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A Structure Theorem for Neighborhoods of Compact Complex Manifolds

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Abstract
We construct an injective map from the set of holomorphic equivalence classes of neighborhoods M of a compact complex manifold C into $\mathbb{C}^m$ for some $m < \infty$ when $(TM)|_C$ is fixed and the normal bundle of C in M is either *weakly negative* or 2-*positive*.

Keywords Normal forms · Neighborhoods of complex manifolds · Weakly negative or positive normal bundles · Foliations

Mathematics Subject Classification 32Q57 · 32Q28 · 32L10 · 37F50

1 Introduction

Let C be a compact complex manifold. We say that two holomorphic embeddings $f: C \hookrightarrow M$ and $\tilde{f}: C \hookrightarrow \tilde{M}$ are *holomorphic equivalent* if there is a biholomorphic mapping F from a neighborhood of $f(C)$ in M into a neighborhood of $\tilde{f}(C)$ in $\tilde{M}$ such that $Ff = \tilde{f}$. To classify such neighborhoods M, we identify C with $f(C)$ via f. We also fix the normal bundle N_C of C in M and the restriction $T_C M$ of TM on C. To determine the holomorphic equivalence of two neighborhoods, Grauert [10] introduced

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the *formal principle* that asserts two neighborhoods of C are holomorphic equivalent if they are formally equivalent. The formal principle holds for neighborhoods when N_C has nice curvature properties. By definitions in [10, Def. 1, p. 342] and [6, Def. 1, p. 108], N_C is *weakly negative* (resp. q-positive) if the zero section of N_C admits a tubular neighborhood $Tub(C)$ in N_C whose boundary is strictly pseudoconvex (resp. of at least q negative Levi eigenvalues). When N_C is *weakly negative*, Grauert [10] proved that the formal principle holds when C has codimension one. Hironaka and Rossi [15] extended Grauert's result to arbitrary positive codimension as well as to the reduced complex spaces C that are exceptional in the sense of Grauert. Griffiths [13] showed that the formal principle holds when N_C is *sufficiently positive* and $\dim C \geq 3$. This result was improved by Hirschowitz [16] including $\dim C \geq 2$ with *weakly positive* N_C and then by Commichau-Grauert [6] for 1-*positive* normal bundle N_C for $\dim C \geq 1$.

Despite all these positive results on the formal principle, it remains unknown until now if the equivalence classes of neighborhoods of C with a fixed $T_C M$ form a *finite-dimensional* space when the formal principle holds. The main result of this paper provides a structure for the neighborhoods of C as follows.

Theorem 1.1 *Let C be a compact complex manifold. Assume that N_C is either weakly negative or 2-positive. There is an injective mapping from the set of holomorphic equivalence classes of neighborhoods of C into the finite-dimensional space*

$$\mathcal{H}^1(T_C M) := \bigoplus_{\ell \geq 2} H^1(C, T_C M \otimes S^\ell N_C^*),$$

where $S^\ell N_C^*$ is the ℓ-th symmetric power of the dual bundle N_C^* of N_C.

Note that $\dim \mathcal{H}^1(T_C M) < \infty$ follows from the assumption on N_C. Indeed, it is known that $\dim \mathcal{H}^1(T_C M)$ is finite, when the zero section of N_C admits a relatively compact neighborhood W in N_C that is either weakly negative or 2-positive; see Lemma 5.1.

The formal principle does not rule out the existence of an infinite-dimensional moduli space for equivalence classes. In fact, given a (compact and smooth) Riemann surface C and a *positive* line bundle N_C, Morrow-Rossi [26, p. 323] constructed a complete set of the equivalence classes of holomorphic transverse foliations of neighborhoods of such C, under the smaller group that preserves the foliations, showing that the moduli space for the equivalence classes is infinite dimensional. See Proposition 6.1 for details.

For other closely related results, we mention that when C is the Riemann sphere and rank $N_C = 1$, Hurtubise-Kamran [17] showed that the space of neighborhoods with 1-positive N_C is infinite dimensional, and Mishustin [25] constructed a normal form with infinitely many invariants. See Ilyashenko [19] when C is elliptic curve and N_C is a positive line bundle and recent results of Falla Luza and Loray [7] on the Riemann sphere.

On the other hand, when C is an elliptic curve embedded in a complex surface with a topologically trivial N_C, Arnol'd [4] showed that the formal principle holds under an extra *Diophantine condition* on $T_C M$ when it splits as the direct sum $TC \oplus N_C$. Furthermore, Arnol'd showed for the first time that the formal principle *fails* when N_C is topologically trivial for an elliptic curve C and a certain Diophantine condition is violated (see [9, sect. 5.4] for construction of counter-examples). Arnold's theorem was extended by Ilyashenko and Pjartli [20] to the case when C is the product of finitely many elliptic curves and N_C is the direct sum of line bundles, and by the authors [8] for a complex torus with a Hermitian flat N_C satisfying a Diophantine condition. Of course, the study of holomorphic neighborhoods of embedded compact complex manifolds has a long history. The reader is referred to [9] and references therein on neighborhoods of compact manifolds. See also recent work of Hwang [18], Koike [22], and Loray-Thom-Touzet [23].

The paper is organized as follows.

In Sect. 2, we construct a formal normal form for holomorphic neighborhoods of C that is realized as a subset of $\mathcal{H}^1(T_C M)$. Our injectivity assertion in Theorem 1.1 remains true if N_C is merely 1-positive and $H^0(C, T_C M \otimes S^\ell N_C^*) = 0$ for all $\ell > 1$. In Sect. 3, we find a formal normal form for *tangential* foliations of neighborhoods of C that contain C as a leaf. In Sect. 4, we find a formal normal form for transverse foliations of neighborhoods of C. In Sect. 5, we apply the theorems in [6, 10] to show Theorem 1.1 and analogous classification for transverse foliations. In Sect. 6, we use a theorem of Camacho-Movasati-Sad [5] to show that when the genus of the compact Riemann surface C is bigger than one, there are neighborhoods of C that are not linearizable. Therefore, the equivalence classes contain at least two elements.

2 A Formal Normal Form to Classify Neighborhoods

We have mentioned Grauert's formal principle asserting that two neighborhoods are holomorphic equivalent if they are formally equivalent. To further motivate our results, let us describe the following problems about the classifications of neighborhoods. We recall the *Kuranishi problem* mentioned in 1982 by Morrow-Rossi [26] for the study of neighborhoods of C, which is to construct a *moduli space* or a parametrization that classifies neighborhoods of C completely and describes the structures of the moduli space. This paper provides partial answers to this problem. However, we should mention that a deformation theory for a *family* of neighborhoods of reduced complex spaces with isolated singularity was achieved in 1972 by Grauert [11] and it is evident that such a deformation theory for the family does not provide a complete classification for an *individual* member in the family, as shown by the finite dimensionality of the Mather-Yau classification [24] for complex hypersurfaces in $\mathbb{C}^n$ with isolated singularity. An interested reader can consult the book of Greuel, Loosen, and Shustin [12] for deformation theory of neighborhoods of germs of complex spaces with isolated singularity and other important results.

With the above introduction, we now turn to the construction of a formal normal form for formal equivalence classes of holomorphic neighborhoods. We will also study the analogous problems for the tangential and transversal foliations of neighborhoods of C. Certain features of the normal forms will be described in details as they will be useful in the convergent proof.

To study classifications, we will use transition functions for various vector bundles in coordinate charts. Let C be a compact complex manifold embedded in a complex manifold M. We cover a neighborhood of C in M by open sets V_j and choose coordinate charts (z_j, w_j) on V_j for M such that

$$U_j := C \cap V_j = \{w_j = 0\}.$$

To have Leray coverings, we may assume for instance that U_j, V_j are biholomorphic to polydiscs. Indeed, the existence of such covering $\{V_j\}$ with a *prescribed* covering $\{U_j\}$ of C is ensured by the triviality of holomorphic vector bundle on the unit ball and a result of Siu [27, Cor. 1] saying that a Stein manifold in a complex manifold M admits a neighborhood that is biholomorphic to a neighborhood of the zero section of the normal bundle of C in M.

Let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a finite open covering of C with coordinate charts $z_i = \varphi_i(p) = (z_i^1, \dots, z_i^n)$ defined on U_i. Furthermore, we may assume that $U_1, \dots, U_m$ are in general position (see [9, Def. A.15]); in particular, the intersection of any number of $U_1, \dots, U_m$ has finitely many connected components. For the existence of such covering $\{U_1, \dots, U_m\}$, see [9, Prop. A.19 (a)]. Let

$$z_k = \varphi_{kj}(z_j) = \varphi_k \varphi_j^{-1}(z_j) \tag{2.1}$$

be the transition function of C on $U_{kj} := U_k \cap U_j$. Thus the neighborhood M has transition functions on $V_{kj} := V_j \cap V_k$

$$\Phi_{kj}\colon \begin{aligned} z_k &= \Phi_{kj}^{\tau}(z_j, w_j) := \varphi_{kj}(z_j) + l_{kj}(z_j)w_j + \phi_{kj}^{\tau}(z_j, w_j),\\ w_k &= \Phi_{kj}^{\nu}(z_j, w_j) := t_{kj}(z_j)w_j + \phi_{kj}^{\nu}(z_j, w_j). \end{aligned} \tag{2.2}$$

Here and in what follows, h^τ (resp. h^ν) stands for n-tuple (resp. d-tuple) of holomorphic functions $(h^1, \dots, h^n)$ (resp. $(h^{n+1}, \dots, h^{n+d})$). Also, $\phi_{kj}^\tau, \phi_{kj}^\nu$ vanish to order ≥ 2 along $w_j = 0$:

$$\phi_{kj}^{\tau}(z_j, w_j) = O(|w_j|^2), \quad \phi_{kj}^{\nu}(z_j, w_j) = O(|w_j|^2).$$

For abbreviation, we also call $\Phi := \{\Phi_{kj}\}$ a neighborhood of C and $O(2)$ denotes such a ϕ_{kj}. Throughout the paper, we identify the restrictions $z_j|_{U_j}, \varphi_{kj}(z_j)|_{U_j}$ of $z_j, \varphi_{kj}(z_j)$ in (2.2) with $z_j, \varphi_{kj}(z_j)$ in (2.1), respectively.

Note that the transition functions of N_C are

$$N_{kj}(z_j, v_j) = (\varphi_{kj}(z_j), t_{kj}(z_j)v_j) \tag{2.3}$$

and the transition matrices of $T_C M := (TM)|_C$ are

$$g_{kj} := \begin{pmatrix} s_{kj} & l_{kj} \\ 0 & t_{kj} \end{pmatrix}(z_j) \quad \text{on } U_j \cap U_k \tag{2.4}$$

for some $n \times d$ matrix l_{kj}, while s_{kj} is the Jacobian matrix of φ_{kj}. When $l_{kj} = 0$, $T_C M$ splits as $TC \oplus N_C$ (see [26, Prop. 2.9]). With g_{kj}, $T_C M$ has a basis $\{\tilde{e}_k^\mu; \mu = 1, \dots, n+d\}$. Set

$$\tilde{e}_k^\tau = \{\tilde{e}_k^i : i = 1, \dots, n\}, \quad \tilde{e}_k^\nu = \{\tilde{e}_k^{n+j} : j = 1, \dots, d\} \tag{2.5}$$

with $\tilde{e}_k^\nu$ being the basis of N_C on U_k. To study the foliations, we will choose $\tilde{e}_k^\nu$ to be a *flat basis* when N_C is *flat*, i.e., its transition functions are locally constant.

Throughout the paper, we fix

$$N_{kj}^0(z_j, w_j) = (\varphi_{kj}(z_j) + l_{kj}(z_j)w_j, t_{kj}(z_j)w_j).$$

We emphasize that $\{N_{kj}^0\}$ does not necessarily define a neighborhood of C as it may not be a cocycle. Thus, we introduce the following.

Definition 2.1 Fix $T_C M$. Fix a holomorphic neighborhood $N^1 = \{N_{kj}^1\}$ of C such that $N_{kj}^1 = N_{kj}^0 + O(2)$. If $T_C M$ splits, we always take $N_{kj}^0 = N_{kj}$ defined by (2.3). Let $\mathcal{M}$ be the set of germs of holomorphic neighborhoods Φ of C in complex manifolds $\tilde{M}$ such that $T_C M = T_C \tilde{M}$ and $\Phi = N^1 + O(2)$.

Throughout the paper, we assume that $\mathcal{M}$ is non-empty. Therefore the existence of N^1 is ensured.

For the holomorphic (resp. formal) equivalence of two neighborhoods of C, we restrict to biholomorphic (resp. formal) mappings F transforming Φ into $\tilde{\Phi}$ fixing C pointwise. We now express F in local coordinates. Suppose that $\{\Phi_{kj}\}$ and $\{\tilde{\Phi}_{kj}\}$ are transition functions for neighborhoods $M, \tilde{M}$ of C. They have local coordinates (z_k, w_k) and $(\tilde{z}_k, \tilde{w}_k)$ on $V_k, \tilde{V}_k$, respectively, where $\{V_k\}$ and $\{\tilde{V}_k\}$ are open coverings of $M, \tilde{M}$. We may assume that $V_k \cap C = \tilde{V}_k \cap C = U_k$ are mapped onto polydiscs in $\mathbb{C}^n$ via $z_k, \tilde{z}_k$, respectively. Thus $\{U_k\}$ is a Leray covering of C such that $H^q(\mathcal{U}, \mathcal{F}) = H^q(C, \mathcal{F})$ for any coherent analytic sheaf $\mathcal{F}$ on C.

For convenience, we also assume that coordinates mappings (z_k, w_k), $(\tilde{z}_k, \tilde{w}_k)$ are biholomorphic in neighborhoods of the closures of $V_k, \tilde{V}_k$. Since F fixes C pointwise, then F can be expressed as

$$F_k : \begin{array}{l} \tilde{z}_k = F_k^\tau(z_k, v_k) := z_k + f_k^\tau(z_k, w_k), \\ \tilde{w}_k = F_k^\nu(z_k, w_k) := w_k + f_k^\nu(z_k, v_k) \end{array} \tag{2.6}$$

with $f_k(z_k, 0) = 0$ such that $F_k \Phi_{kj} F_j^{-1} = \tilde{\Phi}_{kj}$ or $F \Phi F^{-1} = \tilde{\Phi}$ for short. By formal equivalence, we mean that $f_k(z_k, w_k)$ is a formal power series in the variables w_k with holomorphic coefficients in the variables $z_k \in \hat{U}_k$, where the latter is $z_k(U_k)$. In components, we have $F_k^\tau = (F_k^1, \dots, F_k^n)$ and $F_k^\nu = (F_k^{n+1}, \dots, F_k^{n+d})$. We will apply the notation to components of Φ_{kj}^τ, Φ_{kj}^ν and Φ_{kj}.

Definition 2.2 We say that $F = \{F_j\}$ defined by (2.6) is *tangent to the identity* along C and write $F = \mathrm{I} + f$ with $f = O(2)$, where the latter means that $f_k^\tau(z_k, w_k)$ and $f_k^\nu(z_k, w_k)$ are holomorphic functions vanishing to order ≥ 2 at $w_k = 0$ and the identity mapping is denoted by I. Let $\mathcal{M}/\sim$ (resp. $\mathcal{M}/\overset{f}{\sim}$) be the set of equivalence classes of *holomorphic* neighborhoods of C under all such biholomorphisms (resp. formal) F.

Next, we identify sections in symmetric powers of N_C^* with coordinate changes. We will associate

$$[f_k]^m(z_k, w_k) := \left\{\left(\sum_{|Q|=m} f_{k,Q}^{\tau}(z_k)w_k^Q, \sum_{|Q|=m} f_{k,Q}^{\nu}(z_k)w_k^Q\right)\right\} \tag{2.7}$$

with a 0-th cochain $[\tilde{f}]^m \in C^0(\mathcal{U}, T_C M \otimes S^m N_C^*)$ defined by

$$[\tilde{f}_k]^m(p) := \sum_{\mu=1}^{n+d} \sum_{|Q|=m} f_{k,Q}^{\mu}(z_k(p), v_k(p))\tilde{e}_k^{\mu}(p) \otimes (w_k^*(p))^{\otimes Q}. \tag{2.8}$$

Here and in what follow, $Q \in \mathbb{N}^d$. We will also associate

$$[\phi_{kj}]^m(z_j, w_j) := \sum_{|Q|=m} \phi_{kj;Q}(z_j)w_j^Q \tag{2.9}$$

with the 1-cochain $\{[\tilde{\phi}_{kj}]^m\} \in C^1(\{U_j\}, \mathcal{O}(T_C M \otimes S^m(N_C^*)))$ defined by

$$[\tilde{\phi}_{kj}]^m(p) = \sum_{\mu=1}^{n+d} \sum_{|Q|=m} \phi_{kj;Q}^{\mu}(z_k(p), v_k(p))\tilde{e}_k^{\mu}(p) \otimes (w_j^*(p))^{\otimes Q}, \tag{2.10}$$

where $\{w_{j,\mu}^*(p)\}_\mu$ denotes the basis of N_C^* that is dual to $\tilde{e}_j^{\nu}(p)$ and $(w_j^*(p))^{\otimes Q} = w_{j,1}^*(p)^{\otimes q_1} \otimes \cdots \otimes w_{j,n}^*(p)^{\otimes q_n}$. Both associations are $\mathbb{C}$-linear, one-to-one and onto. By abuse of notation, we drop tildes in $\tilde{\phi}_{kj}, \tilde{f}_k$ and we interchange (2.7) and (2.8) (resp. (2.9) and (2.10)) for computation as we wish.

We also need to identify the standard Čech coboundary operator δ for cochain (2.10) with a coboundary operator for (2.9). By Lemma 2.7 in [9], applied to $E = T_C M$, we have $\widetilde{\delta[f]^m} = [\tilde{\phi}]^m$ being equivalent to $\delta[f]^m = [\phi]^m$. Writing in a column vector $[f_i]^m := ([f_i^1]^m, \dots, [f_i^{n+d}]^m)^t$ and recalling (2.4), this reads

$$(\delta[f]^m)_{ij}(z_j, w_j) := g_{ij}[f_j]^m - [f_i^m] \circ N_{ij} = [\phi_{ij}]^m(z_j, w_j). \tag{2.11}$$

To construct a formal normal form for $\mathcal{M}$, recall that $\dim H^q(C, V) < \infty$ for $q > 0$ and any holomorphic vector bundle V on C; see [21, Thm. 3.20 and Cor., p. 161].

Note that δ defined by (2.11) is *not* the standard Čech coboundary operator. We will denote by $Z^q(\mathcal{U}, T_C M \otimes S^m N_C^*)$ the kernel of the standard Čech coboundary operator from $C^q(\mathcal{U}, T_C M \otimes S^m N_C^*)$ to $C^{q+1}(\mathcal{U}, T_C M \otimes S^m N_C^*)$.

Definition 2.3 Let C be a compact complex manifold and fix a holomorphic vector bundle N_C on C. Fix a basis $[e^m] = ([e_1^m], \dots, [e_{k_m}^m])$ for $H^1(C, T_C M \otimes S^m N_C^*)$ that is not zero. We also fix a representative $e_i^m \in Z^1(\mathcal{U}, T_C M \otimes S^m N_C^*)$ for $[e_i^m]$. Define $c^m \cdot e^m := \sum c_i^m e_i^m$ for $c_i^m \in \mathbb{C}$ and

$$(c^2, \dots, c^m) \cdot (e^2, \dots, e^m) = \sum_{j=2}^m c^j \cdot e^j.$$

For convenience, if $H^1(C, T_C M \otimes S^m N_C^*) = 0$, set $c^m = 0$ and $c^m \cdot e^m = 0$.

Lemma 2.4 *Fix $m > 1$. Let $\Phi, \tilde{\Phi} \in \mathcal{M}$ and let $F = \mathrm{I} + f$ be a formal mapping satisfying $f = O(m)$.*

(a) *If $\tilde{\Phi} - \Phi = O(m)$, then $[\tilde{\Phi}]^m - [\Phi]^m \in Z^1(\mathcal{U}, T_C M \otimes S^m N_C^*)$; in particular,*

$$[\Phi - N^1]^2 \in Z^1(\mathcal{U}, T_C M \otimes S^2 N_C^*), \quad \forall \Phi \in \mathcal{M}.$$

(b) *$F \Phi F^{-1} = \tilde{\Phi} + O(m)$ holds if and only if $\delta [f]^m = [\Phi]^m - [\tilde{\Phi}]^m$.*
(c) *If $\tilde{\Phi} = \Phi + O(m)$, there exist a unique $c^m \cdot e^m$ and some $\tilde{F} = \mathrm{I} + O(m)$ such that $\tilde{F} \tilde{\Phi} \tilde{F}^{-1} = \Phi + c^m \cdot e^m + O(m+1)$.*

Proof (a) Recall that $N_{kj}^0(z_j, v_j) = (\varphi_{kj}(z_j) + l_{kj}(z_j) v_j, t_{kj}(z_j) v_j)$ and

$$\Phi_{kj}(z_j, w_j) = N_{kj}^0(z_j, w_j) + \sum_{\ell=2}^m [\phi_{kj}]^\ell (z_j, w_j) + O(m+1).$$

The Jacobian of N_{kj}^0 at (z_j, v_j) applied to functions $(\tilde{z}_j, \tilde{v}_j)$ is given by

$$\begin{aligned} DN_{kj}^0(z_j, v_j)(\tilde{z}_j, \tilde{v}_j) = (&s_{kj}(z_j)\tilde{z}_j + l_{kj}(z_j)\tilde{v}_j \\ &+ \partial_{z_j} l_{kj}(z_j) \tilde{z}_j v_j, t_{kj}(z_j)\tilde{v}_j + \partial_{z_j} t_{kj}(z_j) \tilde{z}_j v_j). \end{aligned}$$

If functions $(\tilde{z}_j, \tilde{v}_j) = O(|v_j|)$, we simplify it as

$$DN_{kj}^0(z_j, v_j)(\tilde{z}_j, \tilde{v}_j) = DN_{kj}^0(z_j, 0)(\tilde{z}_j, \tilde{v}_j) + O(|v_j|^2).$$

Let us use transition functions g_{kj} of $T_C M$ to write

$$DN_{kj}^0(z_j, 0)(\tilde{z}_j, \tilde{v}_j) = g_{kj}(z_j)(\tilde{z}_j, \tilde{v}_j) = (s_{kj}(z_j)\tilde{z}_j + l_{kj}(z_j)\tilde{v}_j, t_{kj}(z_j)\tilde{v}_j).$$

Then for $[\phi_{kj}]^{\le m} = \sum_{\ell=2}^{m}[\phi_{kj}]^{\ell}$, we have

$$\begin{aligned}\Phi_{kj}\Phi_{ji}(z_i,w_i) - N_{kj}^0 N_{ji}^0(z_i,w_i) &= [\phi_{kj}]^{\le m}(\varphi_{ji}(z_i) + l_{ji}(z_i)w_i, t_{ji}(z_i)w_i)\\ &\quad + DN_{kj}^0(N_{ji}^0(z_i,w_i))[\phi_{ji}]^{\le m} + O(m+1)\\ &= [\phi_{kj}]^m(N_{ji}(z_i,w_i)) + g_{kj}(\varphi_{ji}(z_i))[\phi_{ji}]^m(z_i,w_i)\\ &\quad + R_{kji}(z_i,w_i,[\phi_\bullet]^2,\dots,[\phi_\bullet]^{m-1}) + O(m+1)\\ &= \Phi_{ki}(z_i,w_i) - N_{kj}^0 N_{ji}^0(z_i,w_i)\end{aligned}$$

where function R_{kji} is independent of $[\phi_\bullet]^\ell$ for $\ell \ge m$. Applying the same computation to $\tilde{\Phi} = \Phi + O(m)$ and subtracting them, we obtain for $\psi_{kj} := \phi_{kj} - \tilde{\phi}_{kj}$,

$$[\psi_{kj}]^m(N_{ji}(z_i,w_i)) + g_{kj}(\varphi_{ji}(z_i))[\psi_{ji}]^m(z_i,w_i) - [\psi_{ki}]^m(z_i,w_i) = 0.$$

According to [9, Lemma 2.7], this is equivalent to

$$\{[\phi_{kj} - \tilde{\phi}_{kj}]^m\}_{kj} \in Z^1(\mathcal{U}, T_C M \otimes S^m N_C^*).$$

(b) Note that when $f = O(m)$ and $\tilde{\Phi} = F\Phi F^{-1}$, we have $\tilde{\Phi} = \Phi + O(m)$. Then $[\tilde{\Phi}]^m = [F\Phi F^{-1}]^m$ is equivalent to $\delta[f]^m = [\phi]^m - [\tilde{\phi}]^m$ according to [9, Lemma 2.16 (2.34)].

(c) By (a), we have $\delta([\tilde{\Phi}]^m - [\Phi]^m) = 0$. By the definition of the basis e^ℓ, we can find $[f]^m$ such that $[\Phi]^m + c^m \cdot e^m - [\tilde{\Phi}]^m = \delta[f]^m$ for some $[f]^m = \{[f_k]^m\}$. Set $F_k = I + [f_k]^m$. We get $F^{-1}\tilde{\Phi}F = \Phi + c^m \cdot e^m + O(m+1)$ by (b). □

To construct normal forms, we must refine the above order-by-order normalization. Define the groups of formal approximate automorphisms

$$\mathrm{Aut}^m(\Phi) = \{F = \mathrm{I} + O(2)\colon\ F^{-1}\Phi F = \Phi + O(m+1)\}, \quad m = 2, 3, \dots.$$

Here it is important that we allow F admits lower-order terms. Since

$$F(\Phi + O(m+1)) = F(\Phi) + O(m+1),$$

then $F \in \mathrm{Aut}^m(\Phi)$ if and only if

$$F\Phi - \Phi F = O(m+1).$$

The latter implies that $\Phi F^{-1} = F^{-1}\Phi + O(m+1)$. Thus, $\mathrm{Aut}^m(\Phi)$ is indeed a group. The group structure implies that the conjugacy by elements in $\mathrm{Aut}^m(\Phi)$ induces an equivalence relation on

$$\mathcal{M}^m(\Phi) := \{\tilde{\Phi} \in \mathcal{M}\colon\ \tilde{\Phi} = \Phi + c^m \cdot e^m + O(m+1)\}.$$

Namely, if $\Psi, \tilde{\Psi} \in \mathcal{M}^m(\Phi)$, define the equivalence relation $\sim_{\mathrm{Aut}^m(\Phi)}$ such that

$$\Psi \sim_{\mathrm{Aut}^m(\Phi)} \tilde{\Psi}$$

if and only if there is an $F \in \mathrm{Aut}_m(\Phi)$ such that $F\Phi F^{-1} = \tilde{\Phi} + O(m+1)$. Let $\mathcal{M}^m(\Phi)/\,\mathrm{Aut}^m(\Phi)$ be the set of equivalence classes. For each equivalence class, *fix* a representative.

Define $\hat{\mathcal{C}}^m(\Phi)$ to be the set of elements $c^m \cdot e^m$ such that $\Phi(c^m) := \Phi + c^m \cdot e^m + O(m+1)$ are among the (chosen) representatives. It will be clear from the context that $\Phi(c^m)$, determined by Φ and c^m, does not stand for the evaluation of Φ at c^m. Also, we have used the identification via (2.9) and (2.10). Thus, we can express the set $\mathcal{N}^m(\Phi)$ of representatives as

$$\{\Phi(c^m) := \Phi + c^m \cdot e^m + O(m+1) \colon c^m \cdot e^m \in \hat{\mathcal{C}}^m(\Phi)\}.$$

Obviously, $\mathcal{N}^m(\Phi)$ is a subset of $\mathcal{M}^m(\Phi)$. To ensure stability, Φ is always a representative, i.e.,

$$0 \cdot e^m \in \hat{\mathcal{C}}^m(\Phi), \quad \Phi(0) = \Phi.$$

Using this equivalence relation, we define convergent *partial normal forms* N^m and formal normal forms N^∞ as follows.

Definition 2.5 Fix transition functions $N^1 = \{N^1_{kj}\}$ for all neighborhoods of C with a given $T_C M$ as in Definition 2.1.

(i) Define $\mathcal{N}^1 = \{N^1\}$, which has one element. Let $\mathcal{N}^2(N^1)$ be the set of representatives $N^2(c^2) = N^1 + c^2 \cdot e^2 + O(3)$ for elements in $\mathcal{M}^2(N^1)/\,\mathrm{Aut}^2(N^1)$, which are determined by

$$c^2 \cdot e^2 \in \hat{\mathcal{C}}^2(N^1).$$

Inductively, let $\mathcal{N}^m(N^{m-1}(c^2, \dots, c^{m-1}))$ be the set of representatives

$$N^m(c^2, \dots, e^m) := N^{m-1}(c^2, \dots, c^{m-1})(c^m)$$

for elements in $\mathcal{M}^m(N^{m-1}(c^2, \dots, c^{m-1}))/\,\mathrm{Aut}^m(N^{m-1}(c^2, \dots, c^{m-1}))$, which are determined by

$$c^m \cdot e^m \in \hat{\mathcal{C}}^m(N^{m-1}(c^2, \dots, c^{m-1})).$$

(ii) Define $N^\infty(c^2, c^3, \dots)$ to be the formal transition functions such that

$$N^\infty(c^2, c^3, \dots) = N^m(c^2, c^3, \dots, c^m) + O(m+1), \quad \forall m.$$

Let $\mathcal{N}^\infty$ be the set of all such formal transition functions.

By definition, $N^{m-1}(c^2,\dots,c^{m-1})(0) = N^{m-1}(c^2,\dots,c^{m-1})$. Thus, we have

Proposition 2.6 *Assume that* $H^1(C, T_CM \otimes S^m N_C^*) = 0$ *for all* $m > m_0$ *and* m_0 *is finite. Then all* $N^\infty = N^{m_0}$ *defines convergent neighborhoods and* $\mathcal{N}^\infty$ *is finite dimensional.*

Recall that

$$\mathcal{H}^q(T_CM) := \bigoplus_{\ell \geq 2} H^q(C, T_CM \otimes S^\ell N_C^*).$$

We now prove a formal version of Theorem 1.1.

Theorem 2.7 *There exists a mapping* $\mathfrak{C}^f$ *from* $\mathcal{M}/\overset{f}{\sim}$ *into* $\mathcal{H}^1(T_CM)$*. Furthermore, the constructed mapping* $\mathfrak{C}^f$ *is injective and there are formal mappings transforming* $\Phi \in \mathcal{M}$ *into* $N^\infty(\Phi) \in \mathcal{N}^\infty$*, provided*

$$\dim \mathcal{H}^1(T_CM) < \infty, \quad \textit{or} \tag{2.12}$$

$$\mathcal{H}^0(T_CM) = 0. \tag{2.13}$$

Proof Recall that we fix a representant basis e^ℓ for a basis $[e^\ell]$ of $H^1(C, T_CM \otimes S^\ell N_C^*)$. Let $\Phi = N^1 + \phi$ as in Definition 2.1 define a neighborhood. We have $\delta[\phi]^2 = 0$. Applying Lemma 2.4, there is a unique constant vector $c_0^2(\Phi)$ such that for some $f^2 =: \{f_j^2\} \in C^0(\mathcal{U}, T_CM \otimes S^2 N_C^*)$,

$$[\Phi - N^1]^2 = c_0^2(\Phi) \cdot e^2 + \delta f^2.$$

Set $F_j^2 = \mathrm{I} + f_j^2$ and $F_2 = \{F_j^2\}$. Then $F_2 \Phi F_2^{-1} = N^1 + c_0^2(\Phi) \cdot e^2 + O(3)$. Recall the definition

$$\mathcal{M}^2(N^1) := \{\widetilde{\Phi} \in \mathcal{M} \colon \widetilde{\Phi}_{kj} = N_{kj}^1 + c^2 \cdot e^2 + O(3)\}.$$

In other words, we have achieved $F_2 \Phi F_2^{-1} \in \mathcal{M}^2(N^1)$. We now apply the refinement. We find a unique element $c^2 \cdot e^2 \in \hat{\mathcal{C}}^2(N^1)$ and $\tilde{F}_2 \in \mathrm{Aut}^2(N^1)$ such that

$$\Phi_2 := \tilde{F}_2 F_2 \Phi F_2^{-1} \tilde{F}_2^{-1} = N^2(c^2) + O(3).$$

We denote this c^2 by $c^2(\Phi)$ and we will show that $c^2(\Phi)$ is a formal invariant. We need to normalize the third-order term in Φ_2 by the above two-step normalization in higher orders.

Therefore, we repeat the above two-step normalization.

Inductively, using Lemma 2.4, we find $F_\ell = \mathrm{I} + f^\ell$ for $\ell > 2$ with $f^\ell =: \{f_j^\ell\} \in C^0(\mathcal{U}, T_CM \otimes S^\ell N_C^*)$ such that

$$F_\ell \Phi_{\ell-1} F_\ell^{-1} \in \mathcal{M}^\ell(N^{\ell-1}(c^2,\dots,c^{\ell-1})).$$

We then find $\tilde{F}_\ell \in \mathrm{Aut}^\ell(N^{\ell-1}(c^2, \dots, c^{\ell-1}))$ such that

$$\Phi_\ell := \tilde{F}_\ell F_\ell \Phi_{\ell-1} F_\ell^{-1} \tilde{F}_\ell^{-1} = N^\ell(c^2, \dots, c^\ell) + O(\ell+1), \quad c^\ell \in \hat{\mathcal{C}}_\ell(N^{\ell-1}(c^2, \dots, c^{\ell-1})).$$

Again, we denote c^ℓ by $c^\ell(\Phi)$. Let $\ell \to \infty$. Since

$$N^{\ell+1}(c^2, \dots, c^{\ell+1}) = N^\ell(c^2, \dots, c^\ell) + O(\ell+1)$$

it is clear that the sequence defines formal transition functions $N^\infty(c^2(\Phi), c^2(\Phi), \dots)$, denoted by $N^\infty(\Phi)$, by an abuse of notation.

We have constructed a normal form $N^\infty(\Phi)$ without using (2.12) or (2.13). We now use the latter to show that $N^\infty(\Phi)$ is achieved by a formal mapping $F = \mathrm{I} + O(2)$.

We know that $F_\ell = \mathrm{I} + O(\ell)$. Since a subsequence of $\tilde{F}_\ell$ may contain terms of the *same* order, we need to verify that the sequence $\hat{F}_\ell := \tilde{F}_\ell F_\ell \cdots \tilde{F}_2 F_2$ still defines a formal change of coordinates as $\ell \to \infty$. When $H^1(T_C M \otimes S^\ell N_C^*) = 0$ for all $\ell > \ell_0$, we have $N^\infty(c^2, \dots) = N^{\ell_0}(c^2, \dots, c^{\ell_0})$. Note that all neighborhoods $N^{\ell_0}(c^2, \dots, c^{\ell_0}) + O(\ell_0 + 1)$ are formally equivalent to $N^{\ell_0}(c^2, \dots, c^{\ell_0})$. In our two-step normalization, we have $F_\ell = \mathrm{I} + O(\ell)$ by definition and $\tilde{F}_\ell = \mathrm{I}$ (since $H^1(T_C M \otimes S^\ell N_C^*) = 0$). Hence, $\hat{F}_\ell$ converges to a formal transformation.

When $\mathcal{H}^0(T_C M) = 0$, we can show inductively that $\tilde{F}_\ell = \mathrm{I} + O(\ell+1)$ for all $\ell \geq 2$. Indeed, assume that $\tilde{F}_\ell = \mathrm{I} + \tilde{f}$ and $\tilde{f} = O(\ell')$ with $2 \leq \ell' \leq \ell$. Then we have

$$\tilde{F}_\ell N^{\ell'}(c^2, \dots, c^{\ell'}) = N^{\ell'}(c^2, \dots, c^{\ell'}) \tilde{F}_\ell + O(\ell'+1).$$

Collecting terms of order ℓ', we conclude $\delta[\tilde{f}]^{\ell'} = 0$. Then $H^0(C, T_C M \otimes S^{\ell'} N_C^*) = 0$ implies that $[\tilde{f}]^{\ell'} = 0$. This shows that the sequence of coordinate changes define a formal transformation F and $F \Phi F^{-1} = N^\infty(\Phi)$.

We observe that (2.12) or (2.13) is only needed to show that there is a formal mapping $F = \mathrm{I} + O(2)$ that transforms Φ into the normal form $N^\infty(\Phi)$. Once we achieve the latter, (2.12) or (2.13) is not required to finish the proof, including the assertion that the mapping $\mathfrak{C}^f$ is well defined. This observation will also apply to the proofs of Theorems 3.6 and 4.5 below.

Next we want to show that $\mathfrak{C}^f$ is well defined, that is that $c^2(\Phi), \dots, c^\ell(\Phi)$ are uniquely determined by the equivalence class of Φ under formal changes of coordinates that are tangent to the identity. Suppose that $\tilde{\Phi} = G \Phi G^{-1}$ with $G = \mathrm{I} + O(2)$. We have $c^2(\Phi) = c^2(\tilde{\Phi})$ immediately. Suppose that $c^\ell(\Phi) = c^\ell(\tilde{\Phi}) = c^\ell$ for $\ell < m$. Then we can find $F, \tilde{F}$ so that

$$\begin{aligned} F \Phi F^{-1} &= N^{m-1}(c') + c^m \cdot e^m + O(m+1), \\ \tilde{F} \tilde{\Phi} \tilde{F}^{-1} &= N^{m-1}(c') + \tilde{c}^m \cdot e^m + O(m+1) \end{aligned}$$

with $c' = (c^2, \dots, c^{m-1})$. Then $K := \tilde{F} G F^{-1}$ satisfies

$$N^{m-1}(c') + \tilde{c}^m \cdot e^m + O(m+1) = K(N^{m-1}(c') + c^m \cdot e^m + O(m+1))K^{-1} \in \mathcal{M}.$$

Thus $K \in \mathrm{Aut}^m(N^{m-1}(c'))$. In summary, $\tilde{c}^m$, c^m are in $\mathcal{C}^m(N^{m-1}(c'))$ and

$$N^{m-1}(c') + \tilde{c}^m \cdot e^m + O(m+1), \quad N^{m-1}(c') + \tilde{c}^m \cdot e^m + O(m+1)$$

are equivalent module $O(m+1)$ by $K \in \mathrm{Aut}^m(N^{m-1}(c'))$. The equivalence class under the conjugacy of $\mathrm{Aut}^m(N^{m-1}(c'))$ can only be represented by a unique element in $\hat{\mathcal{C}}^m(N^{m-1}(c'))$. We obtain $c^m(\Phi) = \tilde{c}^m(\tilde{\Phi})$. This shows that $N^\infty(\Phi) = N^\infty(c^2(\Phi), \dots)$ is well defined on $\mathcal{M}/\overset{f}{\sim}$. Define $N_\infty(\Phi)$ to be the image under $\mathfrak{C}^f$ for the equivalence class $\Phi \mod \overset{f}{\sim}$ of Φ in $\mathcal{M}/\overset{f}{\sim}$.

Finally, we show that if $c^\ell(\Phi) = c^\ell(\tilde{\Phi})$ for all ℓ, then $\tilde{\Phi}$ and Φ are equivalent. Indeed, we have $F\Phi F^{-1} = N^\infty(c^2(\Phi), \dots) = \tilde{F}\tilde{\Phi}\tilde{F}^{-1}$. This shows that $\tilde{\Phi} = \tilde{F}^{-1}F\Phi F^{-1}\tilde{F}$. □

Remark 2.8 We remark that there exist second-order obstructions to realize an element in $H^1(C, T_C M \otimes S^2 N_C^*)$ by a neighborhood of C. See Griffiths [13] and Morrow-Rossi [26, Prop. 3.4].

Our classification is achieved under a group of biholomorphisms F that is smaller that the whole group of biholomorphisms, by restricting F to be tangent to the identity along C. Therefore, the equivalence classes under unrestricted biholomorphisms might be a smaller set; in fact by a simple dilation, we can further reduce the set of equivalence classes to be a *compact* but possibly a non-Hausdorff set as the case of $\dim M = 2$ in [26, p. 323]. For instance, let us consider the case of N_C is a line bundle. Let t_{kj} be the transition functions of N_C. An isomorphism of N_C is given by $g_k^{-1} t_{kj} g_j = t_{kj}$. Thus g_j defines a global holomorphic function on C without zero. Since C is compact, then the function must be constant. Now it is easy to see that the transition function for the neighborhood $\Phi_{kj}(z_j, v_j)$ is transformed into $(\Phi_{kj}^\tau(z_j, cv_j), c^{-1}\Phi_{kj}^v(cv_j))$.

We will give examples in Sect. 6 for (2.12) and (2.13).

3 A Formal Normal Form for Tangential Foliations

In this section, we study *tangential* foliations of neighborhoods of C. A neighborhood M of C admits a tangential foliation, denoted by $(\mathcal{T}, M, C)$, if on each V_k, there are d holomorphic functions $v_k^1, \dots, v_k^d$ such that $dv_k^1 \wedge \dots \wedge dv_k^d \neq 0$ while $v_k = c_k$ and $v_j = c_j$ define the same foliation $\mathcal{T}$ on $V_k \cap V_j$ and $v_k = 0$ defines U_k. The set of tangential foliations of C will be denoted by $\mathcal{M}_\tau$. A biholomorphism F sends $(\mathcal{T}, M, C)$ into $(\tilde{\mathcal{T}}, \tilde{M}, C)$, if it sends leaves of $\mathcal{T}$ into leaves of $\tilde{\mathcal{T}}$ and fixes C pointwise and consequently it sends M into $\tilde{M}$. By an abuse of notation, we write $(\mathcal{T}, M, C)$ as M_τ. The set of such transformations F that are tangent to the identity is denoted by $\mathcal{T}(M_\tau, \tilde{M}_\tau)$, which depends on $M_\tau, \tilde{M}_\tau$.

Clearly, the equivalence of foliations of neighborhoods of C implies the equivalence of the neighborhoods. Therefore, the classification of tangential foliations is a refinement to that of neighborhoods. This should be reflected in the construction of our normal forms for tangential foliations.

Definition 3.1 Let M_τ be a tangential foliation. Coordinate system $\{(z_j, w_j)\}$ is said to *tangential* if $w_j = cst$ define the foliation and consequently the transition functions satisfy $\Phi_{kj}^\nu(z_j, w_j) = \Phi_{kj}^\nu(w_j)$. Such transition functions if they exist for M_τ, denoted by Φ_τ, are called *tangential transition functions*.

Lemma 3.2 (a) *A tangential foliation M_τ admits tangential (or foliated) coordinates $\{(z_j, w_j)\}$. That $\{(z_j, w_j)\}$ are tangential coordinates for M_τ if and only if the transverse components Φ_{kj}^ν of their transition functions Φ_{kj} depend only on w_j.*

(b) *Suppose $F = \mathrm{I} + O(2)$. Then F sends a tangential foliation Φ_τ into another tangential foliation $\tilde{\Phi}_\tau$ if and only if $F\Phi_\tau F^{-1} = \tilde{\Phi}_\tau$ and $F_j^\nu(z_j, w_j) = F_j^\nu(w_j)$ depends only on w_j, in which case*

$$(\tilde{\Phi}_{\tau,kj})^\nu = F_k^\nu(\Phi_{\tau,kj})^\nu(F_j^\nu)^{-1}, \quad (\tilde{\Phi}_{\tau,kj})^\tau = F_k^\tau \Phi_{\tau,kj} F_j^{-1}.$$

With (a) of the lemma, we denote a tangential foliation by Φ_τ.

Proof (a) Suppose that M_τ is defined on each V_j by $v_j = cst$ such that $v_j = 0$ defines U_j and $dv_j^1 \wedge \cdots \wedge dv_j^d \neq 0$ on C. By assumption, we have $v_j^\ell(z_j, w_j) = \sum_{\ell'=1}^d a_{j,\ell'}^\ell(z) w_j^{\ell'} + O(2)$. Since $\det(a_{j,\ell'}^\ell)_{1\leq \ell,\ell'\leq d}$ is non-singular, then (z_j, v_j) form new coordinates. Consequently, for $\Phi_j(z_j, w_j) := (z_j, v_j)$, we get $\Phi_k \Phi_j^{-1}(z_j, v_j) = (\Phi_{kj}^\tau(z_j, v_j), \Phi_{kj}^\nu(v_j))$. Rename v_j as w_j. Then (z_j, w_j) are tangential coordinates w.r.t. M_τ.

(b) Suppose that $(z_j, w_j), (\tilde{z}_j, \tilde{w}_j)$ are tangential coordinates for tangential foliations $M_\tau, \tilde{M}_\tau$ with transition functions $\Phi_\tau, \tilde{\Phi}_\tau$, respectively. Suppose that F sends M_τ into $\tilde{M}_\tau$. Write $F_j = (F_j^\tau, F_j^\nu)$. Then $F_j^\nu(z_j, w_j)$ must be constant if w_j is constant, i.e., $\tilde{w}_j = F_j^\nu(z_j, w_j) = F_j^\nu(w_j)$. Combining it with $\tilde{w}_k = (\tilde{\Phi}_{\tau,kj})^\nu(\tilde{w}_j)$ and $w_k = (\Phi_{\tau,kj})^\nu(w_j)$, we obtain $(\tilde{\Phi}_{\tau,kj})^\nu = F_k^\nu(\Phi_{\tau,kj})^\nu(F_j^\nu)^{-1}$. Of course, we still have $(\tilde{\Phi}_{\tau,kj})^\tau = F_k^\tau \Phi_{\tau,kj} F_j^{-1}$. □

Definition 3.3 (a) The set of transformations $F = \mathrm{I} + O(2)$ from a neighborhood of C into another neighborhood of C satisfying $F^\nu(z_j, v_j) = F_j^\nu(v_j)$ is denoted by $\mathcal{T}_\tau$. The set of transformations satisfying additional condition $F = \mathrm{I} + O(m)$ is denoted by $\mathcal{T}_\tau^m$.

(b) Two tangential foliations defined by Φ_τ and $\tilde{\Phi}_\tau$ are equivalent holomorphically (resp. formally) if there is a biholomorphism (resp. formal biholomorphism) $F = \mathrm{I} + O(2)$ such that $F\Phi_\tau F^{-1} = \tilde{\Phi}_\tau$ and in tangent coordinates $F_j^\nu(z_j, w_j) = F_j^\nu(w_j)$ depends only on w_j for all j.

It is known that if a tangential foliation exists, then N_C is *flat* (see for instance [9, Prop. 2.6] where such foliations are said to be "horizontal"). We fix a flat basis $\hat{e}_k$ for N_C over U_k. Recall that when we choose the basis $(\tilde{e}_k^\tau, \tilde{e}_k^\nu)$ in (2.5) for $T_C M$ on $U_k \subset C$, we have decided $\tilde{e}_k^\nu$ to be a flat basis of N_C when N_C is flat. Let w_k^* be the dual of $\tilde{e}_k^\nu$. When ω is a subset of U_k, denote by $\mathcal{C}_{cst}^\ell(\omega)$ the (continuous) sections of

$T_C M \otimes S^\ell N_C^*$ of the form

$$\sum_{i=1}^{n} \sum_{|Q|=\ell} a_{iQ}(z_k) e_{k,i}^{\tau} \otimes (w_k^*)^{\otimes Q} + \sum_{\alpha=1}^{d} \sum_{|Q|=\ell} b_{\alpha Q} e_{k,\alpha}^{\nu} \otimes (w_k^*)^{\otimes Q}$$

where a_{iQ} are holomorphic functions and $b_{\alpha Q}$ are locally constant functions on ω. Notice that the germs in the first sum define a sheaf $\mathcal{F}$ over C that is coherent and analytic, whereas the germs in the second sum define a sheaf $\mathcal{G}$ that is not an analytic sheaf of modules of holomorphic functions on C. Then δ defined by (2.11) is still a coboundary operator mapping C_{cst}^q, the q-cochains over $\mathcal{U}$ with value in $\bigoplus_{\ell>1} \mathcal{C}_{cst}^\ell(\omega)$ into its image B_{cst}^{q+1}. Denote by $\check{H}^q(\mathcal{U}, \mathcal{C}_{cst}^\ell)$ the Čech cohomology groups for the covering $\mathcal{U}$ and we have a natural homomorphism from $\check{H}^q(\mathcal{U}, \mathcal{C}_{cst}^\ell)$ into $\check{H}^q(\mathcal{U}, T_C M \otimes S^\ell N_C^*)$ and it is an injective when $q = 1$.

Here, $\check{H}^q(C, \mathcal{C}_{cst}^\ell)$ is the standard q-th Čech cohomology group.

Define

$$\check{\mathcal{H}}_\tau^q(T_C M) := \bigoplus_{\ell>1} \check{H}^q(C, \mathcal{C}_{cst}^\ell). \tag{3.1}$$

Recall that we always take a covering $\mathcal{U} = \{U_j\}$ such that each U_j is biholomorphic to a polydisc or ball, which are convex. Then $N_C \otimes S^\ell N_C^*$ is trivial on U_j and $\check{H}^1(U_j, N_C \otimes S^\ell N_C^*) = 0$. Therefore, by [21, Thm. 3.5, p. 121], we have a useful formula for the Čech cohomology group

$$\check{H}^1(C, N_C \otimes S^\ell N_C^*) = \check{H}^1(\mathcal{U}, N_C \otimes S^\ell N_C^*). \tag{3.2}$$

As previously mentioned, the sheaf $\mathcal{F}$ of sections of $F := TC \otimes S^\ell N_C^*$ is coherent and analytic. By Dolbeault's theorem, $\check{H}^q(\omega, F) = H_{\overline{\partial}}^{(0,q)}(\omega, F)$ for any open set ω in C. Since all U_i are Stein, then $\check{H}^q(U_{i_1} \cap \cdots \cap U_{i_p}, F) = 0$ for $q > 0$ by Cartan's theorem B. Therefore, we have

$$\check{H}^q(C, TC \otimes S^\ell N_C^*) = \check{H}^q(\mathcal{U}, TC \otimes S^\ell N_C^*) \tag{3.3}$$

for all q. Using the splitting $T_C M = TC \oplus N_C$, (3.3) for $q = 1$, and (3.2), we obtain

$$\check{\mathcal{H}}_\tau^1(T_C M) = \bigoplus_{\ell>1} \check{H}^1(\mathcal{U}, \mathcal{C}_{cst}^\ell), \quad \dim \check{H}^1(\mathcal{U}, \mathcal{C}_{cst}^\ell) < \infty. \tag{3.4}$$

Here, we see the finiteness as follows: First, dim $\check{H}^q(C, N_C \otimes S^\ell N_C^*)$ is clearly finite since $\mathcal{U} = \{U_0, \ldots, U_m\}$ is a finite covering such that all $U_{i_0} \cap \cdots \cap U_{i_q}$ have only finitely many components, which follows from the existence of covering by generic polyhedrons in the general position (see for instance Def. A. 15 and Prop. A. 19 in [9]), and the sections of the (flat) sheaf $N_C \otimes S^\ell N_C^*$ as $\mathbb{C}$ module are finite dimensional. Second, we already showed that $\check{H}^q(\mathcal{U}, N_C \otimes S^\ell N_C^*) = H_{\overline{\partial}}^{(0,q)}(C, N_C \otimes S^\ell N_C^*)$. The latter is finite dimensional since C is compact, by a theorem of Kodaira [21, Thm. 3.20 and Cor., p. 161].

We now start to adapt the normal forms of neighborhoods of C for the tangential foliations of neighborhoods of C. This will require us to specify a sequence of partial norm forms $N_\tau^m(c_\tau^2,\dots,c_\tau^m)=N_\tau^{m-1}(c_\tau^2,\dots,c_\tau^{m-1})(c_\tau^m)$ for $m=2,3,\dots$. Let us first define N_τ^1 by choosing a representative:

Definition 3.4 Fix T_CM with flat N_C. Fix a tangential foliation $N_\tau^1=\{N_{\tau,kj}^1\}$ of C such that $N_{\tau,kj}^1=N_{kj}^0+O(2)$. If T_CM additionally splits, we take $N_\tau^1=N$ defined by (2.3).

Throughout the paper, we assume that $\mathcal{M}_\tau$ is non-empty. Each element in $\mathcal{M}_\tau$ is given by tangential transitions functions. Therefore the existence of N_τ^1 is ensured regardless if T_CM splits or not.

Definition 3.5 Let $\mathcal{M}_\tau$ be the set of holomorphic tangential foliations containing C as a leaf. Let $\mathcal{M}_\tau/\overset{f}{\sim}$ the set of equivalence classes under formal tangential foliation mappings in $\mathcal{T}_\tau$.

We now state a formal classification for tangential foliations.

Theorem 3.6 *Let N_C be flat. There is a mapping $\mathfrak{C}_\tau^f$ from $\mathcal{M}_\tau/\overset{f}{\sim}$ into $\check{\mathcal{H}}_\tau^1(T_CM)$. Furthermore, if*

$$\dim\check{\mathcal{H}}_\tau^1(T_CM)<\infty,\quad \textit{or}$$
$$\check{\mathcal{H}}_\tau^0(T_CM)=0,$$

then the constructed mapping $\mathfrak{C}_\tau^f$ is injective and there are formal mappings in $\mathcal{T}_\tau$ transforming $\Phi\in\mathcal{M}_\tau$ into $N_\tau^\infty(\Phi)$. In particular, if $\check{\mathcal{H}}_\tau^1(T_CM)=0$, all tangential foliations are formally equivalent.

We fix a basis e_τ^ℓ for $\check{H}^1(\mathcal{U},\mathcal{C}_{cst}^\ell)$ since the latter is finite dimensional by (3.4). The proof is almost identical to the proof for the general case. We will only give an outline below.

Lemma 3.7 *Fix $m>1$. Consider transformations $F=\mathrm{I}+f\in\mathcal{T}_\tau$ with $f=O(m)$. Suppose that $\Phi_\tau,\tilde\Phi_\tau\in\mathcal{M}_\tau$ and*

$$\tilde\Phi_\tau=\Phi_\tau+O(m),\quad \Phi_\tau=N_\tau^1+O(2),\quad \tilde\Phi_\tau=N_\tau^1+O(2).$$

Then we have the following.

(a) $[\tilde\Phi_\tau]^m-[\Phi_\tau]^m\in Z^1(\mathcal{U},\mathcal{C}_{cst}^m)$*; in particular,*

$$[\Phi-N_\tau^1]^2\in Z^1(\mathcal{U},\mathcal{C}_{cst}^2),\quad \forall\Phi\in\mathcal{M}_\tau.$$

(b) $F\Phi_\tau F^{-1}=\tilde\Phi_\tau+O(m)$ *holds if and only if* $\delta[f]^m=[\tilde\Phi_\tau]^m-[\Phi_\tau]^m$.

(c) *There exist a unique* $c_\tau^m \cdot e_\tau^m \in \check{H}^1(\mathcal{U}, \mathcal{C}_{cst}^m)$ *and some* $f = O(m)$ *such that* $\hat{\Phi} := F^{-1}\tilde{\Phi}_\tau F$ *satisfies*

$$\hat{\Phi} = \Phi_\tau + c_\tau^m \cdot e_\tau^m + O(m+1), \quad \hat{\Phi} \in \mathcal{M}_\tau.$$

For $\Phi \in \mathcal{M}_\tau$, define

$$\begin{aligned} \mathcal{M}_\tau^m(\Phi) &:= \{\tilde{\Phi} \in \mathcal{M}_\tau : \tilde{\Phi} = \Phi + c_\tau^m \cdot e_\tau^m + O(m+1)\}, \\ \operatorname{Aut}_\tau^m(\Phi) &= \{F \in \mathcal{T}_\tau : F^{-1}\Phi F = \Phi + O(m+1)\}. \end{aligned}$$

Then $\operatorname{Aut}_\tau^m(\Phi)$ is a group inducing an equivalence relation on $\mathcal{M}_\tau^m(\Phi)$. Fix an element in each equivalence class. Thus the set

$$\mathcal{N}_\tau^m(\Phi) := \mathcal{M}_\tau^m(\Phi)/\operatorname{Aut}_\tau^m(\Phi)$$

of equivalence classes can be written as

$$\Phi(c_\tau^m) = \Phi + c_\tau^m \cdot e_\tau^m + O(m+1) \in \mathcal{M}_\tau$$

with $c_\tau^m \cdot e_\tau^m \in \check{H}^1(\mathcal{U}, \mathcal{C}_{cst}^m)$. The set of all such $c_\tau^m \cdot e_\tau^m$ will be denoted by $\hat{\mathcal{C}}_\tau^m(\Phi)$. Thus $\mathcal{N}_\tau^m(\Phi)$ is identified with $\hat{\mathcal{C}}_\tau^m(\Phi)$. For stability, we always choose $\Phi(0) = \Phi$.

We now construct two sequences of coordinate changes $F_m, \tilde{F}_m$, and define the mapping $\mathfrak{C}_\tau^f$ in Theorem 3.6. Let $\Phi \in \mathcal{M}_\tau$. Then we proceed as following.

(i) By Lemma 3.7, we find $F_2 = \mathrm{I} + O(2) \in \mathcal{T}_\tau$ such that $F_2 \Phi F_2^{-1} = N_\tau^1 + \tilde{c}_\tau^2 \cdot e_\tau^2 + O(3)$, i.e.,

$$F_2 \Phi F_2^{-1} \in \mathcal{M}_\tau^2(N_\tau^1).$$

Take $\tilde{F}_2 \in \operatorname{Aut}_2^\tau(N_\tau^1)$ such that

$$\Phi_2 := \tilde{F}_2 F_2 \Phi F_2^{-1} \tilde{F}_2^{-1} = N_\tau^1(c_\tau^2) + O(3), \quad c_\tau^2 \cdot e_\tau^2 \in \hat{\mathcal{C}}_\tau^2(N_\tau^1).$$

(ii) Let $m > 2$. Find $F_m = \mathrm{I} + O(m) \in \mathcal{T}_\tau$ such that

$$F_m \Phi_{m-1} F_m^{-1} \in \mathcal{M}_\tau^m(N_\tau^{m-1}(c_\tau^2, \ldots, c_\tau^{m-1})).$$

Choose $\tilde{F}_m \in \operatorname{Aut}_\tau^m(N_\tau^{m-1}(c_2^\tau, \ldots, c_\tau^{m-1}))$ such that

$$\begin{aligned} \tilde{F}_m F_m \Phi_{m-1} F_m^{-1} \tilde{F}_m^{-1} = N_\tau^{m-1}(c_\tau^2, \ldots, c_\tau^{m-1})(c_\tau^m) + O(m+1), \\ c_\tau^m \cdot e_\tau^m \in \hat{\mathcal{C}}_\tau^m(N_\tau^{m-1}(c_\tau^2, \ldots, c_\tau^{m-1})). \end{aligned}$$

(iii) Set $N_\tau^m(c_\tau^2, \dots, c_\tau^m) = N_\tau^{m-1}(c_\tau^2, \dots, c_\tau^{m-1})(c_\tau^m)$. The formal normal form of Φ is N_τ^∞ with

$$N_\tau^\infty = N_\tau^m(c_\tau^2, \dots, c_\tau^m) + O(m+1), \quad m = 1, 2, \dots.$$

Define $(c_\tau^2 \cdot e_\tau^2, \dots) \in \check{\mathcal{H}}_\tau^1(T_C M)$ to be $\mathfrak{C}_\tau^f(\Phi)$ for the equivalence class of Φ under $\mathcal{T}_\tau$.

We can check that the mapping $\mathfrak{C}_\tau^f$ is well defined. As in the proof of Theorem 2.7, by the conditions on cohomology groups, we can verify that the sequence $\tilde{F}_m F_m \cdots \tilde{F}_2 F_2$ converges to a formal mapping F that transforms Φ into $\mathfrak{C}_\tau^f(\Phi)$. We leave the rest of details for the proof of Theorem 3.6 to the reader.

We refer to [9] for a different approach to the existence of holomorphic tangential foliation when the formal obstructions are absent in a stronger sense that the normal component of Φ is formally linearizable. Under a small divisor condition on cohomology groups depending only on a (flat) unitary N_C, a convergence for linearizing the normal components is achieved in [9].

4 A Formal Normal Form of Transverse Foliations

In this section, we will use normal forms in Sect. 2 to describe neighborhoods that admit transverse foliations.

By a holomorphic *transverse* foliation $(\mathcal{V}, M, C)$ of (M, C), we mean that on a neighborhood of C in M there a smooth holomorphic foliation $\mathcal{V}$ with all leaves are holomorphic submanifolds of dimension d that intersect C transversely. First-order obstructions to transverse foliations for (M, C) in higher dimensions or higher codimension were considered in [13, 26]; however, they did not settle the existence of transverse foliations when formal obstructions vanish except for the case of dim $M = 2$ mentioned earlier. In this section, we will obtain an existence result on transverse foliations and the classification on them under suitable conditions on $T_C M$.

One can see that a neighborhood admits a transverse foliation, if and only if in coordinates (z_k, w_k) there are n holomorphic functions $\tilde{z}_k^1, \dots, \tilde{z}_k^n$ such that

$$d\tilde{z}_k^1 \wedge \cdots \wedge d\tilde{z}_k^n \wedge dw_k^1 \wedge \cdots \wedge dw_k^d \neq 0$$

while $\tilde{z}_k = c_k$ and $\tilde{z}_j = c_j$ define the same foliation on $V_k \cap V_j$ on C. By an abuse of notation, we still denote the transverse foliation on a neighborhood (M, C) by M_ν. In the previous section, we have seen that for a neighborhood to admit tangential foliation, N_C must be flat. A transverse foliation does not impose conditions on N_C as N_C as a holomorphic vector bundle is already foliation by the fibers (of the bundle). It, however, imposes a useful condition that $T_C M$ must split [26]. The formal obstructions for transverse foliations were obtained in [13, 26]. In this section, we obtain a normal form for transverse foliations.

Definition 4.1 Let $M_\nu = (\mathcal{V}, M, C)$ be a transverse foliation. We say that (z_j, w_j) are transverse (or foliated) coordinates for M_ν if $z_j = cst$ defines the foliation and consequently the transition functions satisfy $\Phi^{\tau}_{kj}(z_j, w_j) = \Phi^{\tau}_{kj}(z_j)$, in which case Φ is denoted by Φ_ν and we call $\Phi_\nu = \{\Phi_{\nu,kj}\}$ a transverse foliation for abbreviation.

Definition 4.2 Two transverse foliations $(\mathcal{V}, M, C)$, $(\tilde{\mathcal{V}}, \tilde{M}, C)$ are equivalent by a biholomorphic mapping F if $F = I + O(2)$ and it sends each leave of $(\mathcal{V}, M, C)$ into a leaf of $(\tilde{\mathcal{V}}, \tilde{M}, C)$; consequently, F sends the neighborhood (M, C) into $(\tilde{M}, C)$.

Lemma 4.3 (a) *Let $M_\nu = (\mathcal{V}, M, C)$ be a transverse foliation. Then the foliation admits transverse coordinates $\{(z_j, w_j)\}$, of which the transition functions are denoted by Φ_ν and $(\Phi_{\nu,kj})^\tau(z_j, w_j) = \varphi_{kj}(z_j)$ as in (4.1).*

(b) *There is a biholomorphic mapping $F = \mathrm{I} + O(2)$ sending a transverse foliation $\Phi_{\nu,kj}(z_j, w_j)$ into another transverse foliation $\tilde{\Phi}_{\nu,kj}(\tilde{z}_j, \tilde{w}_j)$ if and only if $F^{-1}\Phi_\nu F = \tilde{\Phi}_\nu$ and $F^\tau_j(z_j, w_j) = z_j$, in which case*

$$(\tilde{\Phi}_{\nu,kj})^\tau = F^\tau_k \Phi_{\nu,kj} F^{-1}_j.$$

The set of all transformations $F = I + O(2)$ with $F^\tau_j(z_j, w_j) = z_j$ is denoted by $\mathcal{T}_\nu$.

Proof The proof is almost verbatim to Lemma 3.2, which is leave to the reader. □

Motivated by Lemma 4.3, we now define the following.

Definition 4.4 Two transverse foliations $\tilde{\Phi}_\nu$, Φ_ν, are formally equivalent by a formal biholomorphic mapping F if $F = \mathrm{I} + O(2)$ and in the transverse coordinates of the two foliations, $F^\tau_j(z_j, w_j) = z_j$ $\tilde{\Phi}_\nu = F\Phi_\nu F^{-1}$.

Note that the transformations $F_j = \mathrm{I} + f_j$ with $f_j = O(2)$ that preserve transverse foliations are rather restrictive as $f^\tau_j = 0$. This is quite different from the study of tangential foliations in the previous section whereas transformations that preserve the tangential foliations can be higher-order perturbations in both tangential and normal components. The advantage is that the formal classification of transverse foliations is almost identical to the formal classification of the neighborhoods.

Define

$$\mathcal{H}^q_\nu(T_C M) := \bigoplus_{\ell > 1} H^q(C, N_C \otimes S^\ell N^*_C).$$

Note that $N_C = T_C M / TC$, which justifies that the right-hand side depends on $T_C M$. Using Lemma 4.3, we can obtain the following formal normal form.

Theorem 4.5 *Let $\mathcal{M}_\nu$ be the set of holomorphic transverse foliations of C. There is a mapping $\mathfrak{C}^f_\nu$ from $\mathcal{M}_\nu / \overset{f}{\sim}$ into $\mathcal{H}^1_\nu(T_C M)$. Furthermore, if*

$$\dim \mathcal{H}^1_\nu(T_C M) < \infty, \quad \textit{or}$$
$$\mathcal{H}^0_\nu(T_C M) = 0,$$

then the constructed mapping $\mathfrak{C}_\nu^f$ is injective and there are formal mappings in $\mathcal{T}_\nu$ transforming $\Phi \in \mathcal{M}_\nu$ into $N_\nu^\infty(\Phi)$. In particular, if $\mathcal{H}_\nu^1(T_C M) = 0$, all transverse foliations are formally equivalent.

Proof Given a transverse foliation, we know that $T_C M$ splits. Let $\Phi \in \mathcal{M}_\nu$. Define

$$\mathrm{Aut}_\nu^m(\Phi) = \{F \in \mathcal{T}_\nu \colon F\Phi F^{-1} = \Phi + O(m+1)\}.$$

By conjugacy, the group yields an equivalence relation on

$$\mathcal{M}_\nu^m(\Phi) := \{\Phi + c_\nu^m \cdot e_\nu^m + O(m+1)\} \cap \mathcal{M}_\nu.$$

Select representatives for the equivalence classes and denote the set of corresponding elements $c_\nu^m \cdot e_\nu^m$ by $\mathcal{H}_\nu^m(\Phi)$. Thus the set of equivalence classes is given by

$$\mathcal{N}_\nu^m(\Phi) = \{\Phi(c_\nu^m) = \Phi + c_\nu^m \cdot e_\nu^m + O(m+1) \colon c_\nu^m \\ \cdot e_\nu^m \in H^1(\mathcal{U}, N_C \otimes S^m N_C^*)\} \subset \mathcal{M}_\nu.$$

The set of all such $c_\nu^m \cdot e_\nu^m$ will be identified with $\hat{\mathcal{C}}_\nu^m(\Phi)$.

By Lemma 4.3, we find $F_2 = \mathrm{I} + O(2) \in \mathcal{T}_\nu$ such that $F_2 \Phi F_2^{-1} = N_\nu^1 + c_\nu^2 \cdot e_\nu^2 + O(3)$ with $N_\nu^1 = N^1$. Thus

$$F_2 \Phi F_2^{-1} \in \mathcal{M}_\nu^2(N_\nu^1).$$

Take $\tilde{F}_2 \in \mathrm{Aut}_\nu^2(N^1)$ such that

$$\Phi_2 := \tilde{F}_2 F_2 \Phi F_2^{-1} \tilde{F}_2^{-1} = N_\nu^1(c_\nu^2) + O(3), \quad c_\nu^2 \cdot e_\nu^2 \in \hat{\mathcal{C}}_\nu^2(N_\nu^1).$$

Set $N_\nu^2(c_\nu^2 \cdot e_\nu^2) := N_\nu^1(c_\nu^2 \cdot e_\nu^2)$.

Let $m > 2$. Find $F_m = I + O(m) \in \mathcal{T}_\nu(\mathcal{M}_\nu)$ such that

$$F_m \Phi_{m-1} F_m^{-1} \in \mathcal{M}_\nu^m(N_\nu^{m-1}(c_\nu^2, \dots, c_\nu^{m-1})).$$

Choose $\tilde{F}_m \in \mathrm{Aut}_\nu^m(N_\nu^{m-1}(c_2^\nu, \dots, c_\nu^{m-1}))$ such that

$$\tilde{F}_m F_m \Phi_{m-1} F_m^{-1} \tilde{F}_m^{-1} = N_\nu^{m-1}(c_\nu^2, \dots, c_\nu^{m-1})(c_\nu^m) + O(m+1), \\ c_\nu^m \cdot e_\nu^m \in \hat{\mathcal{C}}_\nu^m(N^{m-1}(c_\nu^2, \dots, c_\nu^{m-1})).$$

Set $N_\nu^m(c_\nu^2, \dots, c_\nu^m) = N_\nu^{m-1}(c_\nu^2, \dots, c_\nu^{m-1})(c_\nu^m)$. The formal normal form of Φ is N_ν^∞ with

$$N_\nu^\infty = N_\nu^m(c_\nu^2, \dots, c_\nu^m) + O(m+1), \quad m = 1, 2, \dots.$$

Define $\mathfrak{C}_\nu^f(\Phi)$ to be $(c_\nu^2 \cdot e_\nu^2, \dots) \in \mathcal{H}_\nu^1(T_C M)$ for equivalence class of Φ under $\mathcal{T}_\nu$.

We have defined the normal forms $N_\nu^\infty(c_\nu^2, \dots)$ for $\Phi \in \mathcal{M}_\nu$. The rest is similar to the proof of Theorem 2.7. We leave the details to the reader. □

For the rest of the section, we deal with the existence of transverse foliations using the normal form for the neighborhoods in Sect. 2. Let us first improve Definition 2.5 as follows.

Definition 4.6 In Definition 2.5, we select $N^\ell(c^2,\dots,c^\ell)$ satisfying

$$(N^\ell_{kj})^\tau(c^2,\dots,c^\ell)=\varphi_{kj}(z_j),$$

if the set of neighborhoods Φ satisfies

$$\Phi=FN^{\ell-1}(c^2,\dots,c^{\ell-1})F^{-1}+O(\ell)$$

for some $F\in \mathrm{Aut}^{\ell-1}(N^{\ell-1}(c^2,\dots,c^{\ell-1}))$ has a neighborhood that admits a holomorphic transverse foliation.

Proposition 4.7 *Let $\ell_0>1$ be an integer. Assume that*

$$H^1(C,T_CM\otimes S^\ell N_C^*)=0,\quad \forall \ell>\ell_0.$$

Then Φ admits a formal transverse foliation if and only if there is a formal mapping $F\in\mathcal{T}$ such that $F\Phi F^{-1}=N^{\ell_0}(c^2,\dots,c^{\ell_0})+O(\ell_0+1)$ with $(N^{\ell_0}(c^2,\dots,c^{\ell_0}))^\tau_{kj}=\varphi_{kj}(z_j)$.

Proposition 4.8 *Assume that $H^0(C,TC\otimes S^\ell N_C^*)=0$ for all $\ell>1$. Then two transverse foliations of neighborhoods of C are equivalent if (and only if) the neighborhoods are equivalent.*

Proof One implication is trivial. Suppose that $F=\mathrm{I}+f$ with $f=O(2)$ sends a neighborhood M that admits a transverse foliation Φ into a neighborhood $\tilde M$ that admits a transverse foliation $\tilde\Phi$. We must show that $f^\tau=0$. We have $\tilde\Phi_{kj}F_j=F_k\Phi_{kj}$. Suppose that $F_j^\tau=\mathrm{I}+[f_j^\tau]^m+O(m+1)$. We want to show that $[f_j^\tau]^m=0$. We know that $\Phi^\tau_{kj}(z_j,v_j)=\phi_{kj}(z_j)$ and $\tilde\Phi^\tau_{kj}(\tilde z_j,\tilde v_j)=\varphi_{kj}(\tilde z_j)$. We have

$$\varphi_{kj}(\mathrm{I}+[f_j^\tau]^m+O(m+1))=\varphi_{kj}(z_j)+[f_k^\tau(\Phi_{kj})]^m+O(m+1).$$

Collecting terms of order m in v_j, we obtain $\delta^\tau\{[f^\tau]^m\}=0$. Thus $[f^\tau]^m$ is a global section of $TC\otimes S^mN_C^*$. We conclude $[f^\tau]^m=0$. □

When C is a compact Riemann surface with genus g, $H^0(C,TC\otimes S^\ell N_C^*)=0$ if $\deg N_C>\max\{0,g-1\}$ and $\ell>1$; see Section 6.

We mentioned that a necessary condition for the existence of transverse foliation is the splitting of T_CM into $TC\oplus N_C$. There are generalizations for splitting of T_CM (as 2-splitting) to k-splitting by Abate-Bracci-Torven [1]; see for instance Thm. 2.1, Cor. 3.4 and Thm. 4.1 therein.

5 Convergence for Two Classifications and Criteria for Transverse Foliations

In this section, we establish the convergence results for normalizations of full neighborhoods and transverse foliations by applying convergent results of Grauert [10], Griffiths [13], and Commichau-Grauert [6]. We also obtain a criteria for the existence of transverse foliations using our normal forms.

The first-order obstructions for the existence of transverse foliations were studied in [13], Morrow-Rossi [26], where the convergence of transverse foliations are not addressed except for the case $\dim M = 2$ in [26] for which the classification of the foliations was also addressed.

According to Grauert [10, Def. 1, p. 342], we say that E is weakly negative if its zero section has a relatively compact strictly pseudoconvex neighborhood. We say that E is weakly positive if E^* is weakly negative [10, Def. 2, p. 342].

Grauert [10, Satz 1, p. 341] proved that if the zero section of a vector bundle E is exceptional, then E is weakly negative. According to Grauert [10, Def. 3, p. 339], a connected compact complex manifold C of positive dimension in M is called *exceptional*, if there is a complex manifold M' and a proper surjective holomorphic map $\phi\colon M \to M'$ such that $\phi(C)$ is a point p, $\phi\colon X\setminus C \to M'\setminus\{p\}$ is biholomorphic. In [10, Thm. 5, p. 340], Grauert proved that C is exceptional, if C has a basis of strongly pseudoconvex neighborhoods. It was proved by Grauert for codimension one compact complex manifold C and by Hironaka and Rossi for higher codimension that the formal principle holds for exceptional sets.

Throughout the paper if a domain W is defined by a C^2 function $r < 0$, we say that ∂W has (at least) q positive Levi eigenvalues if the Levi-form Lr, which is the restriction of complex hessian Hr to the complex tangent space $T^{(1,0)}\partial W$, is positive definite on a q-dimensional subspace of $T_x^{(1,0)}\partial W$ for each $x \in \partial W$. Following Commichau-Grauert [6], we say that a vector bundle V on C is 1-*positive*, if there is a tube neighborhood W of the zero section C of V such that the Levi-form of ∂W has at least 1-negative eigenvalue and $W \cap V_x$ is star-shaped for each $x \in C$ and V_x intersects ∂W transversely. We say that V is q-*positive*, if the ∂W has q negative Levi eigenvalues. We recall a lemma mentioned by Griffiths [13].

Lemma 5.1 ([13], Lemma 2.1) *Let $\pi : N_C \to C$ be the normal bundle of C. Let W be a neighborhood of C in N_C. Let F be any holomorphic vector bundle on C and let $\pi^*F|_W$ be the pull-back bundle on W. Then*

$$\sum_{\ell\geq 0} \dim H^q(C, F \otimes S^\ell N_C^*) \leq \dim H^q(W, \pi^*F|_W). \tag{5.1}$$

Proof We provide a proof using our notation for $q = 1$ only. Let $\mathcal{U}$ be a covering of C by open sets U_j. We may assume that there are open sets V_j in W so that $U_j = V_j \cap C$. Using additional open subset of W, we assume that $\{V_j\}$ is a covering $\mathcal{V}$ for W. Furthermore, we may assume that U_j and V_k are Stein, E and F holomorphically trivial on U_j.

The vector bundles F and N_C play different roles. For F, we simply pull it back as π^*F as vector bundle on N_C. This turns sections of $F\otimes S^\ell N_C^*$ into π^*F valued homogeneous polynomials on the manifold N_C. To be specific, let us identify a section of $F\otimes S^\ell N_C^*$ by a $\mathbb{C}$-linear injection

$$\iota\colon C^q(\mathcal{U}, F\otimes S^\ell N_C^*)\to C^q(\mathcal{V}, \mathcal{O}(\pi^*F|_W)).$$

Recall that $\pi^{-1}(U_j)$ has coordinates (z_j, v_j). Let $\{\tilde e_j^\mu\}$ (resp. $\{e_j^\mu\}$) be a basis of $F_{|U_j}$ (resp. N_C). Let $\{f_{kj}\}\in C^1(\mathcal{U}, F\otimes S^m N_C^*)$. Then by (2.10),

$$f_{kj}(p)=\sum_{\mu=1}^{n+d}\sum_{|Q|=m} f_{kj,Q}^\mu(z_k(p), v_k(p))\tilde e_k^\mu(p)\otimes(w_j^*(p))^{\otimes Q}.$$

Let $\tilde p\in\pi^{-1}(p)$ and $p\in U_j\cap U_k$. In coordinates, we have $\tilde p=(p, e_j(p)\cdot v_j(p))$. Define

$$(\iota f)_{kj}(\tilde p)=\sum_{\mu=1}^{n+d}\sum_{|Q|=m} f_{kj,Q}^\mu(z_k(p), v_k(p))v_j^Q(p)\tilde e_k^\mu(p).$$

Now $\{\pi^{-1}(U_j)\}$ is an open covering $\hat{\mathcal{V}}$ of F as a complex manifold and $\iota f\in C^1(\hat{\mathcal{V}}, \mathcal{O}(F))$. Let $\tilde{\mathcal{V}}$ be the covering of W defined by $\{V_i\cap\pi^{-1}U_j\}$. Note that $V_i\cap\pi^{-1}U_j$ are still Stein. Analogously, if $\{u_j\}\in C^0(\mathcal{U}, F\otimes S^m N_C^*)$, then

$$u_k(p)=\sum_{\mu=1}^{n+d}\sum_{|Q|=m} u_{k,Q}^\mu(z_k(p), v_k(p))\tilde e_k^\mu(p)\otimes(w_k^*(p))^{\otimes Q}.$$

Define $(\iota u)_k(\tilde p):=\sum_{\mu=1}^{n+d}\sum_{|Q|=m} u_{k,Q}^\mu(z_k(p), v_k(p))v_k^Q(p)\tilde e_k^\mu(p)$. In the other way around, to such an expression $U_k(q)$, we associate a 0-cochain $\tilde U=\{\tilde U_j\}\in C^0(\mathcal{U}, F\otimes S^m N_C^*)$. Then $\delta\iota=\iota\delta$.

We want to show that ι induces an injection

$$\iota\colon \check H^1(\mathcal{U}, F\otimes S^\ell N_C^*)\to \check H^1(\tilde{\mathcal{V}}, \mathcal{O}(\pi^*F|_W)).$$

Suppose that $f\in Z^1(\mathcal{U}, F\otimes S^\ell N_C^*)$ and $\iota f=\delta u$ with $u\in C^0(\tilde{\mathcal{V}}, \mathcal{O}(\pi^*F|_W))$. If $V_j\cap C$ is non-empty, we have $V_j\cap C=U_j$ and we can expand $u=\sum u_m$, where u_m is a homogeneous polynomial in v_j of degree m. Then $f=\delta\tilde u_\ell$. This gives us (5.1). □

It is known from the Andreotti-Grauert theory [2] that if ∂W has $(q+1)$ negative Levi eigenvalue or $(n+d-q)$ positive Levi eigenvalues at each boundary point, then

$$\dim H^q(W, \mathcal{F})<\infty$$

for any coherent analytic sheaf $\mathcal{F}$ on W. Our convergent classifications are based on the following theorems, which can be regarded as a strong form of the formal principle.

Theorem 5.2 ([10], Satz 7, p. 363) *Let N_C be negative. Then there exists an integer ℓ_0 such that $H^1(C, T_C M \otimes S^\ell N_C^*) = 0$ for $\ell > \ell_0$. Furthermore, if $\hat{F}_j$ are holomorphic mappings such that $\tilde{\Phi}_{kj} = \hat{F}_k^{-1}\Phi_{kj}\hat{F}_j + O(\ell_0 + 1)$, then there are holomorphic mappings F_j such that $F_j = \hat{F}_j + O(\ell + 1)$ and $F_k^{-1}\Phi_{kj}F_j = \tilde{\Phi}_{kj}$.*

Remark 5.3 Let C be a compact Riemann surface and let N_C be a line bundle with $\deg N_C < 0$, By Riemann-Roch (see (6.1)–(6.2) below), we have

$$\lim_{\ell\to\infty} \ell^{-1} \dim H^0(C, T_C M \otimes S^\ell N_C^*) = -\deg N_C.$$

Using this, one can prove that there are divergent formal mappings that preserve the germs of neighborhoods of the zero section of N_C; see Proposition 6.2 below.

On the other hand, with 1-positivity, we have the following *strong* formal principle.

Theorem 5.4 ([6], Satz 4, p. 119) *Let N_C be 1-positive. Let F_j be formal biholomorphic mappings such that $\tilde{\Phi}_{kj} = F_k^{-1}\Phi_{kj}F_j$. Then F_j must converge.*

Commichau-Grauert [6] proved the theorem for codimension 1 case first. Their proof for the higher codimensions follows from a blow-up [6, p. 115 and p. 126] along C. Indeed, one can verify easily as follows. By blowing up a neighborhood M along C, we obtain $\tilde{M}$ and $\tilde{C}$ such that $\tilde{C}$ has codimension 1 while $\partial\tilde{M}$ is biholomorphic to ∂M. If $F\colon (M, C) \to (M', C)$ is a formal mapping that is tangent to the identity along C, then the blow-up induces a formal mapping $\tilde{F}$ from $(\tilde{M}, \tilde{C})$ to $(\widetilde{M'}, \tilde{C})$ (that may not be tangent to the identity along $\tilde{C}$). However, the theorem for codimension 1 case, which is proved for any formal mapping that preserves C, implies that $\tilde{F}$ is convergent. Consequently, F is also convergent. We leave the reader to verify this outline.

We now prove the convergence classification for the neighborhoods.

Corollary 5.5 *Let N_C be weakly negative or 2-positive. Then the set of holomorphic equivalence classes of neighborhoods of C is identified with a subset of finite-dimensional space $\mathcal{H}^1(T_C M) < \infty$. Furthermore, a neighborhood Φ admits a transverse foliation if and only if its normal form $N^\infty(\Phi)$ satisfies $(N_{kj}^\infty(\Phi))^\tau = \varphi_{kj}$.*

Proof Let us summarize the proof of Theorem 2.7. Let Φ be a holomorphic (convergent) neighborhood M of C such that $\Phi = N^1 + O(2)$, where N^1 is a convergent neighborhood. Then we find a (convergent) biholomorphism $G_2 = \tilde{F}_2 F_2$ such that $G_2\Phi G_2^{-1} = N^1 + c^2(\Phi)\cdot e^2 + O(3) \in \mathcal{N}^2(N^1)$. By selection, $G_2\Phi G_2^{-1} = N^1(c^2(\Phi)) + O(3)$. Inductively, we find biholomorphism $G_{\ell_0} = \tilde{F}_{\ell_0}F_{\ell_0}\cdots\tilde{F}_2F_2$ such that

$$\begin{aligned} G_{\ell_0}^{-1}\Phi G_{\ell_0} &= N^{\ell_0}(c^2(\Phi), \dots, c^{\ell_0}(\Phi)) + O(\ell_0 + 1)\\ &\in \mathcal{N}^{\ell_0}(N^{\ell_0 - 1}(c^2(\Phi), \dots, c^{\ell_0 - 1}(\Phi)). \end{aligned}$$

Since $H^1(C, T_C M \otimes S^\ell N_C^*) = 0$ for $\ell > \ell_0$, then we find ($F_\ell = \mathrm{I}$ and) $\tilde{F}_\ell = \mathrm{I} + O(\ell)$ such that the formal mapping $\tilde{G}_{\ell_0} = \lim_{\ell\to\infty} \tilde{F}_\ell \cdots \tilde{F}_{\ell_0+1}$ satisfies $\tilde{G}_{\ell_0} G_{\ell_0} \Phi G_{\ell_0}^{-1} \tilde{G}_{\ell_0} = N^{\ell_0}(c^2(\Phi), \ldots, c^{\ell_0}(\Phi))$. When N_C is weakly negative, Theorem 5.2 says that there exists a convergent $G_{\ell_0+1} = \tilde{G}_{\ell_0} + O(\ell_0 + 1)$ such that

$$G_{\ell_0+1} G_{\ell_0} \Phi G_{\ell_0}^{-1} G_{\ell_0+1}^{-1} = N^{\ell_0}(c^2(\Phi), \ldots, c^{\ell_0}(\Phi)).$$

When N_C is 1-positive, Theorem 5.4 says that $G_{\ell_0+1} = \tilde{G}_{\ell_0}$ converges. Since $F_\ell, \tilde{F}_\ell, G_{\ell_0+1}$ are tangent to the identity, then $G_{\ell_0+1,j} G_{\ell_0,j}$ is a biholomorphism that is well defined in a neighborhood of $\tilde{U}_j$ in M, where each $\tilde{U}_j$ is relatively compact in U_j and their union covers C.

By our selection, $c^2(\Phi) \cdot e^2 + \cdots + c^{\ell_0}(\Phi) \cdot e^{\ell_0}$ and hence the normal form $N^{\ell_0}(c^2(\Phi), \ldots, c^{\ell_0}(\Phi))$ of Φ is identified with an element in $\mathcal{H}^1(T_C M)$.

Finally, when Φ admits a transverse foliation, we can start with the transition functions Φ_{kj} such that $\Phi_{kj}^\nu = \varphi_{kj}$. By the stability in our choice of normal forms, $(N_{kj}^\infty(\Phi))^\tau = \varphi_{kj}$ still holds. □

Note that the above proof of Corollary 5.5 remains true provided that N_C is 1-positivity and $H^1(C, T_C M \otimes S^\ell N_C^*) = 0$ for $\ell > \ell_0$. We obtain the classification for transverse foliations.

Corollary 5.6 *Let C be a compact complex manifold.*

(i) *Suppose that N_C is 1-positive. If two transverse foliations of neighborhoods are equivalent by a formal biholomorphic mapping F preserving the foliations, then F is actually convergent.*
(ii) *Suppose that N_C is 1-positive. Assume further that $\dim \mathcal{H}_\nu^1(T_C M) < \infty$. There is an injective mapping from the set $\mathcal{M}_\nu/\sim$ of holomorphic equivalence classes of the transverse foliations into the finite-dimensional space $\mathcal{H}_\nu^1(T_C M)$.*

Proof Assertion (i) is a consequence of Theorem 5.4. Assertion (ii) follows from (i) and the formal classification by Theorem 4.5. □

When C is a compact Riemann surface and N_C is a positive line bundle, Morrow-Rossi, using the theorem of Commichau-Grauert, showed that the equivalence classes of transverse foliations under foliation-preserving transformations are actually infinite dimensional.

We conclude this section by considering the classification of neighborhoods of C under biholomorphic mappings that are not necessarily tangent to the identity. We will, however, use coordinate changes that fix C pointwise and preserve N_C. We consider the case of a line bundle N_C. Let t_{kj} be the transition functions of N_C. An isomorphism of N_C is given by $g_k^{-1} t_{kj} g_j = t_{kj}$. Thus g_j define a global holomorphic function on C without zero. Since C is compact, then the function must be constant. Now it is easy to see that the transition functions for the neighborhood $\Phi_{kj}(z_j, v_j)$ are transformed into $(\Phi_{kj}^\tau(z_j, cv_j), c^{-1}\Phi_{kj}^\nu(cv_j))$. Note this non-homogenous dilation is used by Morrow-Rossi [26] to get a complete set of moduli spaces when $\dim C$ and $\operatorname{codim} C$ are 1, and N_C is negative.

6 Riemann Surfaces in Complex Surfaces

We illustrate Grauert's result by showing that the obstructions exist for a neighborhood with negative normal line bundle being not holomorphically equivalent to a neighborhood of its zero section; compare [3, p. 211] on Grauert's work. We also prove the assertion in Remark 5.3. This explains why we cannot apply Grauert's formal principle for weakly negative N_C to the classification of transverse foliations, and it seems to the authors that an application of a formal principle to the transverse foliations needs a statement stronger than Theorem 5.2. An interested reader should consult Morrow-Rossi [26, p. 323, line 3, and Thm. 6.3] for negative N_C, as Corollary 5.6 (ii) addresses only the case of 1-positive normal bundle.

Let us first recall some facts on the dimensions of the zero-th and first cohomology groups of line bundles. Note that a line bundle on a compact Riemann surface is positive if and only if its degree is positive. Let C be a compact Riemann surface with genus g. When $L\to C$ is a line bundle with degree $\nu_L<0$, L has no global section. Recall the duality

$$H^q(C,\Omega^p(E))=H^{1-q}(C,\Omega^{1-p}(E^*))^*,\quad q=0,1.$$

We have $\deg TC=2-2g$ and $\deg K_C=2g-2$, where K_C is the canonical line bundle of C. The Riemann-Roch theorem says that if $h^0(L)$ is the dimension of $H^0(C,\mathcal{O}(L))$ then

$$h^0(L)-h^0(K_C\otimes L^{-1})=\deg L+1-g.$$

Thus

$$h^0(L)\geq \deg L-g+1.$$

This provides the following useful estimates for positive N_C:

$$\dim H^1(C,N_C\otimes S^\ell N_C^*)\geq g-1+(\ell-1)\deg N_C, \tag{6.1}$$

$$\dim H^1(C,TC\otimes S^\ell N_C^*)\geq 3g-3+\ell\deg N_C. \tag{6.2}$$

6.1 Normal forms on neighborhoods

6.1.1 Negative N_C

In this case, the formal principle and (2.12) hold. We have

$$H^1(C,(T_C\oplus N_C)\otimes S^\ell N_C^*)=(H^0(C,K_C\otimes(K_C\otimes S^\ell N_C+N_C^*\otimes S^\ell N_C))^*.$$

Now $d_\ell^\tau := \deg K_C \otimes (K_C \otimes S^\ell N_C) = 4(g-1) + \ell \deg N_C$ and $d_\ell^\nu := \deg K_C \otimes (N_C^* \otimes S^\ell N_C) = 2(g-1) + (\ell - 1) \deg N_C$. Thus

$$\dim H^1(C, TC \otimes S^2 N_C^*) \geq 3g - 3 + 2 \deg N_C,$$
$$\dim H^1(C, N_C \otimes S^2 N_C^*) \geq d_2^\nu - g + 1 = g - 1 + 2 \deg N_C.$$

By Morrow-Rossi [26, Thm. 6.3], each element in $H^1(C, N_C \otimes S^2 N_C^*)$ can be realized by transverse foliations. See Camacho-Movasati-Sad [5] and Abate-Bracci-Tovena [1] for different approaches. This shows that an element corresponding to a non-zero element in $H^1(C, N_C \otimes S^2 N_C^*)$ is not equivalent to a neighborhood of the zero section of N_C.

6.1.2 Positive N_C and Negative $T_C M \otimes S^2 N_C^*$

This occurs if and only if $\deg N_C > \max\{0, g-1\}$, in which case (2.13) holds. Then the formal principle holds. Also, both

$$\oplus_{\ell>1} H^1(TC \otimes S^\ell N_C^*), \quad \oplus_{\ell>1} H^1(C, N_C \otimes S^\ell N_C^*)$$

are infinite dimensional.

Proposition 6.1 *Let C be a compact Riemann surface with genus g. Suppose N_C is a line bundle. For any finite r, the following hold:*

$$\oplus_{m=2}^r H^1(C, T_C M \otimes S^m N_C^*) \subset \mathfrak{C}(\mathcal{M}/\sim) \subset \oplus_{m=2}^\infty H^1(C, T_C M \otimes S^m N_C^*) \quad (6.3)$$

if $\deg N_C > \max\{0, g-1\}$*; if* $\deg N_C > 0$ *and $T_C M$ splits then*

$$\oplus_{m=2}^r H^1(C, N_C \otimes S^m N_C^*) \subset \mathfrak{C}_\nu(\mathcal{M}_\nu/\sim) \subset \oplus_{m=2}^\infty H^1(C, N_C \otimes S^m N_C^*). \quad (6.4)$$

Proof Our proof is based on a construction in [26] to realize $\oplus_{\ell>1} H^1(C, N_C \otimes S^\ell N_C^*)$ for transverse foliations. Thus (6.4) is essentially in [26]. As indicated in [26], the construction applies to

$$\oplus_{\ell>1} H^1(C, T_C M \otimes S^\ell N_C^*)$$

as well, which we show below. We will need

$$H^0(C, T_C M \otimes S^\ell N_C^*) = 0, \quad \forall \ell > 0$$

which holds for $\deg N_C > \max\{0, g-1\}$. Thus $\mathrm{Aut}_m(N^m(c^2, \ldots, c^m))$, defined as

$$\{F = \mathrm{I} + O(2) \colon F N^m(c^2, \ldots, c^m) = N^m(c^2, \ldots, c^m) F + O(m+1)\},$$

consists of mappings of the form $F = \mathrm{I} + O(m+1)$.

Take a holomorphic disk U_0 in C and let U_0' be a smaller disk in U_0. Then $U_0, U_1 := C\setminus U_0'$ form a Leray covering of C as both U_0, U_1 are Stein. $U_0 \cap U_1$ is an annulus biholomorphic to $\{z \in \mathbb{C}\colon r < |z| < 1/r\}$. Since C is covered by two sets, then

$$Z^1(\mathcal{U}, L) = C^1(U_0 \cap U_1, L).$$

Since a line bundle on an open Riemann surface is holomorphically trivial [14, p. 52], then N_C is completely determined by

$$(\varphi_{10}(z_0), t_{10}(z_0)v_0)$$

where $t_{10}(z)$ is a non-vanishing holomorphic function on $U_0 \cap U_1$ and φ_{10} is holomorphic and injective on $U_0 \cap U_1$. Also deg N_C is the winding number of t_{10} on the unit circle. A neighborhood of C with $N_C|_{U_1}$ being trivial is precisely given by

$$\Phi_{10}(z_0, v_0) := (\varphi_{10}(z_0) + l_{10}(z_0)v_0, t_{10}(z_0)v_0) + \sum_{\ell>1} \Phi_{10,\ell}(z_0)v_0^\ell$$

by patching $(U_0 \sqcup U_1) \times \Delta_\epsilon/\sim$ with $(z_1, v_1) \sim \Phi_{10}(z_0, v_0)$. Here $\Phi_{10,\ell}$ are holomorphic functions on $U_{01} \times \Delta_\epsilon$ subject to the only condition that

$$\sum_{\ell>1} \sup_K |\Phi_{10,\ell}| < C_K < \infty \tag{6.5}$$

for any compact subset K of $U_{01} \times \Delta_\epsilon$.

When $l_{10} = 0$, $T_C M$ splits. When $\Phi^\tau_{10,\ell} = 0$, the neighborhood admits a transverse foliation. When t_{10}, l_{10} and $\Phi^\nu_{10,\ell}$ are constant, the neighborhood admits tangential foliations. The degree of N_C is the winding number of t_{10} on the unit circle.

To realize each element in (6.3), we recall briefly the construction in Theorem 2.7. Since C is covered by two sets, we can take

$$N^1 = (\varphi_{10}(z_0) + l_{10}(z_0)v_0, t_{10}(z_0)v_0).$$

We now take the advantage that $\Phi_{01,\ell}$ need only to satisfy (6.5) and are otherwise arbitrary. Fix a finite basis e^2 for $H^1(\mathcal{U}, T_C M \times S^2 N_C^*)$. Let e_j^2 be represented by $C^1(\mathcal{U}, T_C M \times S^2 N_C^*)$, still denoted by e_j^2. We also use the identification via (2.9) and (2.10). Thus each element $c^2 \cdot e^2 \in H^1(\mathcal{U}, T_C M \times S^2 N_C^*)$ is associated with

$$N^2(c^2) = (\varphi_{10}(z_0) + l_{10}(z_0)v_0, t_{10}(z_0)v_0) + c^2 v_0^2.$$

Since C is covered by two open sets, $N^2(c^2)$ is indeed a cocycle for transition functions of a neighborhood of C.

For finitely many $c^2, \dots, c^r$, $\sum_{\ell=2}^r c^\ell \cdot e^\ell$ is associated to $N^r(c^2, \dots, c^r)$ satisfying the convergence constrain (6.5). ◻

6.2 Verifying Remark 5.3

Proposition 6.2 *Assume that T_CM splits. Assume that $H^1(C, T_CM \otimes S^\ell N_C^*) = 0$ for $\ell \geq \kappa$ for a finite κ.*

(a) *Suppose $H^0(C, T_CM \otimes S^\ell N_C^*) \neq 0$ for a sequence $\ell = \ell_j \to \infty$. Then there are divergent formal mappings that preserve the germs of neighborhoods of the zero section of N_C.*
(b) *Suppose $H^0(C, N_C \otimes S^\ell N_C^*) \neq 0$ for some $\ell > 1$. Then there is a possibly divergent mapping F preserving the germ of the zero section of N_C while F does not preserve the transverse foliation of N_C.*

Proof If T_CM has locally constant transition functions, the proof is straightforward as each global section $[\phi]^m$ of $T_CM \otimes S^m N_C^*$ gives us an automorphism of T_CM of the form $F = \mathrm{I} + f$ where f is homogeneous of degree m. Then $F = \mathrm{I} + \sum_{\ell > 1} f_m$ diverges for suitable choices of coefficients of the global sections of $T_CM \otimes N_c^\ell$ for $\ell > 1$.

When T_CM is not flat, we verify the assertion by a *reversing* linearization procedure. Take j such that $\ell_j \geq \kappa$. Using a global section $f_{\ell_j}^*$ of $T_CM \otimes S^{\ell_j} N_C^*$, we find $F_{\ell_j} = \mathrm{I} + f_{\ell_j}$ where f_{k,ℓ_j} is homogeneous of degree ℓ_j in v_k variables. Then

$$F_{\ell_j} N^1 F_{\ell_j}^{-1} = N^1 + O(\ell_j + 1). \tag{6.6}$$

By condition on H^1 and (6.6), we find a formal mapping $\tilde{F}_{\ell_j+1} = \mathrm{I} + O(\ell_j + 1)$ such that $\tilde{F}_{\ell_j+1} F_{\ell_j} N^1 F_{\ell_j}^{-1} \tilde{F}_{\ell_j+1}^{-1} = N^1$. We repeat this and find F_{ℓ_i}, $\tilde{F}_{\ell_i+1}$ for $i > j$ such that $\tilde{F}_{\ell_i+1} F_{\ell_i}$ and N^1 commute. Then

$$F_\infty = \lim_{i \to \infty} \tilde{F}_{\ell_i+1} F_{\ell_i} \circ \cdots \circ \tilde{F}_{\ell_j+1} F_{\ell_j}$$

commutes with N^1. We have

$$F_\infty = F_{\ell_i} \cdots \tilde{F}_{\ell_j+1} F_{\ell_j} + O(\ell_i + 2), \quad F_\infty = \tilde{F}_{\ell_i+1} F_{\ell_i} \cdots \tilde{F}_{\ell_j+1} F_{\ell_j} + O(\ell_i + 2).$$

In other words,

$$[F_\infty]_{\ell_i} = f_{\ell_i} + R(f_{\ell_j}, \ldots, f_{\ell_{i-1}}).$$

When the coefficients of $f_{\ell_i}^*$ grow sufficiently fast as $j \to \infty$, we get a divergent F_∞. This proves the first assertion.

The second assertion can be proved by a similar argument. □

As an example, the condition in Proposition 6.2 (a) is satisfied when C is the Riemann sphere and N_C is a negative line bundle.

In closing, the main purpose of this paper is to show that the holomorphic equivalence classes of neighborhoods can be realized as a subset of a *finitely dimensional* space $\mathcal{H}^1(T_C M)$. An interesting question is if there is a fine structure on a possible subset in $\mathcal{H}^1(T_C M)$ that describes the moduli space of the manifolds completely. However, this theory remains to be further developed.

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