac{\mathrm{i}}{2}\alpha}\bar{C}_{l,0}}{2(e^{\mathrm{i}(l+1)\alpha} - e^{\mathrm{i}\alpha})} = 0, \tag{167}
$$

$$
e^{-\frac{\mathrm{i}}{2}\alpha}\cdot\frac{(\xi\eta)e^{\frac{\mathrm{i}}{2}(l-1)\alpha}C_{0,l}}{2(e^{\mathrm{i}(l-1)\alpha} - e^{-\mathrm{i}\alpha})} + (\xi\eta)e^{\frac{\mathrm{i}}{2}\alpha}\cdot\frac{-e^{\frac{\mathrm{i}}{2}(l+1)\alpha}C_{0,l}}{2(e^{\mathrm{i}(l+1)\alpha} - e^{\mathrm{i}\alpha})} = 0, \tag{168}
$$

$$
e^{-\frac{\mathrm{i}}{2}\alpha}\cdot\frac{-(\xi\eta)e^{-\frac{\mathrm{i}}{2}(l-1)\alpha}e^{-\mathrm{i}\alpha}\bar{C}_{0,l}}{2(e^{\mathrm{i}(l-1)\alpha} - e^{-\mathrm{i}\alpha})} + (\xi\eta)e^{\frac{\mathrm{i}}{2}\alpha}\cdot\frac{e^{-\frac{\mathrm{i}}{2}(l+1)\alpha}e^{\mathrm{i}\alpha}\bar{C}_{0,l}}{2(e^{\mathrm{i}(l+1)\alpha} - e^{\mathrm{i}\alpha})} = 0. \tag{169}
$$

In view of the definition of coefficients of $\hat{p}$ and $\hat{q}$ in (157)–(160), we have, for $1 \leq l \leq K$,

$$
\begin{aligned}
&e^{-\frac{\mathrm{i}}{2}\alpha}\hat{p}_{l-1,0} + (\xi\eta)e^{\frac{\mathrm{i}}{2}\alpha}\hat{q}_{l+1,0} \\
&= \frac{e^{-\frac{\mathrm{i}}{2}\alpha}}{2}(p_{l-1,0} + e^{-\frac{\mathrm{i}}{2}(l-2)\alpha}q_{0,l-1}) + \frac{(\xi\eta)e^{\frac{\mathrm{i}}{2}\alpha}}{2}(q_{l+1,0} + e^{-\frac{\mathrm{i}}{2}(l+2)\alpha}p_{0,l+1}) \\
&\quad + (\xi\eta)e^{-\frac{\mathrm{i}}{2}\alpha}\cdot\frac{-e^{-\frac{\mathrm{i}}{2}\alpha}C_{l,0} + e^{\mathrm{i}(l-1)\alpha}e^{\frac{\mathrm{i}}{2}\alpha}\bar{C}_{l,0} + e^{\frac{\mathrm{i}}{2}(l-1)\alpha}C_{0,l} - e^{-\frac{\mathrm{i}}{2}(l-1)\alpha}e^{-\mathrm{i}\alpha}\bar{C}_{0,l}}{2(e^{\mathrm{i}(l-1)\alpha} - e^{-\mathrm{i}\alpha})}
\end{aligned}
\tag{170}
$$

$$
+ (\xi\eta)e^{\frac{\mathrm{i}}{2}\alpha}\frac{e^{\frac{\mathrm{i}}{2}\alpha}C_{l,0} - e^{\mathrm{i}(l+1)\alpha}e^{-\frac{\mathrm{i}}{2}\alpha}\bar{C}_{l,0} - e^{\frac{\mathrm{i}}{2}l\alpha}C_{0,l} + e^{-\frac{\mathrm{i}}{2}(l+1)\alpha}e^{\mathrm{i}\alpha}\bar{C}_{0,l}}{2(e^{\mathrm{i}(l+1)\alpha} - e^{\mathrm{i}\alpha})}. \tag{171}
$$

Combining (165)–(169), we have (170) + (171)= 0, so that

$$
\|e^{-\frac{\mathrm{i}}{2}\alpha}\hat{p}_{l-1,0} + (\xi\eta)e^{\frac{\mathrm{i}}{2}\alpha}\hat{q}_{l+1,0}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{31}{16}}}{2(r^{(7)})^l}, \quad 1 \leq l \leq K+1.
$$

Similarly, we have

$$\|(\xi\eta)e^{-\frac{\mathrm{i}}{2}\alpha}\hat{p}_{0,j+1}+e^{\frac{\mathrm{i}}{2}\alpha}\hat{q}_{0,j-1}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\frac{\varepsilon^{\frac{31}{16}}}{2(r^{(7)})^j}\quad 1\le j\le K+1.$$

Then, according to (141), we obtain (164) by

$$\begin{aligned}
&\|e^{-\frac{\mathrm{i}}{2}\alpha}\xi\hat{p}+e^{\frac{\mathrm{i}}{2}\alpha}\eta\hat{q}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}\\
&\le\sum_{l=1}^{K+1}|e^{-\frac{\mathrm{i}}{2}\alpha}\hat{p}_{l-1,0}+(\xi\eta)e^{\frac{\mathrm{i}}{2}\alpha}\hat{q}_{l+1,0}|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}(r^{(7)})^l\\
&\quad+\sum_{j=1}^{K+1}|(\xi\eta)e^{-\frac{\mathrm{i}}{2}\alpha}\hat{p}_{0,j+1}+e^{\frac{\mathrm{i}}{2}\alpha}\hat{q}_{0,j-1}|_{\mathcal{O}_\delta^{(6)},8\beta_+}(r^{(7)})^j\\
&<2(K+1)\varepsilon^{\frac{31}{16}}<\frac{\varepsilon^{\frac{61}{32}}}{40}.
\end{aligned}$$

In view of (149) and (150), and recalling that $\|f\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\frac{\varepsilon}{2}$ in Corollary 6.15, we have

$$\|\hat{u}_{l,j}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\frac{101}{100}\delta^{-1}\cdot\left(1+\frac{101}{100}\right)\frac{\varepsilon}{2(r^{(7)})^{l+j}}<\frac{6\delta^{-1}\varepsilon}{5(r^{(7)})^{l+j}}.$$

Hence, recalling that $\delta>80\varepsilon^{\frac{1}{60s}}$ and (141), we have

$$\begin{aligned}
\|\hat{u}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}&\le\sum_{l=2}^{K}\|\hat{u}_{l,0}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}(r^{(7)})^l+\sum_{j=0}^{K}\|\hat{u}_{0,j}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}(r^{(7)})^j\\
&<2K\cdot\frac{6}{5}\delta^{-1}\varepsilon<\frac{\varepsilon^{\frac{49}{50}}}{20}.
\end{aligned}$$

We have similarly, $\|\hat{v}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\dfrac{\varepsilon^{\frac{49}{50}}}{20}$.

In $\eta\hat{u}+\xi\hat{v}$, $(\eta\hat{u}+\xi\hat{v})_{0,0}=0$ since $\hat{u}_{1,0}=\hat{v}_{0,1}=0$. For other terms, we have

$$\begin{aligned}
&\sum_{l=1}^{K+1}\left\|(\eta\hat{u}+\xi\hat{v})_{l,0}\xi^l\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}\\
&=\sum_{l=1}^{K+1}\left\|\left(\hat{u}_{l+1,0}\cdot\xi\eta+\hat{v}_{l-1,0}\right)\xi^l\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}\\
&=\frac{1}{2}\sum_{l=1}^{K+1}\left\|\frac{f_{l+1,0}-e^{\mathrm{i}(l+2)\alpha}\bar{f}_{l+1,0}}{e^{\mathrm{i}(l+1)\alpha}-e^{\mathrm{i}\alpha}}\xi\eta+\frac{g_{l-1,0}-e^{\mathrm{i}(l-2)\alpha}\bar{g}_{l-1,0}}{e^{\mathrm{i}(l-1)\alpha}-e^{-\mathrm{i}\alpha}}\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}(r^{(7)})^l
\end{aligned}$$

$$= \frac{1}{2}\sum_{l=1}^{K+1}\left\|\frac{((\xi\eta)e^{-\mathrm{i}\alpha}f_{l+1,0}+e^{\mathrm{i}\alpha}g_{l-1,0})-e^{\mathrm{i}l\alpha}((\xi\eta)e^{\mathrm{i}\alpha}\bar{f}_{l+1,0}+e^{-\mathrm{i}\alpha}\bar{g}_{l-1,0})}{e^{\mathrm{i}l\alpha}-1}\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}(r^{(7)})^l,$$

and similarly,

$$\sum_{j=1}^{K+1}\left\|(\eta\hat{u}+\xi\hat{v})_{0,j}\eta^j\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}$$
$$= \frac{1}{2}\sum_{j=1}^{K+1}\left\|\frac{(e^{-\mathrm{i}\alpha}f_{0,j-1}+(\xi\eta)e^{\mathrm{i}\alpha}g_{0,j+1})-e^{-\mathrm{i}j\alpha}(e^{\mathrm{i}\alpha}\bar{f}_{0,j-1}+(\xi\eta)e^{-\mathrm{i}\alpha}\bar{g}_{0,j+1})}{e^{-\mathrm{i}j\alpha}-1}\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}(r^{(7)})^j.$$

Note that $(\xi\eta)e^{-\mathrm{i}\alpha}f_{l+1,0}+e^{\mathrm{i}\alpha}g_{l-1,0}$ and $e^{-\mathrm{i}\alpha}f_{0,j-1}+(\xi\eta)e^{\mathrm{i}\alpha}g_{0,j+1}$ are respectively the coefficients of ξ^l and η^j in $e^{-\mathrm{i}\alpha}\eta f+e^{\mathrm{i}\alpha}\xi g$. Hence, in view of Lemma 6.20, we have

$$\begin{aligned}
&\sum_{l=1}^{K+1}\left\|(\eta\hat{u}+\xi\hat{v})_{l,0}\xi^l\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}+\sum_{j=1}^{K+1}\left\|(\eta\hat{u}+\xi\hat{v})_{0,j}\eta^j\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}\\
&\quad< 2(K+1)\delta^{-1}\left(1+\frac{101}{100}\right)\|e^{-\mathrm{i}\alpha}\eta f+e^{\mathrm{i}\alpha}\xi g\|_{\mathcal{O},\beta,r}\\
&\quad< 2(K+1)\delta^{-1}\left(1+\frac{101}{100}\right)\left(\frac{\varepsilon^{\frac{31}{16}}}{40}+2\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q+e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}\right)\\
&\quad< \frac{\varepsilon^{\frac{61}{32}}}{16}+5(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q+e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}.
\end{aligned}$$

This finishes the proof of Lemma 7.5. □

By Lemma 7.5, we see that $\phi=\mathrm{Id}+\hat{\mathcal{U}}$ is invertible on $\mathcal{C}^{r^{(7)}}_{\omega,\tilde{\beta}}$ and $\phi(\mathcal{C}^{r^{(7)}}_{\omega,\tilde{\beta}})\subset\mathcal{C}^{r}_{\omega,\beta}$ for $\omega\in\mathcal{O}_\delta(r^{(7)},\tilde{\beta})=\mathcal{O}_\delta\cap\,]-(r^{(7)})^2+\tilde{\beta},(r^{(7)})^2-\tilde{\beta}[$. Indeed, according to (144), (145) and (141), we have

$$\begin{aligned}
&\left\|\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v}\right\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}\\
&\quad< \frac{\varepsilon^{\frac{61}{32}}}{16}+5(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q+e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}+\frac{\varepsilon^{\frac{49}{25}}}{400}\\
&\quad< \frac{\varepsilon^{\frac{61}{32}}}{16}+5(K+1)\delta^{-1}\cdot\frac{101}{200}\cdot\frac{\varepsilon}{10}+\frac{\varepsilon^{\frac{49}{25}}}{400}\ <\ \varepsilon^{\frac{49}{50}}.
\end{aligned}$$

Then, in view of the definition of the set given in (15), we have for any $(\xi,\eta)\in\mathcal{C}^{r^{(7)}}_{\omega,\tilde{\beta}}$ with $\omega\in\mathcal{O}_\delta(r^{(7)},\tilde{\beta})$,

$$\begin{aligned}
|(\xi+\hat{u}(\xi,\eta))(\eta+\hat{v}(\xi,\eta))-\omega| &\le |\xi\eta-\omega|+|\eta\hat{u}(\xi,\eta)+\xi\hat{v}(\xi,\eta)+\hat{u}(\xi,\eta)\hat{v}(\xi,\eta)|\\
&< \tilde{\beta}+\|\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}<\tilde{\beta}+\varepsilon^{\frac{49}{50}}<\beta,
\end{aligned}$$

$$|\xi + \hat{u}(\xi,\eta)|,\ |\eta + \hat{v}(\xi,\eta)| < r^{(7)} + \varepsilon^{\frac{49}{50}} < r,$$

noting that, under the condition (84),

$$\varepsilon^{\frac{49}{50}} < \varepsilon^{\frac{1}{40s}} \le \frac{\beta}{2} < \beta - \tilde{\beta}, \quad \varepsilon^{\frac{49}{50}} < \varepsilon^{\frac{1}{2400s^2}} < \frac{r - r_+}{8} = r - r^{(7)}.$$

Moreover, since $\hat{u}, \hat{v} \in \mathcal{A}^{\mathbb{R}}_{\tilde{\beta}, r^{(7)}}(\mathcal{O}_\delta)$, we have, by Lemma 3.1, $\rho \circ \phi = \phi \circ \rho$.

With $\phi = \mathrm{Id} + \hat{\mathcal{U}}$ constructed above, we define $\tilde{\tau}_1 := \phi^{-1} \circ \tau_1 \circ \phi$. For any $h = h(\xi\eta)$, we define the linear operator L_h by:

$$\left[L_h \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right](\xi,\eta) := e^{-\mathrm{i}h(\xi\eta)} \xi\, p_1(\xi,\eta) + e^{\mathrm{i}h(\xi\eta)} \eta\, p_2(\xi,\eta). \tag{172}$$

Lemma 7.7 *For $\tilde{\tau}_1 = \phi^{-1} \circ \tau_1 \circ \phi$, we have*

$$\tilde{\tau}_1(\xi,\eta) - \begin{pmatrix} (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} + p_{0,1})\eta \\ (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} + p_{0,1})^{-1}\xi \end{pmatrix} \in (\mathcal{A}_{2\beta_+, \tilde{r}}(\mathcal{O}_\delta))^2,$$

satisfying that

$$\left\| \tilde{\tau}_1(\xi,\eta) - \begin{pmatrix} (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} + p_{0,1})\eta \\ (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} + p_{0,1})^{-1}\xi \end{pmatrix} \right\|_{\mathcal{O}_\delta, 2\beta_+, \tilde{r}} < \frac{\varepsilon^{\frac{61}{32}}}{3} + \frac{22(K+1)}{\delta} \| e^{\frac{\mathrm{i}}{2}\alpha} \eta q + e^{-\frac{\mathrm{i}}{2}\alpha} \xi p \|_{\mathcal{O},\beta,r}, \tag{173}$$

$$\left\| L_{\frac{\alpha}{2}} \left(\tilde{\tau}_1(\xi,\eta) - \begin{pmatrix} (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} + p_{0,1})\eta \\ (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} + p_{0,1})^{-1}\xi \end{pmatrix} \right) \right\|_{\mathcal{O}_\delta, 2\beta_+, \tilde{r}} < \frac{\varepsilon^{\frac{61}{32}}}{2}. \tag{174}$$

Proof A direct computation yields that

$$\begin{aligned} \tilde{\tau}_1 &= \phi^{-1} \circ \tau_1 \circ \phi \\ &= \begin{pmatrix} e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta + \eta\hat{u} + \xi\hat{v} + \hat{u}\hat{v})}\eta + e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta + \eta\hat{u} + \xi\hat{v} + \hat{u}\hat{v})}\hat{v} + p \circ \phi - \hat{u} \circ \tau_1 \circ \phi \\ e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta + \eta\hat{u} + \xi\hat{v} + \hat{u}\hat{v})}\xi + e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta + \eta\hat{u} + \xi\hat{v} + \hat{u}\hat{v})}\hat{u} + q \circ \phi - \hat{v} \circ \tau_1 \circ \phi \end{pmatrix} + \tilde{\mathcal{U}} \circ \tau_1 \circ \phi, \end{aligned}$$

with $\tilde{\mathcal{U}} = \phi^{-1} - \mathrm{Id} + \hat{\mathcal{U}}$ defined as in the proof of Lemma 3.8. Recalling (142), we see that

$$\begin{aligned} &\tilde{\tau}_1(\xi,\eta) - \begin{pmatrix} (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} + p_{0,1})\eta \\ (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} + p_{0,1})^{-1}\xi \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \left((e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)} + q_{1,0}) - (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} + p_{0,1})^{-1} \right) \xi \end{pmatrix} \end{aligned} \tag{175}$$

$$+\begin{pmatrix} e^{\frac{i}{2}\alpha(\xi\eta)}\hat{v}-\hat{u}(e^{\frac{i}{2}\alpha}\eta,e^{-\frac{i}{2}\alpha}\xi)+p_K-p_{0,1}\eta \\ e^{-\frac{i}{2}\alpha(\xi\eta)}\hat{u}-\hat{v}(e^{\frac{i}{2}\alpha}\eta,e^{-\frac{i}{2}\alpha}\xi)+q_K-q_{1,0}\xi \end{pmatrix} \tag{176}$$

$$+\begin{pmatrix} p-p_K \\ q-q_K \end{pmatrix}+\begin{pmatrix} p\circ\phi-p \\ q\circ\phi-q \end{pmatrix} \tag{177}$$

$$+\begin{pmatrix} (e^{\frac{i}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})}-e^{\frac{i}{2}\alpha(\xi\eta)})(\eta+\hat{v}) \\ (e^{-\frac{i}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})}-e^{-\frac{i}{2}\alpha(\xi\eta)})(\xi+\hat{u}) \end{pmatrix} \tag{178}$$

$$-\left(\hat{\mathcal{U}}\circ\tau_1\circ\phi-\hat{\mathcal{U}}(e^{\frac{i}{2}\alpha}\eta,e^{-\frac{i}{2}\alpha}\xi)\right)+\tilde{\mathcal{U}}\circ\tau_1\circ\phi. \tag{179}$$

In what follows, we shall estimate the norms of terms (175)–(179) as well as their image under $L_{\frac{\alpha}{2}}$. We emphasize that if a given term $T\in(\mathcal{A}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta))^2$ satisfies $\|T\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}<D\varepsilon^\varsigma$ with $\varsigma>1$ and some constant $D>0$, then $\|L_{\frac{\alpha}{2}}(T)\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}<\frac{101}{200}D\varepsilon^\varsigma<D\varepsilon^\varsigma$.

- **Terms in** (175)

In view of (120) in Corollary 6.19, we obtain, for $\omega\in\mathcal{O}_\delta(\tilde{r},2\beta_+)$,

$$\begin{aligned}&|(e^{\frac{i}{2}\alpha}+p_{0,1})(e^{-\frac{i}{2}\alpha}+q_{1,0})-1|_{\omega,2\beta_+}\\&\quad=|e^{\frac{i}{2}\alpha}q_{1,0}+e^{-\frac{i}{2}\alpha}p_{0,1}+p_{0,1}q_{1,0}|_{\omega,2\beta_+}<\frac{\varepsilon^{\frac{61}{32}}}{50r^{(7)}}.\end{aligned} \tag{180}$$

Since Corollary 6.5 implies that, for any $(\xi,\eta)\in\mathcal{C}_{\omega,2\beta_+}$, $\omega\in\mathcal{O}_\delta(\tilde{r},2\beta_+)$,

$$|e^{\frac{i}{2}\alpha(\xi\eta)}+p_{0,1}(\xi\eta)|\geq|e^{\frac{i}{2}\alpha(\xi\eta)}|-|p_{0,1}|_{\omega,2\beta_+}>\frac{4}{5},$$

we see that $((e^{-\frac{i}{2}\alpha}+q_{1,0})-(e^{\frac{i}{2}\alpha}+p_{0,1})^{-1})(\omega)$ is analytic on $\mathcal{O}_\delta(\tilde{r},2\beta_+)$, and (180) implies that

$$|(e^{-\frac{i}{2}\alpha}+q_{1,0})-(e^{\frac{i}{2}\alpha}+p_{0,1})^{-1}|_{\omega,2\beta_+}<\frac{\varepsilon^{\frac{61}{32}}}{40r^{(7)}}.$$

Hence, (175)$\in(\mathcal{A}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta))^2$,

$$\left\|\begin{pmatrix}0\\ \left((e^{-\frac{i}{2}\alpha(\xi\eta)}+q_{1,0})-(e^{\frac{i}{2}\alpha(\xi\eta)}+p_{0,1})^{-1}\right)\xi\end{pmatrix}\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}<\frac{\varepsilon^{\frac{61}{32}}}{40}. \tag{181}$$

- **Terms in** (176)

According to Lemma 7.5, $\hat{u},\ \hat{v}\in\mathcal{A}^{\mathbb{R}}_{\tilde{\beta},r^{(7)}}(\mathcal{O}_\delta)$, which implies that

$$\begin{pmatrix} e^{\frac{i}{2}\alpha(\xi\eta)}\hat{v}-\hat{u}(e^{\frac{i}{2}\alpha}\eta,e^{-\frac{i}{2}\alpha}\xi)+p_K-p_{0,1}(\xi\eta)\eta \\ e^{-\frac{i}{2}\alpha(\xi\eta)}\hat{u}-\hat{v}(e^{\frac{i}{2}\alpha}\eta,e^{-\frac{i}{2}\alpha}\xi)+q_K-q_{1,0}(\xi\eta)\xi \end{pmatrix}\in(\mathcal{A}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta))^2.$$

By (146)–(148), we obtain

$$\left\|\begin{pmatrix} e^{\frac{i}{2}\alpha(\xi\eta)}\hat{v}-\hat{u}(e^{\frac{i}{2}\alpha}\eta, e^{-\frac{i}{2}\alpha}\xi)+p_K-p_{0,1}(\xi\eta)\eta \\ e^{-\frac{i}{2}\alpha(\xi\eta)}\hat{u}-\hat{v}(e^{\frac{i}{2}\alpha}\eta, e^{-\frac{i}{2}\alpha}\xi)+q_K-q_{1,0}(\xi\eta)\xi \end{pmatrix}\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}$$
$$< \frac{\varepsilon^{\frac{61}{32}}}{40}+6(K+1)\delta^{-1}\|e^{\frac{i}{2}\alpha}\eta q+e^{-\frac{i}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}, \tag{182}$$

$$\left\|L_{\frac{\alpha}{2}}\begin{pmatrix} e^{\frac{i}{2}\alpha(\xi\eta)}\hat{v}-\hat{u}(e^{\frac{i}{2}\alpha}\eta, e^{-\frac{i}{2}\alpha}\xi)+p_K-p_{0,1}(\xi\eta)\eta \\ e^{-\frac{i}{2}\alpha(\xi\eta)}\hat{u}-\hat{v}(e^{\frac{i}{2}\alpha}\eta, e^{-\frac{i}{2}\alpha}\xi)+q_K-q_{1,0}(\xi\eta)\xi \end{pmatrix}\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} < \frac{\varepsilon^{\frac{61}{32}}}{20}. \tag{183}$$

- **Terms in** (177)

  In view of (143), we have $p-p_K, q-q_K\in\mathcal{A}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta)$, with

$$\left\|\begin{pmatrix} p-p_K \\ q-q_K \end{pmatrix}\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} < \frac{\varepsilon^2}{5}. \tag{184}$$

Since $\tilde{r}=\frac{r+r_+}{2}$ and $\beta\ge\varepsilon^{\frac{1}{40s}}$, we have, by Lemma 6.11,

$$\begin{pmatrix} p\circ\phi-p \\ q\circ\phi-q \end{pmatrix}\in(\mathcal{A}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta))^2,$$

satisfying that

$$\left\|\begin{pmatrix} p\circ\phi-p \\ q\circ\phi-q \end{pmatrix}\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} < \frac{6r}{(r-\tilde{r})\beta}\cdot\frac{\varepsilon^{\frac{49}{50}}}{20}\cdot\frac{\varepsilon}{10}\le\frac{3r\varepsilon^{\frac{99}{50}-\frac{1}{40s}}}{50(r-r_+)}<\frac{\varepsilon^{\frac{61}{32}}}{80}. \tag{185}$$

The last inequality follows from (97).

- **Terms in** (178)

  In view of Lemma 7.5, we see that $\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v}\in\mathcal{A}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta)$ and

$$\|\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v}\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}$$
$$< \frac{\varepsilon^{\frac{61}{32}}}{16}+5(K+1)\delta^{-1}\|e^{\frac{i}{2}\alpha}\eta q+e^{-\frac{i}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}+\frac{\varepsilon^{\frac{49}{25}}}{400}$$
$$< \frac{\varepsilon^{\frac{61}{32}}}{12}+5(K+1)\delta^{-1}\|e^{\frac{i}{2}\alpha}\eta q+e^{-\frac{i}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}. \tag{186}$$

By Lemma 6.7, we obtain $\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})-\alpha(\xi\eta)\in\mathcal{A}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta)$, and for $-1\le b\le 1$, $e^{ib\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})}-e^{ib\alpha(\xi\eta)}\in\mathcal{A}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta)$, with

$$\left\|\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})-\alpha(\xi\eta)\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}$$

$$< \frac{\varepsilon^{\frac{61}{32}}}{8} + 7(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r},$$

$$\left\| e^{\mathrm{i}b\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{\mathrm{i}b\alpha(\xi\eta)} \right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}$$

$$< \frac{\varepsilon^{\frac{61}{32}}}{9} + 7(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}, \quad -1 \le b \le 1.$$

Hence, we have

$$\begin{pmatrix} (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)})(\eta+\hat{v}) \\ (e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)})(\xi+\hat{u}) \end{pmatrix} \in (\mathcal{A}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta))^2,$$

satisfying that

$$\left\| \begin{pmatrix} (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)})(\eta+\hat{v}) \\ (e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)})(\xi+\hat{u}) \end{pmatrix} \right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}$$

$$< \left(\frac{\varepsilon^{\frac{61}{32}}}{9} + 7(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r} \right) \cdot \left(\frac{1}{4} + \frac{\varepsilon^{\frac{49}{50}}}{20} \right)$$

$$< \frac{\varepsilon^{\frac{61}{32}}}{30} + 2(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r} \tag{187}$$

In order to obtain the estimate of the image under $L_{\frac{\alpha}{2}}$ of this term, we shall follow the scheme of the proof of Lemma 6.14. Developing $e^{\frac{\mathrm{i}}{2}\alpha(\cdot)}$ around $\xi\eta$,

$$\begin{aligned} &e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} \\ &= e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} \sum_{k\ge1} \frac{\mathrm{i}^k}{2^k\cdot k!}(\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v}) - \alpha(\xi\eta))^k \\ &= \frac{\mathrm{i}}{2} e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\alpha'(\xi\eta)(\eta\hat{u}+\xi\hat{v}) \\ &\quad + \frac{\mathrm{i}}{2} e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\alpha'(\xi\eta)\hat{u}\hat{v} + e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} \sum_{j\ge2} \frac{\alpha^{(j)}(\xi\eta)}{j!}(\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})^j \\ &\quad + e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} \sum_{k\ge2} \frac{\mathrm{i}^k}{2^k\cdot k!}(\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v}) - \alpha(\xi\eta))^k. \end{aligned} \tag{188}$$

Noting that $\|\hat{u}\hat{v}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{49}{25}}}{400}$, and (186) gives the rough estimates via (144):

$$\|\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v}\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} < \frac{\varepsilon^{\frac{49}{50}}}{20}, \tag{189}$$

$$\left\|\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})-\alpha(\xi\eta)\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}<\frac{\varepsilon^{\frac{49}{50}}}{16}, \tag{190}$$

$$\left\|e^{\mathrm{i}b\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})}-e^{\mathrm{i}b\alpha(\xi\eta)}\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}<\frac{\varepsilon^{\frac{49}{50}}}{15},\quad -1\le b\le 1, \tag{191}$$

we have, in view of (188),

$$\left\|e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})}-e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}-\frac{\mathrm{i}}{2}e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\alpha'(\xi\eta)(\eta\hat{u}+\xi\hat{v})\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}<\frac{\varepsilon^{\frac{61}{32}}}{3}. \tag{192}$$

Indeed, according to Corollary 6.4, we have, for $\omega\in\mathcal{O}_\delta(\tilde{r},2\beta_+)$,

$$|\alpha'(\xi\eta)|_{\omega,2\beta_+}<\frac{6}{5},\qquad |\alpha^{(k)}(\xi\eta)|_{\omega,2\beta_+}<\begin{cases}2\beta^{-\frac{1}{32}}, & 2\le k\le s,\ \text{if } s\ge 2\\ \frac{k!2^{k+5}}{\beta^k}, & k\ge s+1\end{cases}.$$

As a consequence, (188) implies that, if $s\ge 2$, then,

$$\begin{aligned}&\left\|e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})}-e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}-\frac{\mathrm{i}}{2}e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\alpha'(\xi\eta)(\eta\hat{u}+\xi\hat{v})\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}\\
&\le\frac{101}{200}\cdot\frac{6}{5}\cdot\frac{\varepsilon^{\frac{49}{25}}}{400}+\frac{101}{100}\cdot 2\beta^{-\frac{1}{32}}\sum_{k=2}^{s}\frac{1}{k!}\left(\frac{\varepsilon^{\frac{49}{50}}}{20}\right)^k\\
&\quad+\frac{101}{100}\sum_{k\ge s+1}\frac{2^{k+5}}{\beta^k}\left(\frac{\varepsilon^{\frac{49}{50}}}{20}\right)^k+\frac{101}{100}\sum_{k\ge 2}\frac{1}{2^k\cdot k!}\left(\frac{\varepsilon^{\frac{49}{50}}}{16}\right)^k<\frac{\varepsilon^{\frac{61}{32}}}{3}.\end{aligned}$$

Otherwise, for $s=1$, it is bounded by

$$\frac{101}{200}\cdot\frac{6}{5}\cdot\frac{\varepsilon^{\frac{49}{25}}}{400}+\frac{101}{100}\sum_{k\ge 2}\frac{2^{k+5}}{\beta^k}\left(\frac{\varepsilon^{\frac{49}{50}}}{20}\right)^k+\frac{101}{100}\sum_{k\ge 2}\frac{1}{2^k\cdot k!}\left(\frac{\varepsilon^{\frac{49}{50}}}{16}\right)^k<\frac{\varepsilon^{\frac{61}{32}}}{3}.$$

Similarly,

$$\left\|e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})}-e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}+\frac{\mathrm{i}}{2}e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\alpha'(\xi\eta)(\eta\hat{u}+\xi\hat{v})\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}<\frac{\varepsilon^{\frac{61}{32}}}{3}. \tag{193}$$

Noting that $L_{\frac{\alpha}{2}}\begin{pmatrix}-\frac{\mathrm{i}}{2}e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\alpha'(\xi\eta)(\eta\hat{u}+\xi\hat{v})\eta\\ \frac{\mathrm{i}}{2}e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\alpha'(\xi\eta)(\eta\hat{u}+\xi\hat{v})\xi\end{pmatrix}=0$, we have

$$L_{\frac{\alpha}{2}}\begin{pmatrix}\left(e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})}-e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\right)(\eta+\hat{v})\\ \left(e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})}-e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\right)(\xi+\hat{u})\end{pmatrix}$$

$$= L_{\frac{\alpha}{2}} \begin{pmatrix} \left(e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\right)(\eta+\hat{v}) - \frac{\mathrm{i}}{2}e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\alpha'(\xi\eta)(\eta\hat{u}+\xi\hat{v})\eta \\ \left(e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\right)(\xi+\hat{u}) + \frac{\mathrm{i}}{2}e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\alpha'(\xi\eta)(\eta\hat{u}+\xi\hat{v})\xi \end{pmatrix}.$$

Hence, by (192), (193) and Lemma 6.7, we have

$$\begin{aligned} &\left\| L_{\frac{\alpha}{2}} \begin{pmatrix} \left(e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\right)(\eta+\hat{v}) \\ \left(e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\right)(\xi+\hat{u}) \end{pmatrix} \right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} \\ &\quad < \frac{2\tilde{r}}{3}\varepsilon^{\frac{61}{32}} + \frac{\varepsilon^{\frac{49}{25}}}{150} < \frac{\varepsilon^{\frac{61}{32}}}{5}. \end{aligned} \tag{194}$$

- **Terms in** (179)

Since $\|\hat{\mathcal{U}}\|_{\mathcal{O}_\delta,8\beta_+,r^{(7)}} < \frac{\varepsilon^{\frac{49}{50}}}{10}$, and according to Lemma 3.8, $\tilde{\mathcal{U}} = \Phi^{-1} - \mathrm{Id} + \mathcal{U}$ satisfies

$$\|\tilde{\mathcal{U}}\|_{\mathcal{O}_\delta,4\beta_+,r^{(6)}} \le \frac{r^{(7)}\|\hat{\mathcal{U}}\|^2_{\mathcal{O}_\delta,8\beta_+,r^{(7)}}}{(r^{(7)}-r^{(6)})\beta_+} \le \frac{\varepsilon^{\frac{61}{32}}}{8}.$$

Then, by Lemma 6.9, we have that

$$\begin{aligned} \|\hat{\mathcal{U}}(e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta, e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi)\|_{\mathcal{O}_\delta,4\beta_+,r^{(6)}} &< \frac{\varepsilon^{\frac{49}{50}}}{10}, \\ \|\tilde{\mathcal{U}}(e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta, e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi)\|_{\mathcal{O}_\delta,2\beta_+,r^{(5)}} &< \frac{\varepsilon^{\frac{61}{32}}}{8}. \end{aligned}$$

By Lemma 6.11, we have

$$\begin{aligned} &\|p(\xi+\hat{u},\eta+\hat{v})\|_{\mathcal{O}_\delta,8\beta_+,r^{(7)}},\ \|q(\xi+\hat{u},\eta+\hat{v})\|_{\mathcal{O}_\delta,8\beta_+,r^{(7)}} \\ &\quad < \frac{\varepsilon}{10} + \frac{3r}{(r-r^{(7)})\beta}\cdot\frac{\varepsilon^{1+\frac{49}{50}}}{200} < \frac{\varepsilon}{8}, \end{aligned}$$

which, together with (191), implies that

$$\begin{aligned} &\left\| \begin{pmatrix} (e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)})\eta + p(\xi+\hat{u},\eta+\hat{v}) \\ (e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta+\eta\hat{u}+\xi\hat{v}+\hat{u}\hat{v})} - e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)})\xi + q(\xi+\hat{u},\eta+\hat{v}) \end{pmatrix} \right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} \\ &< \frac{2r^{(7)}\varepsilon^{\frac{49}{50}}}{15} + \frac{\varepsilon}{4} < \frac{\varepsilon^{\frac{49}{50}}}{20}. \end{aligned}$$

Therefore, by Lemma 6.11, we obtain

$$\|\hat{\mathcal{U}}\circ\tau_1\circ\phi - \hat{\mathcal{U}}(e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\eta, e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)}\xi)\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}$$

$$< \frac{6r^{(7)}}{8(r^{(7)}-\tilde{r})\beta_+}\cdot\frac{\varepsilon^{\frac{49}{50}}}{10}\cdot\frac{\varepsilon^{\frac{49}{50}}}{20} < \frac{\varepsilon^{\frac{61}{32}}}{200}. \tag{195}$$

In a similar way, we have,

$$\begin{aligned}
&\|\tilde{\mathcal{U}}\circ\tau^{(1)}\circ\phi\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}\\
&\leq \|\tilde{\mathcal{U}}(e^{\frac{i}{2}\alpha(\xi\eta)}\eta, e^{-\frac{i}{2}\alpha(\xi\eta)}\xi)\|_{\omega,4\beta_+,r^{(5)}}\\
&\quad+\|\tilde{\mathcal{U}}\circ\tau^{(1)}\circ\phi-\tilde{\mathcal{U}}(e^{\frac{i}{2}\alpha(\xi\eta)}\eta, e^{-\frac{i}{2}\alpha(\xi\eta)}\xi)\|_{\omega,2\beta_+,\tilde{r}}\\
&< \frac{\varepsilon^{\frac{61}{32}}}{8}+\frac{6r^{(6)}}{4(r^{(6)}-\tilde{r})\beta_+}\cdot\frac{\varepsilon^{\frac{49}{50}}}{20}\cdot\frac{\varepsilon^{\frac{61}{32}}}{8} < \frac{\varepsilon^{\frac{61}{32}}}{6}.
\end{aligned} \tag{196}$$

With (175)–(179) estimated as above, we have (173) by combining (181), (182), (184), (185), (187), (195), (196), and get (174) by combining (181), (183), (184), (185), (194), (195), (196). Hence Lemma 7.7 is shown. □

Proof of Theorem 4.7. By Lemma 7.7, we see that $\tilde{\tau}_1=\phi^{-1}\circ\tau_1\circ\phi$ can be written as

$$\tilde{\tau}_1=\begin{pmatrix}(e^{\frac{i}{2}\alpha(\xi\eta)}+p_{0,1})\eta\\(e^{\frac{i}{2}\alpha(\xi\eta)}+p_{0,1})^{-1}\xi\end{pmatrix}+\begin{pmatrix}\tilde{p}(\xi,\eta)\\\tilde{q}(\xi,\eta)\end{pmatrix}$$

with $\tilde{p},\tilde{q}\in\mathcal{A}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta)$ satisfying that

$$\left\|\begin{pmatrix}\tilde{p}\\\tilde{q}\end{pmatrix}\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} < \frac{\varepsilon^{\frac{61}{32}}}{3}+22(K+1)\delta^{-1}\|e^{\frac{i}{2}\alpha}\eta q+e^{-\frac{i}{2}\alpha}\xi p\|_{\mathcal{O},\beta,r}, \tag{197}$$

$$\left\|L_{\frac{\alpha}{2}}\begin{pmatrix}\tilde{p}\\\tilde{q}\end{pmatrix}\right\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} < \frac{\varepsilon^{\frac{61}{32}}}{2}. \tag{198}$$

Since $\|p\|_{\mathcal{O},\beta,r}$, $\|q\|_{\mathcal{O},\beta,r}<\frac{\varepsilon}{10}$, (197) implies a rough estimate for $\tilde{p}$ and $\tilde{q}$:

$$\|\tilde{p}\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}},\ \|\tilde{q}\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} < \frac{\varepsilon^{\frac{61}{32}}}{3}+\frac{\varepsilon^{\frac{31}{32}}}{20} < \frac{\varepsilon^{\frac{31}{32}}}{16}. \tag{199}$$

As in Sect. 7.1, with the well-defined fourth root

$$\Theta(\xi\eta):=\left((e^{\frac{i}{2}\alpha(\xi\eta)}+p_{0,1}(\xi\eta))(e^{-\frac{i}{2}\alpha(\xi\eta)}+\bar{p}_{0,1}(\xi\eta))\right)^{\frac{1}{4}},$$

we define

$$\varphi:(\xi,\eta)\mapsto\left(\Theta(\xi\eta)\xi,\ \Theta^{-1}(\xi\eta)\eta\right). \tag{200}$$

Since $|p_{0,1}|_{\omega,\beta_+}<\varepsilon$, we can apply Proposition 7.2 with $r=\tilde{r}$, $r'=r_+$, and get

$$(\varphi^{-1}\circ\tilde{\tau}_1\circ\varphi)(\xi,\eta)=\begin{pmatrix}e^{\frac{i}{2}\alpha_+(\xi\eta)}\eta+p_+(\xi,\eta)\\e^{-\frac{i}{2}\alpha_+(\xi\eta)}\xi+q_+(\xi,\eta)\end{pmatrix},$$

with $\alpha_+ \in \mathcal{A}^{\mathbb{R}}_{2\beta_+,\tilde{r}}(\mathcal{O}_\delta)$ satisfying

$$\alpha_+(\xi\eta) - \alpha(\xi\eta) = -\mathrm{i}(e^{-\frac{\mathrm{i}}{2}\alpha(\xi\eta)} p_{0,1}(\xi\eta) - e^{\frac{\mathrm{i}}{2}\alpha(\xi\eta)} \bar{p}_{0,1}(\xi\eta)).$$

By Lemma 6.13, we obtain (64).

With $\psi := \phi \circ \varphi$, which satisfies $\psi \circ \rho = \rho \circ \psi$, let $\mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix} := \psi - \mathrm{Id}$. In view of the definitions of $\phi = \mathrm{Id} + \hat{\mathcal{U}}$ in Lemma 7.5 and φ given in (200), we have that

$$(\psi - \mathrm{Id})(\xi, \eta) = \begin{pmatrix} (\Theta - 1)\xi \\ (\Theta^{-1} - 1)\eta \end{pmatrix} + \begin{pmatrix} \hat{u}(\Theta\xi, \Theta^{-1}\eta) \\ \hat{v}(\Theta\xi, \Theta^{-1}\eta) \end{pmatrix}.$$

Since $\|\hat{u}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}}, \|\hat{v}\|_{\mathcal{O}_\delta,\tilde{\beta},r^{(7)}} < \frac{\varepsilon^{\frac{49}{50}}}{20}$, by Lemma 7.4, we have

$$\|\hat{u}(\Theta\xi, \Theta^{-1}\eta)\|_{\mathcal{O}_\delta,\beta_+,r_+}, \ \|\hat{v}(\Theta\xi, \Theta^{-1}\eta)\|_{\mathcal{O}_\delta,\beta_+,r_+} < \frac{\varepsilon^{\frac{49}{50}}}{20}, \tag{201}$$

Moreover, by Lemma 7.1, we obtain that

$$\|\Theta - 1\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}, \ \|\Theta^{-1} - 1\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} < \frac{3}{4}\|p_{0,1}\|_{\mathcal{O}_\delta,\beta,r} < \frac{3\varepsilon}{40r}. \tag{202}$$

Combining (201) and (202), we obtain $\|u\|_{\mathcal{O}_\delta,\beta_+,r_+}, \ \|v\|_{\mathcal{O}_\delta,\beta_+,r_+} < \frac{\varepsilon^{\frac{49}{50}}}{2}$.

It remains to prove (65) and (66). By (131), (132) in Proposition 7.2, we obtain that $p_+, q_+ \in \mathcal{A}_{\beta_+,r_+}(\mathcal{O}_\delta)$, and

$$\begin{aligned}
\|p_+\|_{\mathcal{O}_\delta,\beta_+,r_+} &< \left(1 + \frac{3}{4}\|p_{0,1}\|_{\mathcal{O},\beta,r}\right)\|\tilde{p}\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} + \|p_{0,1}\|^2_{\mathcal{O},2\beta,r}, \\
&< \left(1 + \frac{3\varepsilon}{40r}\right)\left(\frac{\varepsilon^{\frac{61}{32}}}{3} + 22(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\tilde{\beta},r^{(7)}}\right) + \varepsilon^2 \\
&< \frac{\varepsilon^{\frac{61}{32}}}{2} + 24(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\tilde{\beta},r^{(7)}}, \\
\|q_+\|_{\omega,\beta_+,r_+} &< \frac{\varepsilon^{\frac{61}{32}}}{2} + 24(K+1)\delta^{-1}\|e^{\frac{\mathrm{i}}{2}\alpha}\eta q + e^{-\frac{\mathrm{i}}{2}\alpha}\xi p\|_{\mathcal{O},\tilde{\beta},r^{(7)}}.
\end{aligned}$$

Moreover, by (133) in Proposition 7.2 and (198), (199), we have

$$\begin{aligned}
&\|e^{-\frac{\mathrm{i}}{2}\alpha_+}\xi p_+ + e^{-\frac{\mathrm{i}}{2}\alpha_+}\eta q_+\|_{\mathcal{O}_\delta,\beta_+,r_+} \\
&< \|e^{-\frac{\mathrm{i}}{2}\alpha}\xi\tilde{p} + e^{\frac{\mathrm{i}}{2}\alpha}\eta\tilde{q}\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} + \|p_{0,1}\|_{\mathcal{O},\beta,r}(\|\tilde{p}\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}} + \|\tilde{q}\|_{\mathcal{O}_\delta,2\beta_+,\tilde{r}}) + \|p_{0,1}\|^2_{\mathcal{O},\beta,r} \\
&< \frac{\varepsilon^{\frac{61}{32}}}{2} + \frac{\varepsilon}{10r}\cdot\frac{\varepsilon^{\frac{31}{32}}}{8} + \frac{\varepsilon^2}{100r^2} < \varepsilon^{\frac{61}{32}}.
\end{aligned}$$

It remains to show that

$$\psi(\xi,\eta) \in \mathcal{C}^{r}_{\omega,\beta} \quad \text{for } (\xi,\eta) \in \mathcal{C}^{r_+}_{\omega,\beta_+}, \quad \omega \in \mathcal{O}_\delta(r_+,\beta_+). \tag{203}$$

Recalling the definition of the set given in (15), we have

$$\mathcal{C}^{r_+}_{\omega,\beta_+} := \left\{(\xi,\eta) \in \mathbb{C}^2 : |\xi\eta - \omega| < \beta_+, \quad |\xi|, |\eta| < r_+\right\}.$$

Since $\|u\|_{\mathcal{O}_\delta,\beta_+,r_+}, \|v\|_{\mathcal{O}_\delta,\beta_+,r_+} < \dfrac{\varepsilon^{\frac{49}{50}}}{2}$, we have that

$$\begin{aligned} |(\xi + u(\xi,\eta))(\eta + v(\xi,\eta)) - \omega| &< \beta_+ + \|\eta u + \xi v + uv\|_{\omega,\beta_+,r_+} \\ &< \beta_+ + \varepsilon^{\frac{49}{50}} + \varepsilon^{\frac{49}{25}} < \beta, \\ |\xi + u(\xi,\eta)|, \; |\eta + v(\xi,\eta)| &< r_+ + \varepsilon^{\frac{49}{50}} < r. \end{aligned}$$

(203) is shown. □

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Appendix A: Proof of Lemma 6.11

Let $\varsigma := \max\{\|f_1 - f_2\|_{\mathcal{O},\beta'',r''}, \; \|g_1 - g_2\|_{\mathcal{O},\beta'',r''}\}$, which is, by (98), smaller than $\frac{\beta'^2}{16}$. Let $\omega \in \mathcal{O}(r'',\beta'')$. In order to estimate its norm, let us first decompose the following expression:

$$\begin{aligned} & h(e^{\mathrm{i}b\alpha(\xi\eta)}\xi + f_1, e^{-\mathrm{i}b\alpha(\xi\eta)}\eta + g_1) - h(e^{\mathrm{i}b\alpha(\xi\eta)}\xi + f_2, e^{-\mathrm{i}b\alpha(\xi\eta)}\eta + g_2) \\ & \quad = \sum_{l\ge 0}\Big(h_{l,0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1) \\ & \qquad -h_{l,0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\Big) \qquad (204) \\ & \qquad \cdot \left(e^{\mathrm{i}b\alpha}\xi + f_2\right)^l \\ & \qquad + \sum_{l\ge 1} h_{l,0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1)\left((e^{\mathrm{i}b\alpha}\xi + f_1)^l - (e^{\mathrm{i}b\alpha}\xi + f_2)^l\right) \qquad (205) \\ & \qquad + \text{similar expressions involving } h_{0,j} \text{ instead of } h_{l,0}. \qquad (206) \end{aligned}$$

- **Terms in** (204)

Expanding $h_{l,0}$ around $\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2$ in (204), we obtain:

$$\begin{aligned}
&h_{l,0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1) \\
&\quad - h_{l,0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2) \\
&= \sum_{k\geq 1} \frac{1}{k!} h_{l,0}^{(k)}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2) \\
&\quad \cdot \Big((e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1) - (e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\Big)^k .
\end{aligned} \tag{207}$$

Combining Corollary 6.5 together with Remark 6.6 and (98), we obtain, for $(\xi, \eta) \in \mathcal{C}^{r''}_{\omega,\beta''}$ and $m = 1, 2$:

$$\begin{aligned}
&|\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_m + e^{\mathrm{i}b\alpha}\xi g_m + f_m g_m - \omega| \\
&\quad \leq |\xi\eta - \omega| + |e^{-\mathrm{i}b\alpha}\eta f_m + e^{\mathrm{i}b\alpha}\xi g_m + f_m g_m| \\
&\quad < \beta'' + 2e^{\frac{9}{8}\tilde{\beta}} r'' \cdot \frac{\beta'^2}{16} + \frac{\beta'^4}{256} \\
&\quad < \frac{\beta'}{2} + \frac{101}{200} \cdot \frac{\beta'}{16} + \frac{\beta'}{256} < \beta'.
\end{aligned}$$

By Cauchy's inequality, we have, for all $(\xi, \eta) \in \mathcal{C}_{\omega,\beta''}$ and $\tilde{k} \geq 1$,

$$\frac{1}{\tilde{k}!} \left| h_{l,0}^{(\tilde{k})}(\xi\eta) \right| \leq \sup_{|z-\xi\eta| = \frac{\beta'}{2}} |h_{l,0}(z)| \left(\frac{2}{\beta'}\right)^{\tilde{k}} \leq \left(\frac{2}{\beta'}\right)^{\tilde{k}} |h_{l,0}|_{\omega,\beta'}. \tag{208}$$

We recall that, for $|z| < 1$,

$$\sum_{\tilde{k}\geq k} C_{\tilde{k}}^k z^{\tilde{k}-k} = \frac{1}{k!} \sum_{\tilde{k}\geq 0} \frac{d^k}{dz^k}\left(z^{\tilde{k}}\right) = \frac{1}{k!} \frac{d^k}{dz^k}\left(\frac{1}{1-z}\right) = (1-z)^{-(k+1)}.$$

Hence, developing $h_{l,0}^{(k)}$ around $\xi\eta$, we have, for $k \geq 1$,

$$\begin{aligned}
&\frac{1}{k!} \left\| h_{l,0}^{(k)}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2) \right\|_{\omega,\beta'',r''} \\
&\quad \leq \sum_{\tilde{k}\geq k} \frac{\left| h_{l,0}^{(\tilde{k})} \right|_{\omega,\beta''}}{k!(\tilde{k}-k)!} \| e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2 \|^{\tilde{k}-k}_{\omega,\beta'',r''} \\
&\quad = \sum_{\tilde{k}\geq k} \frac{\tilde{k}!}{(\tilde{k}-k)! \cdot k!} \cdot \frac{\left| h_{l,0}^{(\tilde{k})} \right|_{\omega,\beta''}}{\tilde{k}!} \| e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2 \|^{\tilde{k}-k}_{\omega,\beta'',r''}
\end{aligned}$$

$$\leq |h_{l,0}|_{\omega,\beta'} \sum_{\tilde{k}\geq k} C_{\tilde{k}}^{k} \left(\frac{2}{\beta'}\right)^{\tilde{k}} \left(\frac{\beta'^2}{16}\right)^{\tilde{k}-k}$$
$$= \left(\frac{2}{\beta'}\right)^{k} |h_{l,0}|_{\omega,\beta'} \sum_{\tilde{k}\geq k} C_{\tilde{k}}^{k} \left(\frac{\beta'}{8}\right)^{\tilde{k}-k}$$
$$= \left(\frac{2}{\beta'}\right)^{k} |h_{l,0}|_{\omega,\beta'} \left(1-\frac{\beta'}{8}\right)^{-(k+1)}. \tag{209}$$

Recalling that $|\omega| < r''^2 - \beta''$, by Lemma 3.3, we have, for $-1 \leq b \leq 1$,

$$\begin{aligned}
&\left\|(e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1) - (e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\right\|_{\omega,\beta'',r''} \\
&\leq \left\|e^{-\mathrm{i}b\alpha}\eta(f_1-f_2)\right\|_{\omega,\beta'',r''} + \left\|e^{\mathrm{i}b\alpha}\xi(g_1-g_2)\right\|_{\omega,\beta'',r''} \\
&\quad + \|(f_1-f_2)g_1 - f_2(g_1-g_2)\|_{\omega,\beta'',r''} \\
&< e^{\frac{9}{8}\tilde{\beta}} r'' \left(\|f_1-f_2\|_{\omega,\beta'',r''} + \|g_1-g_2\|_{\omega,\beta'',r''}\right) \\
&\quad + \|f_1-f_2\|_{\omega,\beta'',r''}\|g_1\|_{\omega,\beta'',r''} + \|f_2\|_{\omega,\beta'',r''}\|g_1-g_2\|_{\omega,\beta'',r''} \\
&< \frac{101}{50} r''\varsigma + \frac{\beta'^2}{8}\varsigma < \frac{13}{25}\varsigma.
\end{aligned} \tag{210}$$

Since $\frac{\varsigma}{\beta'} < \frac{\beta'}{8}$ and according to (85), we have $1-\frac{\beta'}{8} > \frac{99}{100}$ and $1-\frac{26\varsigma}{25\beta'}\left(1-\frac{\beta'}{8}\right)^{-1} > \frac{99}{100}$, so that $\frac{26}{25}\left(1-\frac{\beta'}{8}\right)^{-2}\left(1-\frac{26\varsigma}{25\beta'}\left(1-\frac{\beta'}{8}\right)^{-1}\right)^{-1} < \frac{27}{25}$. Combining (209) and (210), together with (19), we obtain

$$\begin{aligned}
&\left\|h_{l,0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1) - h_{l,0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\right\|_{\omega,\beta'',r''} \\
&\leq \sum_{k\geq 1}\frac{1}{k!}\|h_{l,0}^{(k)}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\|_{\omega,\beta'',r''} \\
&\quad\cdot \left\|(e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1) - (e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\right\|_{\omega,\beta'',r''}^{k} \\
&< \|h\|_{\omega,\beta',r'} r'^{-l} \cdot \sum_{k\geq 1}\left(\frac{26\varsigma}{25\beta'}\right)^{k}\left(1-\frac{\beta'}{8}\right)^{-(k+1)} \\
&= \|h\|_{\omega,\beta',r'} r'^{-l} \frac{26\varsigma}{25\beta'}\left(1-\frac{\beta'}{8}\right)^{-2}\left(1-\frac{26\varsigma}{25\beta'}\left(1-\frac{\beta'}{8}\right)^{-1}\right)^{-1} \\
&< \frac{27\varsigma}{25\beta'}\|h\|_{\mathcal{O},\beta',r'} r'^{-l}.
\end{aligned} \tag{211}$$

Hence, according to Lemma 3.3, for $l \geq 0$,

$$\left\|\left(h_{l,0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1) - h_{l,0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\right)\right.$$

$$\cdot \left(e^{\mathrm{i}b\alpha}\xi + f_2\right)^l \Big\|_{\omega,\beta'',r''}$$
$$< \frac{27\varsigma}{25\beta'}\|h\|_{\omega,\beta',r'} r'^{-l}\left(e^{\frac{9}{8}\tilde{\beta}}r'' + \frac{\beta'^2}{16}\right)^l.$$

On the other hand, (95) implies $2r''\tilde{\beta} < \frac{r'-r''}{8}$. Indeed, since $0 < r'' < r' < \frac{1}{4}$ and $0 < \beta < 1$, then $8\beta^{\frac{1}{2}} < (r'-r'')r''$ implies

$$2r''\tilde{\beta} = 2r'' \cdot 16\beta^{\frac{5}{4}} < 8\beta^{\frac{5}{4}} < 8\beta < 8\beta^{\frac{1}{2}} \cdot \frac{(r'-r'')r''}{8} < \frac{r'-r''}{8}. \tag{212}$$

Therefore, according to (86), we have

$$r' - e^{\frac{9}{8}\tilde{\beta}}r'' - \frac{\beta'^2}{16} > r' - r'' - 2\tilde{\beta}r'' > r' - r'' - \frac{r'-r''}{8} = \frac{7}{8}(r'-r'').$$

As a consequence, we have

$$\sum_{k\geq 0} r'^{-k}\left(e^{\frac{9}{8}\tilde{\beta}}r'' + \frac{\beta'^2}{16}\right)^k = \frac{r'}{r' - e^{\frac{9}{8}\tilde{\beta}}r'' - \frac{\beta'^2}{16}} < \frac{8r'}{7(r'-r'')}.$$

Thus, under (95), the $\|\cdot\|_{\mathcal{O},\beta'',r''}$-norm of (204) is bounded by

$$\frac{27\varsigma}{25\beta'}\|h\|_{\mathcal{O},\beta',r'}\sum_{l\geq 0} r'^{-l}\left(e^{\frac{9}{8}\tilde{\beta}}r'' + \frac{\beta'^2}{16}\right)^l < \frac{27}{25}\cdot\frac{8r'}{7(r'-r'')}\frac{\varsigma}{\beta'}\|h\|_{\mathcal{O},\beta',r'}$$
$$< \frac{5r'}{4(r'-r'')}\frac{\varsigma}{\beta'}\|h\|_{\mathcal{O},\beta',r'}. \tag{213}$$

To show (204)$\in \mathcal{A}_{\beta'',r''}(\mathcal{O})$ provided that $f_1, f_2, g_1, g_2 \in \mathcal{A}_{\beta'',r''}(\mathcal{O})$, it remains to verify the analyticity on $\mathcal{O}(r'',\beta'')$ for the coefficients $(204)_{l,j}$, $l, j \geq 0$, $lj = 0$. According to (207), (209) and (210), for $\tilde{l} \geq 0$,

$$h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1) - h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2$$
$$+ e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2) \in \mathcal{A}_{\beta'',r''}(\mathcal{O}),$$

and, by (23) and (211), for $\tilde{l} \geq 0$, for $l, j \geq 0$ with $lj = 0$,

$$\Big|\Big(h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1)$$
$$- h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\Big)_{l,j}\Big|_{\mathcal{O}(r'',\beta'')} \leq \frac{27\varsigma}{25\beta'}\frac{\|h\|_{\mathcal{O},\beta',r'}}{r'^{\tilde{l}}r''^{l+j}}.$$

Note that, for $l \geq 0$,

$$
\begin{aligned}
(204)_{l,0} = &\sum_{\tilde{l}\geq 0}\sum_{0\leq k\leq l}\left((e^{\mathrm{i}b\alpha}\xi + f_2)^{\tilde{l}}\right)_{k,0}\\
&\cdot\Big(h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1)\\
&-h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\Big)_{l-k,0}\\
&+\sum_{\tilde{l}\geq 0}\sum_{k\geq 1}\left((e^{\mathrm{i}b\alpha}\xi + f_2)^{\tilde{l}}\right)_{l+k,0}(\xi\eta)^k\\
&\cdot\Big(h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1)\\
&-h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\Big)_{0,k},
\end{aligned}
$$

and for $j \geq 1$,

$$
\begin{aligned}
(204)_{0,j} = &\sum_{\tilde{l}\geq 0}\sum_{0\leq k\leq j}\left((e^{\mathrm{i}b\alpha}\xi + f_2)^{\tilde{l}}\right)_{0,k}\\
&\cdot\Big(h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1)\\
&-h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\Big)_{0,j-k}\\
&+\sum_{\tilde{l}\geq 0}\sum_{k\geq 1}\left((e^{\mathrm{i}b\alpha}\xi + f_2)^{\tilde{l}}\right)_{0,j+k}(\xi\eta)^k\\
&\cdot\Big(h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1)\\
&-h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\Big)_{k,0},
\end{aligned}
$$

where, by Lemma 6.5 and by (23), for $l, j \geq 0$ with $lj = 0$,

$$
\left|\left((e^{\mathrm{i}b\alpha}\xi + f_2)^{\tilde{l}}\right)_{l,j}\right|_{\mathcal{O}(r'',\beta'')} \leq \left(e^{\frac{9}{8}\tilde{\beta}}r'' + \frac{\beta'^2}{16}\right)^{\tilde{l}} r''^{-(l+j)}.
$$

Then we see that, for $\omega \in \mathcal{O}(r'', \beta'')$,

$$
\begin{aligned}
\sum_{\tilde{l}\geq 0}\sum_{0\leq k\leq l}&\Bigg|\left((e^{\mathrm{i}b\alpha}\xi + f_2)^{\tilde{l}}\right)_{k,0}(\omega)\\
&\cdot\Big(h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_1 + e^{\mathrm{i}b\alpha}\xi g_1 + f_1 g_1)\\
&-h_{\tilde{l},0}(\xi\eta + e^{-\mathrm{i}b\alpha}\eta f_2 + e^{\mathrm{i}b\alpha}\xi g_2 + f_2 g_2)\Big)_{l-k,0}(\omega)\Bigg|
\end{aligned}
$$

$$\leq \frac{27(l+1)\varsigma\|h\|_{\mathcal{O},\beta',r'}}{25\beta' r''^{l}} \sum_{\tilde{l}\geq 0}\left(e^{\frac{9}{8}\tilde{\beta}}\frac{r''}{r'}+\frac{\beta'^2}{16r'}\right)^{\tilde{l}},$$

$$\sum_{\tilde{l}\geq 0}\sum_{k\geq 1}\left|\left((e^{\mathrm{i}b\alpha}\xi+f_2)^{\tilde{l}}\right)_{l+k,0}(\omega)\cdot\omega^k\right.$$
$$\cdot\left(h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\right.$$
$$\left.\left.-h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_2+e^{\mathrm{i}b\alpha}\xi g_2+f_2g_2)\right)_{0,k}(\omega)\right|$$
$$\leq \sum_{\tilde{l}\geq 0}\sum_{k\geq 1}\left(e^{\frac{9}{8}\tilde{\beta}}r''+\frac{\beta'^2}{16}\right)^{\tilde{l}} r''^{-(l+k)}(r''^2-\beta'')^k\cdot\frac{27\varsigma}{25\beta'}\frac{\|h\|_{\mathcal{O},\beta',r'}}{r'^{\tilde{l}}r''^k}$$
$$\leq \frac{27\varsigma\|h\|_{\mathcal{O},\beta',r'}}{25\beta' r''^{l}}\sum_{k\geq 1}\left(1-\frac{\beta''}{r''^2}\right)^k\sum_{\tilde{l}\geq 0}\left(e^{\frac{9}{8}\tilde{\beta}}\frac{r''}{r'}+\frac{\beta'^2}{16r'}\right)^{\tilde{l}},$$

which implies the analyticity of $(204)_{l,0}$ under (95), and it is similar for that of $(204)_{0,j}$. Hence, (204)$\in\mathcal{A}_{\beta'',r''}(\mathcal{O})$.

- **Terms in** (205)

Note that (19) and (208) imply that, for $\omega\in\mathcal{O}(r'',\beta'')$, for $l\geq 0$,

$$\|h_{l,0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\|_{\omega,\beta'',r''}$$
$$\leq\sum_{k\geq 0}\frac{|h_{l,0}^{(k)}|_{\omega,\beta''}}{k!}\|e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1\|^k_{\omega,\beta'',r''}$$
$$\leq |h_{l,0}|_{\omega,\beta'}\sum_{k\geq 0}\left(\frac{2}{\beta'}\right)^k\left(2e^{\frac{9}{8}\tilde{\beta}}r''\frac{\beta'^2}{8}+\frac{\beta'^4}{256}\right)^k\leq\frac{101}{100}\frac{\|h\|_{\mathcal{O},\beta',r'}}{r'^l}. \tag{214}$$

In (205), we have, for $l\geq 1$,

$$\begin{aligned}\|(e^{\mathrm{i}b\alpha}\xi+f_1)^l-(e^{\mathrm{i}b\alpha}\xi+f_2)^l\|_{\omega,\beta'',r''} &\leq\sum_{k=1}^{l}C_l^k\|e^{\mathrm{i}b\alpha}\xi+f_1\|^{l-k}_{\omega,\beta'',r''}\|f_2-f_1\|^k_{\omega,\beta'',r''}\\ &<\sum_{k=1}^{l}C_l^k\left(e^{\frac{9}{8}\tilde{\beta}}r''+\frac{\beta'^2}{16}\right)^{l-k}\varsigma^k\\ &=\left(e^{\frac{9}{8}\tilde{\beta}}r''+\frac{\beta'^2}{16}+\varsigma\right)^l-\left(e^{\frac{9}{8}\tilde{\beta}}r''+\frac{\beta'^2}{16}\right)^l\\ &\leq l\left(e^{\frac{9}{8}\tilde{\beta}}r''+\beta'^2+\varsigma\right)^{l-1}\varsigma.\end{aligned}\tag{215}$$

Furthermore, by (95), we have $\beta^{\frac{5}{4}} < \beta^{\frac{1}{2}} < \frac{r'-r''}{32}$. Recalling that $\varsigma < \frac{\beta'^2}{8} \leq \frac{(16\beta^{\frac{5}{4}})^2}{8}$ and using (212), we have

$$1-\frac{e^{\frac{9}{8}\tilde{\beta}}r''+\beta'^2+\varsigma}{r'} > 1-\frac{(1+2\tilde{\beta})r''}{r'} > \frac{r'-r''}{r'}-\frac{r'-r''}{8r'}=\frac{7(r'-r'')}{8r'}. \quad (216)$$

Therefore, the $\|\cdot\|_{\mathcal{O},\beta'',r''}$-norm of (205) is bounded by

$$\begin{aligned}
&\sum_{l\geq 1}\|h_{l,0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\|_{\omega,\beta'',r''}\\
&\quad\cdot\|(e^{\mathrm{i}b\alpha}\xi+f_1)^l-(e^{\mathrm{i}b\alpha}\xi+f_2)^l\|_{\mathcal{O},\beta'',r''}\\
&< \frac{101\varsigma}{100r'}\|h\|_{\mathcal{O},\beta',r'}\sum_{l\geq 1} l\left(\frac{e^{\frac{9}{8}\tilde{\beta}}r''+\beta'^2+\varsigma}{r'}\right)^{l-1}\\
&= \frac{101\varsigma}{100r'}\|h\|_{\mathcal{O},\beta',r'}\left(1-\frac{e^{\frac{9}{8}\tilde{\beta}}r''+\beta'^2+\varsigma}{r'}\right)^{-2}\\
&< \frac{101\varsigma}{100r'}\|h\|_{\mathcal{O},\beta',r'}\cdot\frac{8^2r'^2}{7^2(r'-r'')^2} < \frac{7r'\varsigma\|h\|_{\mathcal{O},\beta',r'}}{5(r'-r'')^2}.
\end{aligned} \quad (217)$$

For $l \geq 0$, we have that $(205)_{l,0}$ equals to

$$\begin{aligned}
&\sum_{\tilde{l}\geq 1}\sum_{0\leq k\leq l}\Big(h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\Big)_{l-k,0}\\
&\quad\times\Big((e^{\mathrm{i}b\alpha}\xi+f_1)^{\tilde{l}}-(e^{\mathrm{i}b\alpha}\xi+f_2)^{\tilde{l}}\Big)_{k,0}\\
&\quad+\sum_{\tilde{l}\geq 1}\sum_{k\geq 1}\Big(h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\Big)_{l+k,0}\\
&\quad\times\Big((e^{\mathrm{i}b\alpha}\xi+f_1)^{\tilde{l}}-(e^{\mathrm{i}b\alpha}\xi+f_2)^{\tilde{l}}\Big)_{0,k}(\xi\eta)^k\\
&\quad+\sum_{\tilde{l}\geq 1}\sum_{k\geq 1}\Big(h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\Big)_{0,k}\\
&\quad\times\Big((e^{\mathrm{i}b\alpha}\xi+f_1)^{\tilde{l}}-(e^{\mathrm{i}b\alpha}\xi+f_2)^{\tilde{l}}\Big)_{l+k,0}(\xi\eta)^k,
\end{aligned}$$

and for $j \geq 1$, $(205)_{0,j}$ equals to

$$\begin{aligned}
&\sum_{\tilde{l}\geq 1}\sum_{0\leq k\leq j}\Big(h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\Big)_{0,j-k}\\
&\quad\times\Big((e^{\mathrm{i}b\alpha}\xi+f_1)^{\tilde{l}}-(e^{\mathrm{i}b\alpha}\xi+f_2)^{\tilde{l}}\Big)_{0,k}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\tilde{l}\geq 1}\sum_{k\geq 1}\Big(h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\Big)_{0,j+k}\\
&\times\Big((e^{\mathrm{i}b\alpha}\xi+f_1)^{\tilde{l}}-(e^{\mathrm{i}b\alpha}\xi+f_2)^{\tilde{l}}\Big)_{k,0}(\xi\eta)^k\\
&+\sum_{\tilde{l}\geq 1}\sum_{k\geq 1}\Big(h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\Big)_{k,0}\\
&\times\Big((e^{\mathrm{i}b\alpha}\xi+f_1)^{\tilde{l}}-(e^{\mathrm{i}b\alpha}\xi+f_2)^{\tilde{l}}\Big)_{0,j+k}(\xi\eta)^k.
\end{aligned}
$$

If $f_1, f_2, g_1, g_2 \in \mathcal{A}_{\beta'',r''}(\mathcal{O})$, then by (214)–(216) and (19), we see the analyticity of $(205)_{l,0}$, since for $\omega \in \mathcal{O}(r'', \beta'')$,

$$
\begin{aligned}
&\Bigg|\sum_{\tilde{l}\geq 1}\sum_{0\leq k\leq l}\Big(h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\Big)_{l-k,0}(\omega)\\
&\quad\times\Big((e^{\mathrm{i}b\alpha}\xi+f_1)^{\tilde{l}}-(e^{\mathrm{i}b\alpha}\xi+f_2)^{\tilde{l}}\Big)_{k,0}(\omega)\Bigg|\\
&\leq\sum_{\tilde{l}\geq 1}\sum_{0\leq k\leq l}\frac{101}{100}\frac{\|h\|_{\mathcal{O},\beta',r'}}{r'^{\tilde{l}}r''^{l-k}}\cdot\frac{\tilde{l}(e^{\frac{9}{8}\tilde{\beta}}r''+\beta'^2+\varsigma)^{\tilde{l}-1}\varsigma}{r''^k}\\
&=\frac{101}{100}\frac{(l+1)\|h\|_{\mathcal{O},\beta',r'}\varsigma}{r'r''^l}\sum_{\tilde{l}\geq 1}\tilde{l}\left(\frac{e^{\frac{9}{8}\tilde{\beta}}r''+\beta'^2+\varsigma}{r'}\right)^{\tilde{l}-1},\\
&\Bigg|\sum_{\tilde{l}\geq 1}\sum_{k\geq 1}\Big(h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\Big)_{l+k,0}(\omega)\\
&\quad\times\Big((e^{\mathrm{i}b\alpha}\xi+f_1)^{\tilde{l}}-(e^{\mathrm{i}b\alpha}\xi+f_2)^{\tilde{l}}\Big)_{0,k}(\omega)\cdot\omega^k\Bigg|\\
&\leq\sum_{\tilde{l}\geq 1}\sum_{k\geq 1}\frac{101}{100}\frac{\|h\|_{\mathcal{O},\beta',r'}}{r'^{\tilde{l}}r''^{l+k}}\cdot\frac{\tilde{l}(e^{\frac{9}{8}\tilde{\beta}}r''+\beta'^2+\varsigma)^{\tilde{l}-1}\varsigma}{r''^k}\cdot(r''^2-\beta'')^k\\
&=\frac{101}{100}\frac{\|h\|_{\mathcal{O},\beta',r'}\varsigma}{r'r''^l}\sum_{\tilde{l}\geq 1}\tilde{l}\left(\frac{e^{\frac{9}{8}\tilde{\beta}}r''+\beta'^2+\varsigma}{r'}\right)^{\tilde{l}-1}\sum_{k\geq 1}\frac{(r''^2-\beta'')^k}{r''^{2k}},\\
&\Bigg|\sum_{\tilde{l}\geq 1}\sum_{k\geq 1}\Big(h_{\tilde{l},0}(\xi\eta+e^{-\mathrm{i}b\alpha}\eta f_1+e^{\mathrm{i}b\alpha}\xi g_1+f_1g_1)\Big)_{0,k}(\omega)\\
&\quad\times\Big((e^{\mathrm{i}b\alpha}\xi+f_1)^{\tilde{l}}-(e^{\mathrm{i}b\alpha}\xi+f_2)^{\tilde{l}}\Big)_{l+k,0}(\omega)\cdot\omega^k\Bigg|
\end{aligned}
$$

$$\leq \sum_{\tilde{l}\geq 1}\sum_{k\geq 1} \frac{101}{100}\frac{\|h\|_{\mathcal{O},\beta',r'}}{r'^{\tilde{l}}r''^{k}} \cdot \frac{\tilde{l}(e^{\frac{9}{8}\tilde{\beta}}r'' + \beta'^2 + \varsigma)^{\tilde{l}-1}\varsigma}{r''^{l+k}} \cdot (r''^2 - \beta'')^k$$

$$= \frac{101}{100}\frac{\|h\|_{\mathcal{O},\beta',r'}\varsigma}{r'r''^{l}} \sum_{\tilde{l}\geq 1} \tilde{l}\left(\frac{e^{\frac{9}{8}\tilde{\beta}}r'' + \beta'^2 + \varsigma}{r'}\right)^{\tilde{l}-1} \sum_{k\geq 1}\frac{(r''^2 - \beta'')^k}{r''^{2k}}.$$

The proof for $(205)_{0,j}$ is similar, hence (205)$\in \mathcal{A}_{\beta'',r''}(\mathcal{O})$.

Combining (213), (217) and similar estimates obtained for expressions (206), we obtain

$$\|h(e^{\mathrm{i}b\alpha}\xi + f_1, e^{-\mathrm{i}b\alpha}\eta + g_1) - h(e^{\mathrm{i}b\alpha}\xi + f_2, e^{-\mathrm{i}b\alpha}\eta + g_2)\|_{\mathcal{O},\beta'',r''}$$
$$< \frac{1}{r' - r''}\left(\frac{14r'}{5(r' - r'')} + \frac{5r'}{2\beta'}\right)\varsigma\|h\|_{\mathcal{O},\beta',r'} < \frac{3r'}{(r' - r'')\beta'}\varsigma\|h\|_{\mathcal{O},\beta',r'},$$

since (95) implies that $\beta' \leq \tilde{\beta} < \frac{r'-r''}{64}$. This finishes the proof of Lemma 6.11. □

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