ON NEIGHBORHOODS OF EMBEDDED COMPLEX TORI

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Abstract. The goal of the article is to show that an n-dimensional complex torus embedded in a complex manifold of dimensional $n+d$, with a split tangent bundle, has neighborhood biholomorphic a neighborhood of the zero section in its normal bundle, provided the latter has (locally constant) Hermitian transition functions and satisfies a *non-resonant Diophantine* condition.

In memory of Jean-Pierre Demailly

1. Introduction

In this paper we show the following

Theorem 1.1. *Let C be an n-dimensional complex torus embedded in a complex manifold M of dimensional $n+d$. Assume that T_CM, the restriction of TM on C, splits as $TC \oplus N_C$. Suppose that the normal bundle of C in M admits transition functions that are Hermitian matrices and satisfy a non-resonant Diophantine condition* (*see Definition* 4.5). *Then a neighborhood of C in M is biholomorphic to a neighborhood of the zero section in the normal bundle.*

We first describe the organization of the proof of our main theorem.

A complex torus C can be identified with the quotient of $\mathbb{C}^n$ by a lattice Λ spanned by the standard unit vectors $e_1, \dots, e_n$ in $\mathbb{C}^n$ and n additional vectors $e_1', \dots, e_n'$ in $\mathbb{C}^n$, where $\operatorname{Im} e_1', \dots, \operatorname{Im} e_n'$ are linearly independent vectors in $\mathbb{R}^n$. Let Λ' be the lattice in the cylinder $\widetilde{C} := \mathbb{R}^n/\mathbb{Z}^n + i\mathbb{R}^n$ spanned by $e_1', \dots, e_n' \mod \mathbb{Z}^n$. There are two coverings for the torus $C = \mathbb{C}^n/\Lambda = \widetilde{C}/\Lambda'$: the universal covering $\pi\colon \mathbb{C}^n \to C$ and the covering by cylinder, $\pi_{\widetilde{C}}\colon \widetilde{C} \to C$ that extends to a covering $\mathcal{M}$ over M. In section two we recall some facts about factors of automorphy for vector bundles on C via the covering by $\mathbb{C}^n$. In section three, we study the flat vector bundles on C. The pull back of the flat vector bundle N_C to the cylinder $\widetilde{C}$ is the normal bundle $N_{\widetilde{C}}$ of $\widetilde{C}$ in $\mathcal{M}$. We show that $N_{\widetilde{C}}$ is always the holomorphically trivial vector bundle $\widetilde{C} \times \mathbb{C}^d$. By "vertical coordinates", we mean "coordinates on $\mathbb{C}^d$", the normal component of the normal bundle N_C, while "horizontal coordinates" mean the tangential components of N_C.

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Since $\widetilde{C}$ is a Stein manifold, a theorem of Siu [Siu77] says that a neighborhood of $\widetilde{C}$ in $\mathcal{M}$ is biholomorphic to a neighborhood of the zero section in its normal bundle, which is trivial as mentioned above. We show that the holomorphic classification of neighborhoods M of C with flat N_C is equivalent to the holomorphic classification of the family of the deck transformations of coverings $\mathcal{M}$ of M in a neighborhood of $\mathcal{C}$. These deck transformations are "higher-order" (in the vertical coordinates) perturbations $\tau_1, \dots, \tau_n$ of $\hat{\tau}_1, \dots, \hat{\tau}_n$, where the latter are the deck transformations of the covering of $\widetilde{N}_C$ over N_C. In order to find a biholomorphism between a neighborhood of C in M and a neighborhood of its zero section in N_C, it is sufficient to find a biholomorphism that conjugates $\{\tau_1, \dots, \tau_n\}$ to $\{\hat{\tau}_1, \dots, \hat{\tau}_n\}$.

There are two useful features. First, since the fundamental group of C is abelian, the deck transformations $\tau_1, \dots, \tau_n$ commute pairwise. Second, we can also introduce suitable coordinates on $\widetilde{C}$ so that the "horizontal" components of deck transformations have diagonal linear parts. In such a way the classification of neighborhoods of C is reduced to a more attainable classification of deck transformations. While the full theory for this classification is out the scope of this paper, we study the case when N_C admits Hermitian transition functions. Since a Hermitian transition matrix must be locally constant, we call such an N_C *Hermitian flat*. The convergence proof for Theorem 1.1 is given in section four. It relies on a Newton rapid convergence scheme adapted to our situation based on an appropriate Diophantine condition among the lattice and the normal bundle. At step k of the iteration scheme, let δ_k be the error of the deck transformations $\{\tau_1^{(k)}, \dots, \tau_n^{(k)}\}$ defined on domain $D^{(k)}$ to $\hat{\tau}_1, \dots, \hat{\tau}_n$ in suitable norms. By an appropriate transformation $\Phi^{(k)}$, we conjugate to a new set of deck transformations $\{\tau_1^{(k+1)}, \dots, \tau_n^{(k+1)}\}$ of which the error to the linear ones is now δ_{k+1} on a slightly smaller domain $D^{(k+1)}$. Using our Diophantine conditions, related to the lattice Λ and the normal bundle, we show that the sequence $\Phi^{(k)} \circ \cdots \circ \Phi^{(1)}$ converges to a holomorphic transformation Φ on an open domain $D^{(\infty)}$ where we linearize $\{\tau_1, \dots, \tau_n\}$.

We now describe closely related previous results. Our work is motivated by work of Arnold and Ilyashnko-Pyartli. Our main theorem was proved in [Arn76] when C is an elliptic curve ($n = 1$) and N_C has rank one ($d = 1$). Il'yashenko-Pyartli [IP79] extended Arnold's result to the case when the torus is the product of elliptic curves together with a normal bundle which is a direct sum of line bundles, while Theorem 1.1 deals with general complex tori. We also assume that N_C is *non-resonant*, a condition that is weaker than the non-resonant condition used by Il'yashenko-Pyartli. Our small-divisor condition is also weaker. Of course, the study of neighborhood of embedded compact complex manifolds has a long history. Also, see some recent work [Hwa19, Koi21, LTT19]. We refer to [GS22] for some references and a different approach to this range of questions.

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2. Vector bundles on Tori and factors of automorphy

In this section, we identify vector bundles on a complex torus with factors of automorphy. The latter gives us a useful alternative definition of vector bundles on a higher dimensional complex torus C and the isomorphisms of two vector bundles.

General references for line bundle on complex tori are [BL04, Deb05, Ien11] and [GH94, p. 307].

Let Λ be a $2n$-dimensional lattice in $\mathbb{C}^n$. We may assume that Λ is defined by $2n$ vectors $e_1, \dots, e_n, e'_1, \dots, e'_n$ of $\mathbb{C}^n$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 being at the i-th place, $e'_i = (e'_{i,1}, \dots, e'_{i,n})$ and the matrix

$$\operatorname{Im} \tau := (\operatorname{Im} e'_{i,j})_{1 \le i,j \le n} =: (e''_{i,j})_{1 \le i,j \le n}$$

is invertible [BL04, exerc. 2, p. 21]. The compact complex manifold $C := \mathbb{C}^n/\Lambda$ is called an (n-dimensional) complex torus. Unless the lattice is equivalent to another one defined by a diagonal matrix e', C is not biholomorphic to a product of one-dimensional tori. Let $\pi : \mathbb{C}^n \to C$ be the universal cover of C. Its group Γ of deck transformations consists of translations

$$T_\lambda \colon z \to z + \lambda, \quad \lambda \in \Lambda.$$

Note that Γ is abelian and is isomorphic to $\mathbb{Z}^{2n}$. Γ is also isomorphic to $\pi_1(C, 0)$ since $\mathbb{C}^n$ is a universal covering of C.

Next, we consider equivalence relations for holomorphic vectors bundles on C and $\mathbb{C}^n$, following the realization proof of Theorem 3.2 in [Ien11]. Let E be a vector bundle of rank d over C. The pull-back bundle π^*E on $\mathbb{C}^n$ is trivial and has global coordinates $\hat{\xi}$. Let $\{U_j\}$ be an open covering of C so that coordinates $\xi_j = (\xi_{j,1}, \dots, \xi_{j,d})^t$ of E are well-defined (injective) on U_j. Then we have

$$\hat{\xi} = h_j \xi_j(\pi), \tag{2.1}$$

where h_j is a non-singular holomorphic matrix on $\pi^{-1}(U_j)$. The transition functions g_{kj} satisfy

$$g_{kj}(\pi) = h_k^{-1} h_j, \quad \text{on } \pi^{-1}(U_k) \cap \pi^{-1}(U_j). \tag{2.2}$$

For any z, λ, we know that both $\pi(z + \lambda)$ and $\pi(z)$ are in the same U_j for some j. Then we have

$$\hat{\xi}(z + \lambda) = h_j(z + \lambda)\xi_j(\pi(z)) = h_j(z + \lambda)h_j(z)^{-1}\hat{\xi}(z).$$

We can define

$$\rho(\lambda, z) = h_j(z + \lambda)h_j(z)^{-1}, \quad z \in \pi^{-1}(U_j) \tag{2.3}$$

as the latter is independent of the choice of j by (2.2). Therefore,

$$\hat{\xi}(\lambda + z) = \rho(\lambda, z)\hat{\xi}(z), \quad \rho(\lambda, z) \in GL(d, \mathbb{C}), \tag{2.4}$$

$$\rho \colon \Lambda \times \mathbb{C}^n \to GL(d, \mathbb{C}). \tag{2.5}$$

Here ρ is called a *factor of automorphy*. We can verify that

$$\rho(\lambda + \mu, z) = \rho(\lambda, \mu + z)\rho(\mu, z). \tag{2.6}$$

In particular, if all $\rho(\lambda, z) = \rho(\lambda)$ are independent of z, then $\rho(\Lambda)$ is an abelian group.

The above construction from a vector bundle E on C to a factor of automorphy can be reversed. Namely, given (2.4)-(2.6), define the vector bundle E on C as the quotient vector space of $\mathbb{C}^n \times \mathbb{C}^d$ via the equivalence relation

$$(z, \xi) \sim (z + \lambda, \rho(\lambda, z)\xi), \quad z \in \mathbb{C}^n,\ \xi \in \mathbb{C}^d,\ \lambda \in \Lambda. \tag{2.7}$$

We denote the projection from the cylinder $\widetilde{C} := \mathbb{R}^n/\mathbb{Z}^n + i\mathbb{R}^n = \mathbb{C}^n/\mathbb{Z}^n$ onto C by $\pi_{\widetilde{C}}$. Therefore, we can define $\pi^*_{\widetilde{C}}E$ on the cylinder $\widetilde{C}$ by the equivalence relation

$$(z,\xi) \sim (z+\lambda, \rho(\lambda,z)\xi), \quad z \in \mathbb{C}^n,\ \xi \in \mathbb{C}^d,\ \lambda \in \mathbb{Z}^n. \tag{2.8}$$

Of course, global coordinates $\hat{\xi}$, ρ, and g_{jk} are not uniquely determined by E. However, their equivalence classes are determined. Two vector bundles $E, \tilde{E}$ are isomorphic if their corresponding transitions $g_{jk}, \tilde{g}_{jk}$ satisfy $\tilde{g}_{jk} = h_j^{-1} g_{jk} h_k$ where h_j are non-singular holomorphic matrices. Replacing global coordinates $\hat{\xi}$ by $\nu\hat{\xi}$ where $\nu\colon \mathbb{C}^n \to GL(n,d)$ is holomorphic, we can verify that

$$\nu(\lambda+z)\rho(\lambda,z)\nu(z)^{-1} =: \tilde{\rho}(\lambda,z) \tag{2.9}$$

is also a factor of automorphy. Define two factors of automorphy $\rho, \tilde{\rho}$ to be *equivalent* if (2.9) holds. Therefore, the classification of holomorphic vector bundles is identified with the classification of factors of automorphy.

3. Flat vector bundles

In this section, we will show that the pull-back of a flat vector bundle E on C to the cylinder $\widetilde{C} = \mathbb{C}^n/\mathbb{Z}^n$ is always trivial.

When E is flat, we can choose global coordinates as follows. We know that π^*E is also flat and we can choose its global flat basis, or global flat coordinates $\hat{\xi}$ by using analytic continuation on $\mathbb{C}^n$ and pulling back flat local coordinates of E. In other words, in (2.1) h_j are locally constants while ξ_j are locally flat coordinates. Then $\rho(\lambda, z)$ depend only on λ, in which case we write $\rho(\lambda)$ for $\rho(\lambda,z)$. As remarked above, $\rho(\Lambda)$ is abelian. When E is unitary (flat), by the same reasoning π^*E is unitary and we can choose h_j and $\rho(\lambda)$ to be unitary.

A $d \times d$ Jordan block $J_d(\lambda)$ is a matrix of the form $\lambda I_d + N_d$, where I_d is the $d \times d$ identity matrix and N_d is the $d \times d$ matrix with all entries being 0, except all the $(i, i+1)$-th entries being 1. A matrix T commutes with J if and only if

$$T = T_d(a) := a_0 I_d + \sum_{i>0} a_i N_d^i.$$

Note that $N_d^i = 0$ for $i \geq d$. Following [Gan98, p. 218], we call the above T as well as the following two types of matrices, *regular upper triangular* matrices w.r.t. J:

$$A = (0, T_d(a)), \quad \text{or} \quad B = \begin{pmatrix} T_{d'}(a') \\ 0 \end{pmatrix},$$

where 0 denotes in A (resp. B) a 0 matrix of d rows (resp. d' columns). Given a Jordan matrix

$$\tilde{J} = \operatorname{diag}(J_{d_1}(\lambda_1), \ldots, J_{d_k}(\lambda_k)).$$

the matrices that commute with $\tilde{J}$ are precisely the block matrices

$$X = (X_{\alpha\beta})_{m\times m}$$

where $X_{\alpha\beta} = 0$ if $\lambda_\alpha \neq \lambda_\beta$, while $X_{\alpha\beta}$ is a regular upper triangular $(d_\alpha \times d_\beta)$ matrix if $\lambda_\alpha = \lambda_\beta$. Such a matrix X is said to be a *regular upper triangular matrix* w.r.t. J

From the structure of matrices commuting with a Jordan matrix, we can verify the following two results.

Proposition 3.1. *Let $A_1, \dots, A_m$ be 2×2 matrices commuting pairwise. Then there is a non-singular matrix S such that all $S^{-1}A_jS$ are Jordan matrices.*

Example 3.2. *The 3×3 matrices $\lambda I_3 + N_3, \mu I_3 + N_3^2$ commute, but they cannot be transformed into the Jordan normal forms simultaneously.*

The following results on logarithms are likely classical. However, we cannot find a reference. Therefore, we give proofs emphasizing commutativity of logarithms of matrices.

We start with the following.

Lemma 3.3. *Let $A_1, \dots, A_m$ be pairwise commuting matrices. Then there is a non-singular matrix S such that*

$$S^{-1}A_jS = (\hat{A}_{\alpha\beta})_{1 \leq \alpha, \beta \leq s} =: \hat{A}_j, \quad 1 \leq j \leq m$$

where $\hat{A}_1$ is a Jordan matrix and all $\hat{A}_j$ are upper triangular matrices.

Proof. We may assume that A_1 is a Jordan matrix $J = \operatorname{diag}(J_{d_1}(\lambda_1), \dots, J_{d_s}(\lambda_s))$. Then

$$A_j = (X^j_{\alpha\beta})_{1 \leq \alpha, \beta \leq s}$$

are regular w.r.t J. Note that pairwise commuting non-singular matrices have non-trivial common eigenspaces. The eigenspace of J are spanned $e_{d_1'}, \dots, e_{d_s'}$ with $d_1' = 1$ and $d_j' = d_1 + \cdots d_{j-1} + 1$. To simplify the indices, we may assume that e_1 is an eigenvector of all A_j. Then the first column of A_j is a_je_1 with $a_j \neq 0$. The new matrices $\tilde{A}_j$, obtained by removing all first rows and first columns, still commute pairwise. In particular all $\tilde{A}_j$ are regular to the new Jordan matrix $\tilde{J} = \tilde{A}_1$. By induction on d, we can find a non-singular matrix $\tilde{S}$ which is regular to $\tilde{J}$ so that all $\tilde{S}^{-1}\tilde{A}_j\tilde{S}$ are upper-triangular. Let $S = \operatorname{diag}(1, \tilde{S})$. Now $S^{-1} = \operatorname{diag}(1, \tilde{S}^{-1})$. We can check that all $\hat{A}_j := S^{-1}A_jS$ are upper triangular. Then $\hat{A}_1$, J have the same entries, with only one possible exception

$$\hat{A}_{1;12} = s_{11}J_{12}.$$

If $\hat{A}_{1;12} \neq 0$, dilating the first coordinate can transform $\hat{A}_1$ into the original J, while $\hat{A}_j$ remain upper-triangular. If $s_{12}J_{12} = 0$, then J_{12} must be 0, i.e. $\hat{A}_1 = J$, because one cannot transform a Jordan matrix, $A_1 = J$, into a new Jordan matrix, $\hat{A}_1$, by reducing an entry 1 to 0 and keeping other entries unchanged. $\square$

The above simultaneous normalization of upper-triangular matrices allows us to define the logarithms. The construction of logarithms of non-singular matrices can be found in [Gan98, p. 239]. Here we need to find a definition that is suitable to determine the commutativity of the logarithm of pairwise commuting non-singular matrices.

Recall that for a $d \times d$ matrix A, the generalized eigenspace $E_\lambda(A)$ with eigenvalue λ is the kernel of $(A - \lambda I)^d$, while $\mathbb{C}^d$ is the direct sum of all $E_\lambda(A)$. A matrix B that commutes with A leaves each $E_\lambda(A)$ invariant, i.e. $B(E_\lambda(A)) \subset E_\lambda(A)$. Thus if $A_1, \dots, A_m$ commute pairwise, we can decompose $\mathbb{C}^d$ as a direct sum of liner subspaces V_j such that each V_j is invariant by A_i and admits exactly one eigenvalue of A_i. Thus to define $\ln A_j$, we will assume that each A_j has a single eigenvalue on $\mathbb{C}^d$ if we wish.

Given a non-singular matrix A, a logarithm of A is a matrix $\ln A$ satisfying

$$e^{\ln A} = A$$

where the exponential matrix $e^B = \sum \frac{B^n}{n!}$ is always well-defined. However, $\ln A$ is not unique.

For a non-singular upper triangular matrix

$$A = \lambda I_d + a, \quad a := (a_{ij})_{1\le i,j\le d}, a_{ij=0} \quad \forall i \ge j, \tag{3.1}$$

we have $a^k = 0$ for $k \ge d$. Using the identity $e^{B+C} = e^B e^C$ for two commuting matrices B, C, we see that $e^{\ln A} = A$ for

$$\ln A := (\ln \lambda) I_d - \sum_{k>0} \frac{(-\lambda^{-1} a)^k}{k} \tag{3.2}$$

with $0 \le \operatorname{Im} \ln(.) < 2\pi$. For a non-singular Jordan matrix, we can define

$$\ln \operatorname{diag}(J_{d_1}(\lambda_1), \ldots, J_{d_m}(\lambda_m)) = \operatorname{diag}(\ln J_{d_1}(\lambda_1), \ldots, \ln J_{d_m}(\lambda_m)).$$

Note that $\ln \lambda_\alpha = \ln \lambda_\beta$ if and only if $\lambda_\alpha = \lambda_\beta$. Since matrices that commute with a fixed matrix is closed under multiplication by a scalar, addition and multiplication. It is thus clear that if A is an upper triangular matrix that is regular to a non-singular Jordan matrix $J = \operatorname{diag}(J_{d_1}(\lambda_1), \ldots, J_{d_s}(\lambda_s))$, then $\ln A$ remains regular to J. Equivalently and more importantly, $\ln A$ is regular to the Jordan normal form of $\ln J$, which is

$$\operatorname{diag}(J_{d_1}(\ln \lambda_1), \ldots, J_{d_s}(\ln \lambda_s)).$$

Proposition 3.4. *Let $A_1, \ldots, A_m$ be pairwise commuting $d \times d$ matrices. Then there is a non-singular matrix S such that $\hat{A}_j := S^{-1} A_j S$ are block diagonal matrices of the form*

$$\hat{A}_j = \operatorname{diag}(\hat{A}_{j,d_1}, \ldots, \hat{A}_{j,d_k}) \tag{3.3}$$

where all $\hat{A}_{j,d_i}$ are upper triangular $d_i \times d_i$ matrices, and each $\hat{A}_{j,d_i}$ has only one eigenvalue $\lambda_{j,i}$. Assume further that all A_j are non-singular. Then

$$\ln A_j := S \operatorname{diag}(\ln \hat{A}_{j,d_1}, \ldots, \ln \hat{A}_{j,d_k}) S^{-1}, \quad 1 \le j \le m \tag{3.4}$$

commute pairwise and $e^{\ln A_j} = A_j$, where $\ln \hat{A}_{j,d_i}$ are defined by (3.1)-(3.2).

Proof. Note that for pairwise commuting matrices $A_1, \ldots, A_m$, we have a decomposition $\mathbb{C}^d = \bigoplus_{i=1}^t V_i$ where each A_j preserves V_i and has only one eigenvalue $\lambda_{j,i}$. Their restrictions of $A_1, \ldots, A_m$ on V_i remain commutative pairwise. Let $\dim V_i = d_i, d'_0 = 0, d'_{i+1} - d'_i = d_i$. By Lemma 3.3, we can find a basis

$$e^*_{d'_{i-1}+1}, \ldots, e^*_{d'_{i-1}+d_i}$$

for V_i such that $A_1|_{V_i}, \ldots, A_m|_{V_i}$ are upper triangular matrices. Using the new basis $e^*_1, \ldots, e^*_d$ we can find the matrix S for the decomposition (3.3). Therefore, $\hat{A}_{1,d_i}, \ldots, \hat{A}_{m,d_i}$ commute pairwise.

Assume now that all $\lambda_{j,i}$ are non-zero. It remains to show that $\ln \hat{A}_{1,d_i}, \ldots, \ln \hat{A}_{m,d_i}$ commute pairwise. Write

$$\hat{A}_{j,i} = \lambda_{j,i} I_{d_i} + W_{j,i}$$

where $W_{j,i}$ are upper triangular matrices and $W_{j,i}^{d_i} = 0$. By a straightforward computation, we have

$$[W_{j,i}, W_{j',i}] = [\hat{A}_{j,i}, \hat{A}_{j',i}] = 0.$$

Therefore, $W^k_{j,i}$ commutes with $W^\ell_{j',i}$ for any k,ℓ. Consequently, the finite sum

$$\ln \hat{A}_{j,i} = \ln \hat{\lambda}_{j,i} I_{d_i} - \sum_{k>0} \frac{(-\lambda_{j,i}^{-1} W_{j,i})^k}{k}$$

commutes with $\ln \hat{A}_{j',i}$. Therefore, $\ln A_1, \ldots, \ln A_m$ in (3.4) commute pairwise. □

Lemma 3.5. *Let $A_1, \ldots, A_n$ be non-singular upper triangular $d \times d$ matrices. Suppose that $A_1, \ldots, A_n$ commute pairwise. There exists a linear mapping $w \to \tilde{v}^z(w) := v(z)w$ in $\mathbb{C}^d$, entire in $z \in \mathbb{C}^n$ such that $v(z) \in GL_d(\mathbb{C})$, $v(0) = Id$ and $v(e_j) = A_j$ for $j = 1, \ldots, n$. Furthermore, $v(z+z') = v(z)v(z')$ for all $z, z' \in \mathbb{C}^n$.*

Proof. By Proposition 3.4, we define pairwise commuting matrices $\ln A_1, \ldots, \ln A_n$ such that $e^{\ln A_j} = A_j$. Then the Lie brackets of the vector fields for $\dot{w} = \ln A_j w$, $j = 1, \ldots, n$ vanish and their flows $\varphi_j^t(w)$ commute pairwise. Note that

$$\varphi_j^0(w) = w, \quad \varphi_j^1(w) = e^{\ln A_j} w = A_j w.$$

Define

$$\tilde{v}^z(w) = \varphi_1^{z_1} \cdots \varphi_n^{z_n}(w).$$

We conclude $\tilde{v}^z \tilde{v}^{z'}(w) = \tilde{v}^{z+z'}(w)$, that is $v(z+z') = v(z)v(z')$. □

By Proposition 3.4, we define $\ln \rho(e_1), \ldots, \ln \rho(e_{2n})$ and they commute pairwise. We now define $\ln \rho(\lambda)$ for all $\lambda = \sum_{j=1}^{2n} m_j e_j \in \Lambda$ as follows

$$\ln \rho \left(\sum_{j=1}^{2n} m_j e_j \right) := \sum_{j=1}^{2n} m_j \ln \rho(e_j).$$

Thus the matrices $\ln \rho(\lambda)$ for $\lambda \in \Lambda$ commute pairwise.

Proposition 3.6. *Let E be a flat vector bundle on C. Then $\pi^*_{\widetilde{C}} E$ admits a factor of automorphy ρ satisfying $\rho(e_j) = Id$ for $j = 1, \ldots, n$; in particularly, $\pi^*_{\widetilde{C}} E$ is holomorphically trivial.*

Proof. Let d be the rank of E. Let $A_j = \rho(e_j)^{-1}$ for $j = 1, \ldots, n$. With $v(e_j) = A_j$ and $v(0) = Id_d$, we first see that

$$\tilde{\rho}(\lambda, z) := v(z+\lambda)\rho(\lambda)v(z)^{-1}$$

satisfies $\tilde{\rho}(e_j, 0) = Id_d$. We want to show that $\tilde{\rho}(\lambda, z)$ depends only on λ. Fix $\lambda = \sum_{j=1}^{2n} m_j e_j \in \Lambda$. By definition, the matrix $\ln \rho(\lambda)$ commutes with each $\ln \rho(e_j)$, $j = 1, \ldots, 2n$. Thus the flow φ_λ^t of $\dot{w} = \ln \rho(\lambda) w$ commutes with the flows of $\dot{w} = \ln \rho(e_j) w$, $j = 1, \ldots, 2n$. As in the proof of the previous lemma, we know that $\varphi_\lambda^t(w)$ is linear in w and entire in $t \in \mathbb{C}$. For $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^d$, let $\tilde{v}^z(w)$ be as defined in the previous lemma. Thus we have

$$\varphi_\lambda^t \tilde{v}^z(w) = \tilde{v}^z \varphi_\lambda^t(w).$$

Taking derivatives in w and plugging in $t = 1$, we get

$$\exp(\ln \rho(\lambda)) v(z) = v(z) \exp(\ln \rho(\lambda)).$$

Since $\exp(\ln \rho(\lambda)) = \rho(\lambda)$, we have $\rho(\lambda) v(z) = v(z) \rho(\lambda)$ for all $\lambda \in \Lambda$, $z \in \mathbb{C}^n$. Considering $z = z_1 e_1 + \cdots + z_n e_n \in \Lambda$, we have $v(z+\lambda) = v(z)v(\lambda)$. Hence, $\tilde{\rho}(\lambda, z)$ is independent of z.

We have achieved $\tilde{\rho}(\lambda) = \nu(z+\lambda)\rho(z)\nu(z)^{-1}$ and $\tilde{\rho}(e_j) = Id_d$ for $j = 1, \ldots, n$. Therefore, $\pi|^*_{\widetilde{C}}E$ is trivial, by the equivalence relation (2.8). □

It is known that there are Stein manifolds with non-trivial vector bundle [FR68]. Furthermore, we conclude the section by emphasizing that the triviality $\pi^*|_{\widetilde{C}}E$ relies on the extra assumption that it is a pull-back bundle. The following result is likely known, but we include a short proof for completeness.

Proposition 3.7. *The set of holomorphic equivalence classes of flat holomorphic line bundles on $\widetilde{C}$ can be identified with $H^1(\widetilde{C}, \mathbb{C}^*)$. The latter is non-trivial.*

Proof. Each element $\{c_{jk}\}$ in $H^1(\widetilde{C}, \mathbb{C}^*)$ is clearly an element in $H^1(\widetilde{C}, \mathcal{O}^*)$. We want to show that if $\{c_{jk}\}, \{\tilde{c}_{jk}\} \in H^1(\widetilde{C}, \mathbb{C}^*)$ represent the same element in $H^1(\widetilde{C}, \mathcal{O}^*)$, then they are also the same in $H^1(\widetilde{C}, \mathbb{C}^*)$. Indeed, we can cover $\widetilde{C}$ by convex open sets U_1, U_2, U_3, U_4 such that $U_1 \cap U_2 \cap U_3 \cap U_3$ is non empty. Thus $\{U_i\}$ is a Leray covering. If $\tilde{c}_{jk} = h_j c_{jk} h_k^{-1}$, where each h_j is a non-vanishing holomorphic function on U_j. Take p in all U_j. We get $\tilde{c}_{jk} = c_j c_{jk} c_k^{-1}$ for $c_j = h_j(p)$.

Note that $H^1(\widetilde{C}, \mathbb{C}^*)$ is non-trivial. Otherwise, the exact sequences $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$ and $0 \to H^0(\widetilde{C}, \mathbb{Z}) \to H^0(\widetilde{C}, \mathbb{C}) \to H^0(\widetilde{C}, \mathbb{C}^*) \to 0$ would imply that $H^1(\widetilde{C}, \mathbb{Z}) \cong H^1(\widetilde{C}, \mathbb{C})$, a contradiction. □

4. Equivalence of neighborhoods and commuting deck transformations

In this section, we will discuss how the classification of neighhorhoods U of a compact complex manifold C is related to the classification of deck transformations of a holomorphic covering $\tilde{U} \to U$, where $U, \tilde{U}$ are chosen carefully and $\tilde{U}$ contain C^* that covers C. When C^* is additionally Stein, we can choose $\tilde{U}$ to be a neighborhood of C^* in its normal bundle $N_{C^*}(\tilde{U})$ by applying a result of Siu. After preliminary results in Lemma 4.1 and Lemma 4.2, we will return to our previous study where C is a complex torus, and its covering of C is the Stein manifold $C^* = \widetilde{C}$, and $N_C(M)$ is Hermitian flat. We then prove the main result of this paper by using a KAM rapid iteration scheme.

Let us start with $\iota \colon C \hookrightarrow M$, a holomorphic embedding of a compact complex manifold C. We shall still denote $\iota(C)$ by C. Let U be a neighborhood of C in M such that U admits a smooth, possibly non-holomorphic, deformation or strong retract C [Mun00, p. 361]; namely there is a smooth mapping $R \colon U \times [0,1] \to U$ such that $R(\cdot, 0) = Id$ on U, $R(\cdot, t) = Id$ on C, and $R(\cdot, 1)(U) = C$. Thus, $\pi_1(U, x_0) = \pi_1(C, x_0)$ for $x_0 \in C$ (see [Mun00, p. 361]). When M is N_C, we can find a *holomorphic* deformation retraction from a suitable neighborhood of its the zero section onto C, by using a Hermitian metric on N_C.

Let X be a complex manifold and $\mathcal{X}$ be a universal covering of X. Then the group of deck transformations of the covering is identified with $\pi_1(X, x_0)$. The set of equivalence classes of coverings of X is identified with the conjugate classes of subgroups of $\pi_1(X, x_0)$; see [Mun00, Thm. 79.4, p. 492]. Furthermore, $\pi_1(X, x_0)$ acts transitively and freely on each fiber of the covering and X is the quotient of $\mathcal{X}$ by $\pi_1(X, x_0)$.

Lemma 4.1. *Let C be a compact complex manifold. Let $\pi \colon C^* \to C$ be a holomorphic covering and $\pi(x_0^*) = x_0$. Suppose that (M, C) (resp. (M', C)) is a holomorphic neighborhood of C. There is a neighborhood U in M (resp. U' in M') of*

C and a holomorphic neighborhood $\tilde U$ (resp. $\widetilde{U'}$) of C^* such that $p\colon \tilde U \to U$ (resp $p\colon \widetilde{U'} \to U'$) is an extended covering of the covering $\pi\colon C^* \to C$ and C (resp. C^*) is a smooth strong retract of U, U' (resp. $\tilde U, \widetilde{U'}$). Consequently,

$$\pi_1(\tilde U, x_0^*)) = \pi_1(C^*, x_0^*), \quad \pi_1(U, x_0) = \pi_1(C, x_0).$$

Suppose that (M, C) is biholomorphic to (M', C). Then $U, U', \tilde U, \widetilde{U'}$ can be so chosen that there is covering transformation sending $\tilde U$ onto $\widetilde{U'}$ and fixing C^* pointwise. Conversely, if there is a convering transformation sending $\tilde U$ onto $\widetilde{U'}$ fixing C^* pointwise, then $(U, C), (U', C)$ are holomorphically equivalent.

Proof. Since $\pi\colon C^* \to C$ is a covering map, according to [Vic94, Thm. 4.9], it extends to a covering map $p : \tilde U \to U$ such that $\tilde U$ contains C^* and $p|_{C^*} = \pi$. Suppose that R is a strong retraction of U onto C. We can lift $R(z, \cdot)\colon [0,1] \to U$ to a continuous mapping $\tilde R(\tilde z, \cdot)\colon [0,1] \to \tilde U$ such that $\tilde R(\tilde z, 0) = \tilde z$ and $p\tilde R(\tilde z, .) = R(p(\tilde z), .)$ for all $\tilde z \in \tilde U$. One can verify that $\tilde R$ is a strong retraction of $\tilde U$ onto C^*.

Suppose that a biholomorphic map f sends (M, C) onto (M', C) fixing C pointwise. We may assume that f is a biholomorphic mapping from U onto U'. Then we can lift the mapping $f\pi\colon \tilde U \to U'$ to obtain a desired covering biholomorphism F, since $(f\pi)_*\pi_1(\tilde U, x_0^*) = \pi_*\pi_1(C^*, x_0^*)$. Conversely, a covering biholomorphism from $\tilde U$ onto $\tilde U'$ fixing C^* pointwise clearly induces a biholomorphism from U onto U' fixing C pointwise. □

With the covering, we can identity (M, C) with $(\tilde M, C^*)/\sim$ where $\tilde p \sim p$ if and only if $p, \tilde p$ are in the same stack of the covering π.

Applying the above to (N_C, C) and a covering $\pi|_{C^*}\colon C^* \to C$, we have a covering $\hat\pi\colon \widetilde{N_C} \to N_C$ such that

$$C^* \subset \widetilde{N_C}, \quad \pi_1(\widetilde{N_C}, x_0^*) = \pi_1(C^*, x_0^*), \quad \pi_1(N_C, x_0) = \pi_1(C, x_0).$$

To simplify the notation, we denote $U, \tilde U$ by $M, \tilde M$ respectively. Thus we have commuting diagrams for the coverings:

$$\begin{array}{ccccc} \widetilde{N_C} & \hookleftarrow & C^* & \hookrightarrow & \tilde M \\ \hat\pi \downarrow & & \pi_{C^*} \downarrow & & p \downarrow. \\ N_C & \hookleftarrow & C & \hookrightarrow & M \end{array}$$

The set of deck transformations of p (resp. $\hat\pi$) will be denoted by $\{\tau_1, \dots, \tau_n\}$ (resp. $\{\hat\tau_1, \dots, \hat\tau_n\}$). If $\pi\colon \tilde U \to U$ is a covering map, $Deck(\tilde U)$ denotes the set of deck transformations.

Lemma 4.2. *Let C, C^*, M be as in Lemma 4.1. Suppose that C^* is a Stein manifold. Let ω_0^* be an open set of $\widetilde{N_C}$ such that $\hat\pi(\omega^*)$ contains C. Then there is an open subset ω^* of ω_0^* such that $\hat\pi\omega^*$ contains C and (M, C) is holomorphically equivalent to the quotient space of ω^* by $Deck(\widetilde{N_C})$.*

Proof. By a result of Siu [Siu77, Cor. 1], we find a biholomorphism L from a holomorphic strong retraction neighborhood of C^* in $\tilde M$, still denoted by $\tilde M$ into $N_{C^*}(\tilde M)$ and a biholomorphism L' from a strong retraction neighborhood of C^* in $\widetilde{N_C}$, still denoted by $\widetilde{N_C}$ into $N_{C^*}(\widetilde{N_C})$. Furthermore, L, L' fix C^* pointwise. We have

$$p_*\pi_1(N_{C^*}(\tilde M),x_0^*) = p_*L_*^{-1}\pi_1(\tilde M,x_0^*) = \pi_1(C,x_0) = \pi_1(L^{-1}\tilde M,x_0^*)$$
$$= \hat\pi_*(L')_*^{-1}(\pi_1(\widetilde{N_C},x_0^*)).$$

Both $\hat\pi\circ L'^{-1}\colon N_{C^*}(\widetilde{N_C}))\to M$ and $p\circ L^{-1}\colon N_{C^*}(\tilde M)\to M$ are coverings and the above identifications show that the lifts of the two coverings yield a biholomorphism between neighborhoods of C^* in $\tilde M$ and $N_{C^*}(\widetilde{N_C})$ fixing C^* pointwise. □

Here, $z=(z_1,\dots,z_n)$ belongs to the fundamental domain

$$\omega_0=\left\{\sum_{j=1}^{2n} t_je_j\in\mathbb{C}^n\colon t\in[0,1)^{2n}\right\},\quad e_{n+i}:=\tau_i,\quad i=1,\dots,n.$$

Thus h belongs to the fundamental domain Ω_0 defined

$$\Omega_0:=\{(e^{2\pi i\zeta_1},\dots e^{2\pi i\zeta_n})\colon \zeta\in\omega_0\},\quad \Omega_0^+=\{(|z_1|,\dots,|z_n|)\colon z\in\Omega_0\}.$$

Thus Ω_0 is a Reinhardt domain, being $\{(\nu_1R_1,\dots,\nu_nR_n)\colon |\nu_j|=1\colon R\in\Omega_0^+\}$. We have

$$\Omega_0^+=\left\{(e^{-2\pi R_1},\dots,e^{-2\pi R_n})\colon R=\sum_{i=1}^n t_i\operatorname{Im}\tau_i, t\in[0,1)^n\right\}.$$

For $\epsilon>0$, define a (Reinhardt) neighborhood Ω_ϵ of $\overline{\Omega_0}$ by

$$\omega_\epsilon:=\left\{\sum_{j=1}^{2n} t_je_j\colon t\in[0,1)^n\times(-\epsilon,1+\epsilon)^n\right\},\quad \Omega_\epsilon:=\{(e^{2\pi i\zeta_1},\dots e^{2\pi i\zeta_n})\colon\zeta\in\omega_\epsilon\}.$$

For any ℓ-tuple of indices in $\{n+1,\dots,2n\}$, we set

(4.1)
$$\omega_\epsilon^{j_1\cdots j_\ell}:=\left\{\sum_{j=1}^{2n} t_je_j\colon t_k\in(-\epsilon,\epsilon),k-n\in\{j_1,\dots,j_\ell\}; t_k\in(-\epsilon,1+\epsilon),k-n\notin\{j_1,\dots,j_\ell\}\right\},$$

(4.2)
$$\tilde\omega_\epsilon^{j_1\cdots j_\ell}:=\left\{\sum_{j=1}^{2n} t_je_j\colon t_k-1\in(-\epsilon,\epsilon),k-n\in\{j_1,\dots,j_\ell\}; t_k\in(-\epsilon,\epsilon),k-n\notin\{j_1,\dots,j_\ell\}\right\}.$$

Note that $\tilde\omega_\epsilon^{j_1\cdots j_\ell}$ and $\omega_\epsilon^{j_1\cdots j_\ell}$ are subset of ω_ϵ, and $\omega_\epsilon^{1\cdots n}=\{\sum_{j=1}^{2n}t_je_j\in\omega_\epsilon\colon t\in[0,1)^n\times(-\epsilon,\epsilon)^n\}$. Then

$$\Omega_\epsilon^{j_1\cdots j_\ell}:=\Omega_\epsilon\cap\bigcap_{k=1}^{\ell}T_{j_k}^{-1}\Omega_\epsilon=\{(e^{2\pi i\zeta_1},\dots e^{2\pi i\zeta_n})\colon\zeta\in\omega_\epsilon^{j_1\cdots j_\ell}\},\tag{4.3}$$

$$\tilde\Omega_\epsilon^{j_1\cdots j_\ell}:=\Omega_\epsilon\cap\bigcap_{k=1}^{\ell}T_{j_k}\Omega_\epsilon=\{(e^{2\pi i\zeta_1},\dots e^{2\pi i\zeta_n})\colon\zeta\in\tilde\omega^{j_1\cdots j_\ell}\}\tag{4.4}$$

are connected non-empty Reinhardt domains. Moreover, $\Omega_0^{1\cdots n}:=\cap_{\epsilon>0}\Omega_0^{1\cdots n}$ and $\tilde\Omega_0^{1\cdots n}=\cap_{\epsilon>0}\Omega_\epsilon^{1\cdots n}$ are diffeomorphic to the real torus $(S^1)^n$. We remark that T_iT_j maps Ω_ϵ^{ij} into Ω_ϵ for $i\neq j$, while $T_i\circ T_i$ does not map $\Omega_{\epsilon,r}^{ii}$ into Ω_ϵ .

With $\Delta_r = \{z \in \mathbb{C} \colon |z| < r\}$, we also define

$$\omega_{\epsilon,r} := \omega_\epsilon \times \Delta_r^d, \quad \Omega_{\epsilon,r} := \Omega_\epsilon \times \Delta_r^d. \tag{4.5}$$

Throughout the paper, a mapping $(z', v') = \psi^0(z, v)$ from $\omega_{\epsilon,r}$ into $\mathbb{C}^{n+d}$ that commutes with $z_j \to z_j + 1$ for $j = 1, \ldots, n$ will be identified with a well-defined mapping $(h', v') = \psi(h, v)$ from $\Omega_{\epsilon,r}$ into $\mathbb{C}^{n+d}$, where z, h and z', h' are related as in (4.8). A function on $\omega_{\epsilon,r}$ that has period 1 in all z_j is identified with a function on $\Omega_{\epsilon,r}$. We shall use these identifications as we wish.

Proposition 4.3. *Let C be the complex torus and $\pi_{\widetilde{C}} \colon \widetilde{C} = \mathbb{C}^n/\mathbb{Z}^n \to C$ be the covering. Let (M, C) be a neighborhood of C. Assume that N_C is flat.*

(*i*) *Then one can take $\omega_{\epsilon_0,r_0} = \omega_{\epsilon_0} \times \Delta^d_{r_0}$ such that (M, C) is biholomorphic to the quotient of ω_{ϵ_0,r_0} by $\tau_1^0, \ldots, \tau_n^0$. Let τ_j be the mapping defined on Ω_{ϵ_0,r_0} corresponding to τ_j^0. Then $\tau_1, \ldots, \tau_n$ commute pairwise wherever they are defined, i.e.*

$$\tau_i \tau_j(h, v) = \tau_j \tau_i(h, v) \quad \forall i \neq j$$

for $(h, v) \in \Omega_{\epsilon_0,r_0} \cap \tau_i^{-1}\Omega_{\epsilon_0,r_0} \cap \tau_j^{-1}\Omega_{\epsilon_0,r_0}$.

(*ii*) *Let $(\tilde{M}, C)$ be another such neighborhood having the corresponding generators $\tilde{\tau}_1, \ldots, \tilde{\tau}_n$ of deck transformations defined on $\Omega_{\tilde{\epsilon}_0,\tilde{r}_0}$. Then (M, C) and $(\tilde{M}, C)$ are holomorphically equivalent if and only if there is a biholomorphic mapping F from $\Omega_{\epsilon,r}$ into $\Omega_{\tilde{\epsilon},\tilde{r}}$ for some positive $\epsilon, r, \tilde{e}, \tilde{r}$ such that*

$$F\tilde{\tau}_j(h, v) = \tau_j F(h, v), \ \ j = 1, \ldots, n,$$

wherever both sides are defined, i.e. $(h, v) \in \Omega_{\tilde{\epsilon},\tilde{r}} \cap \tilde{\tau}_j^{-1}\Omega_{\epsilon,r} \cap \Omega_{\epsilon,r} \cap F^{-1}\Omega_{\epsilon,r}$.

Proof. We now apply Lemma 4.2, in which C^* is replaced by $\widetilde{C} = \mathbb{R}^n/\mathbb{Z}^n + i\mathbb{R}^n$ is a Stein manifold. Assume that N_C is flat. Then according to Proposition 3.6, $N_{\widetilde{C}}(\widetilde{N_C}) = N_{\widetilde{C}}(\tilde{M}) = \pi^*_{\widetilde{C}}(N_C)$ is the trivial vector bundle $\widetilde{C} \times \mathbb{C}^d$ with coordinates (h, v), while $\widetilde{C} \times \{0\}$ is defined by $v = 0$. □

Set

$$\mathcal{P}_\epsilon^+ := \left\{ \sum_{i=1}^n t_i \operatorname{Im} \tau_i \colon t \in (-\epsilon, 1+\epsilon)^n \right\},$$
$$\Omega_\epsilon^+ := \left\{ (e^{-2\pi R_1}, \ldots, e^{-2\pi R_n}) \colon R \in \mathcal{P}_\epsilon^+ \right\}.$$

Note that $\mathcal{P}_\epsilon^+, \mathcal{P}_0^+$ are n-dimensional parallelotopes, and Ω_ϵ^+ contains $(1, \ldots, 1)$, the image of $0 \in \mathcal{P}_\epsilon^+$, corresponding to the real torus $(S^1)^n$.

Since Ω_ϵ is Reinhardt, we have

$$\Omega_\epsilon \supset \Omega_\epsilon^+, \quad (\partial\Omega_\epsilon)^+ = \partial(\Omega_e^+), \quad (\partial\Omega_\epsilon)^+ := \{(|h_1|, \ldots, |h_n|) \colon h \in \partial\Omega_\epsilon\}. \tag{4.6}$$

We now apply the above general results to the case where C is a complex torus, $C^* = \widetilde{C}$ and $N_C(M)$ is Hermitian flat.

As in [IP79], the deck transformations of $(\tilde{N}_C, \widetilde{C})$ are generated by n biholomorphisms $\hat{\tau}_1, \ldots, \hat{\tau}_n$ that preserve $\widetilde{C}$.

$$\hat{\tau}_j(h, v) = (T_j h, M_j v), \quad M_j := \operatorname{diag}(\mu_{j,1}, \ldots, \mu_{j,d}) \tag{4.7}$$

with h, T_j being defined by :

$$h = (e^{2\pi i z_1}, \cdots, e^{2\pi i z_n}), \quad T_j := \operatorname{diag}(\lambda_{j,1}, \ldots, \lambda_{j,n}), \quad \lambda_{j,k} := e^{2\pi i \tau_{jk}}. \tag{4.8}$$

Here the invertible $(d\times d)$-matrix M_j is the factor of automorphy $\rho(e_{n+j})$ of $\pi_{\tilde C}^* N_C$.

Recall that each deck transformation $\tau_j^0(z,v)$, $j=1,\ldots,n$, is a holomorphic map defined on $\overline{\omega}_{\epsilon,r}$. In coordinates (h,v), τ_j^0 becomes τ_j defined on $\overline{\Omega}_{\epsilon,r}$. Since T_CM splits, it is a higher-order perturbation of $\hat\tau_j$ with

$$\tau_j(h,0)=(T_jh,0)$$

The above computation is based on the assumption that N_C is flat. We now assume that N_C is *Hermitian and flat*, i.e. the N_C admits locally constant Hermitian transition matrices. Then all $\rho(e_{n+j})$ are Hermitian and constant matrices. Since they pairwise commute, they are simultaneously diagonalizable. Hence, we can assume that $M_j=\operatorname{diag}(\mu_{j,1},\ldots,\mu_{j,d})$. We recall from (4.8) that $T_j=\operatorname{diag}(\lambda_{j,1},\ldots,\lambda_{j,n})$ are already diagonal.

Definition 4.4. *The normal bundle N_C is said to be non-resonant if, for each $(Q,P)\in\mathbb{N}^d\times\mathbb{Z}^n$ with $|Q|>1$, each $i=1,\ldots,n$, and each $j=1,\ldots,d$, there exist $i_h:=i_h(Q,P,i)$ and $i_v:=i_v(Q,P,j)$ that are in $\{1,\ldots,n\}$ such that*

$$\lambda_{i_h}^P\mu_{i_h}^Q-\lambda_{i_h,i}\neq 0\quad\text{and}\quad \lambda_{i_v}^P\mu_{i_v}^Q-\mu_{i_v,j}\neq 0.$$

Definition 4.5. *The pullback normal bundle N_C, or N_C, is said to be Diophantine if for all $(Q,P)\in\mathbb{N}^d\times\mathbb{Z}^n$, $|Q|>1$ and all $i=1,\ldots,n$, and $j=1,\ldots,d$,*

$$\max_{\ell\in\{1,\ldots,n\}}\left|\lambda_\ell^P\mu_\ell^Q-\lambda_{\ell,i}\right|>\frac{D}{(|P|+|Q|)^\tau},\tag{4.9}$$

$$\max_{\ell\in\{1,\ldots,n\}}\left|\lambda_\ell^P\mu_\ell^Q-\mu_{\ell,j}\right|>\frac{D}{(|P|+|Q|)^\tau}.\tag{4.10}$$

We shall choose i_h (resp. i_v) to be the index that realizes the maximum of (4.9) *(resp.* (4.10)*).*

Remark 4.6. *If the right-hand sides are replaced by* 0*, then N_C is non-resonant.*

Proposition 4.7. *The properties of being non-resonant and Diophantine is a property of the (abelian) group and not the choice of the generators.*

Proof. Recall that

$$\lambda_\ell=(\lambda_{\ell,1},\ldots,\lambda_{\ell,n})=(e^{2\pi i\tau_{\ell,1}},\ldots,e^{2\pi i\tau_{\ell,n}}).$$

Let G be the group generated by the $\hat\tau_\ell$'s. Then, $\{\tilde\tau_\ell\}_\ell$ defines another set of generators of G if $\tilde\tau_\ell=\hat\tau_1^{a_{\ell,1}}\cdots\hat\tau_n^{a_{\ell,n}}$, $\ell=1,\ldots,n$ where $A=(a_{i,j})_{1\le i,j\le n}\in GL_n(\mathbb{Z})$ with $\det A=\pm1$. Then, the eigenvalues of $\tilde\tau_\ell$ are

$$\tilde\lambda_{\ell,i}=\prod_{k=1}^n\lambda_{k,i}^{a_{\ell,k}},\quad \tilde\mu_{\ell,j}=\prod_{k=1}^d\mu_{k,j}^{a_{\ell,k}}.$$

Hence, we have

$$\tilde\lambda_\ell^P\tilde\mu_\ell^Q\tilde\lambda_{\ell,i}^{-1}=(\lambda_1^P\mu_1^Q\lambda_{1,i}^{-1})^{a_{\ell,1}}\cdots(\lambda_n^P\mu_n^Q\lambda_{n,i}^{-1})^{a_{\ell,n}}.$$

Fix P,Q and i. Taking the logarithm, we have as n-vectors

$$\left(\ln\tilde\lambda_\ell^P\tilde\mu_\ell^Q\tilde\lambda_{\ell,i}^{-1}\right)_{\ell=1,\ldots,n}=A\left(\ln\lambda_\ell^P\mu_\ell^Q\lambda_{\ell,i}^{-1}\right)_{\ell=1,\ldots,n},\quad \mod 2\pi i.\tag{4.11}$$

Since $A,A^{-1}\in GL_n(\mathbb{Z})$, given P,Q and i,

$$\left(\ln\tilde\lambda_\ell^P\tilde\mu_\ell^Q\tilde\lambda_{\ell,i}^{-1}\right)_{\ell=1,\ldots,n}=0\mod 2\pi i$$

iff $\left(\ln \lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1}\right)_{\ell=1,\ldots,n} = 0 \mod 2\pi i$. Similarly, by considering $\ln \tilde\lambda_\ell^P \tilde\mu_\ell^Q \tilde\mu_{\ell,i}^{-1}$, we obtain that the non-resonant condition does not depend on the choice of generators. Given P, Q, i, if one of the $\lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1}$'s is not close to 1, then $\left\|(\ln \tilde\lambda_\ell^P \tilde\mu_\ell^Q \tilde\lambda_{\ell,i}^{-1})_\ell\right\|$ is bounded way from 0. On the other hand, if all $\lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1}$'s are close to 1, then $|\ln \lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1}|$ (with $\operatorname{Im}\ln$ in $(-\pi,\pi]$) is comparable to $|\lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1} - 1|$. Furthermore, taking the module of (4.11), we obtain

$$\|A^{-1}\|^{-1}\left\|(\ln \lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1})_\ell\right\| \leq \left\|(\ln \tilde\lambda_\ell^P \tilde\mu_\ell^Q \tilde\lambda_{\ell,i}^{-1})_\ell\right\| \leq \|A\|\left\|(\ln \lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1})_\ell\right\|,$$

where $\|(a_\ell)_\ell\| = \max_\ell |a_\ell|$. If the latter is bounded below by $\frac{C}{(|P|+|Q|)^\tau}$, so is $\left\|(\ln \tilde\lambda_\ell^P \tilde\mu_\ell^Q \tilde\lambda_{\ell,i}^{-1})_\ell\right\|$. □

Theorem 4.8. *Let C_n be an n-dimensional complex torus, holomorphically embedded into a complex manifold M_{n+d}. Assume that T_CM splits. Assume the normal bundle N_C has (locally constant) Hermitian transition functions. Assume that N_C is Diophantine. Then some neighborhood of C is biholomorphic to a neighborhood of the zero section in the normal bundle.*

Remark 4.9. *When C is a product of 1-dimensional tori with normal bundle which is a direct sum of line bundles, the above result is due to Il'yashenko-Pyartli* [IP79].

We have $\tau_j(h,v) = \hat\tau_j(h,v) + (\tau_j^h(h,v), \tau_j^v(h,v))$. Here, functions

$$\tau_j^\bullet(h,v) = \sum_{Q\in\mathbb{N}^d, |Q|\geq 2} \tau_{j,Q}^\bullet(h)v^Q$$

are holomorphic in (h,v) in a neighborhood of $\Omega_{\epsilon,r}$ with values in $\mathbb{C}^{n+d}$.

Definition 4.10. *Set $\Omega_{\epsilon,r} := \Omega_\epsilon \times \Delta_r^d$, $\tilde\Omega_{\epsilon,r} := \overline{\Omega}_{\epsilon,r} \cup \bigcup_{i=1}^n \hat\tau_i(\overline{\Omega}_{\epsilon,r})$. Denote by $\mathcal{A}_{\epsilon,r}$ (resp. $\tilde{\mathcal{A}}_{\epsilon,r}$) the set of holomorphic functions on $\overline{\Omega_{\epsilon,r}}$ (resp. $\overline{\tilde\Omega}_{\epsilon,r}$). If $f \in \mathcal{A}_{\epsilon,r}$ (resp. $\tilde f \in \tilde{\mathcal{A}}_{\epsilon,r}$), we set*

$$\|f\|_{\epsilon,r} := \sup_{(h,v)\in\Omega_{\epsilon,r}} |f(h,v)|, \quad |||\tilde f|||_{\epsilon,r} := \sup_{(h,v)\in\tilde\Omega_{\epsilon,r}} |\tilde f(h,v)|.$$

As such, each $f \in \mathcal{A}_{\epsilon,r}$ can be expressed as a convergent Taylor-Laurent series

$$f(h,v) = \sum_{P\in\mathbb{Z}^n} f_{Q,P} h^P v^Q$$

for $(h,v) \in \Omega_{\epsilon,r} = \Omega_\epsilon \times \Delta_r^d$. Recall that each holomorphic function on E_j in Lemma 4.11 admits a unique Taylor-Laurent series expansion on $\tilde\Omega_{\epsilon'}^{1\cdots n} \times \Delta_{\epsilon'}^d$ when $\epsilon' > 0$ is sufficiently small.

We can state the following, for later use :

Lemma 4.11. *For $i \neq j$, the set $\hat\tau_j\overline{\Omega}_{\epsilon,r} \cap \hat\tau_i(\overline{\Omega}_{\epsilon,r})$ is a connected Reinhardt domain containing $\tilde\Omega_\epsilon^{1\ldots n} \times \Delta_{r'}^d$ in $\mathbb{C}^{n+d}$ when $r' > 0$ is sufficiently small.*

4.1. **Holomorphic functions on $\Omega_{\epsilon,r}$.** In this section, we study elementary properties and estimate holomorphic functions f on $\overline{\Omega}_{\epsilon,r}$.

Lemma 4.12. *An element $f(h)$ of $\mathcal{A}_\epsilon$, that is a holomorphic function in a neighborhood of $\overline{\Omega}_\epsilon$, admits a Laurent series expansion in h*

$$f(h) = \sum_{P\in\mathbb{Z}^n} c_P h^P. \tag{4.12}$$

The series converges normally on Ω_ϵ. Moreover, the Laurent coefficients

$$c_P = \frac{1}{(2\pi i)^n}\int_{|\zeta_1|=s_1,\dots,|\zeta_n|=s_n} f(\zeta)\zeta^{-P-(1,\dots,1)}\,d\zeta_1\wedge\cdots\wedge d\zeta_n \tag{4.13}$$

are independent of $s\in\Omega_\epsilon^+$ and

$$|c_P| \le \sup_{\Omega_\epsilon}|f| \inf_{s\in\Omega_\epsilon^+} s^{-P}. \tag{4.14}$$

Proof. Obviously, estimate (4.14) follows from (4.13). Define $A(a,b)=\{w\in\mathbb{C}: a<|w|<b\}$. Fix $h\in\Omega_\epsilon$. Then $|h|:=(|h_1|,\dots,|h_n|)\in\Omega_\epsilon^+$. The latter is an open set and we have for a small positive number $\epsilon=\epsilon(h)$,

$$f(h)=\frac{1}{(2\pi i)^n}\int_{\partial A(|h_1|-\epsilon,|h_1|+\epsilon)}\cdots\int_{\partial A(|h_n|-\epsilon,|h_n|+\epsilon)}\frac{f(\zeta)\,d\zeta_1\dots d\zeta_n}{(\zeta_1-h_1)\cdots(\zeta_n-h_n)}.$$

By Laurent expansion in one-variable, we get the expansion (4.12) in which c_P are given by (4.13) if we take $r_j=|h_j|\pm\epsilon$ according to the sign of $p_j\in\mathbb{Z}^{\mp}$.

We want to show that c_P is independent of $s\in\Omega_\epsilon^+$. Note that Ω_ϵ^+ is a connected open set. For any two points $s,\tilde{s}$ can be connected by a union of line segments in Ω_ϵ^+ which are parallel to coordinate axes in $\mathbb{R}^n$. Using such line segments, say a line segment $[a,b]\times(s_2,\dots,s_n)$ in Ω_ϵ^+, we know that $f(\zeta)\zeta^{-P-(1,\dots,1)}$ is holomorphic in ζ_1 in the closure of $A(a,b)$ when $|\zeta_2|=s_2,\dots,|\zeta_n|=s_n$. By Cauchy theorem, the integrals are independent of $s_1\in[a,b]$ for these $s_2,\dots,s_n$. This shows that (4.13) is independent of s.

Finally the series converges uniformly on each compact subset of Ω_ϵ. Indeed, for a small perturbation of h, we can choose $\epsilon(h)$ to be independent of h. Then we can see easily that the series converges locally uniform in h. □

Recall that

$$\mathcal{P}_\epsilon^+ = \left\{\sum_{i=1}^n t_i \operatorname{Im}\tau_i\in\mathbb{R}^n : t\in(-\epsilon,1+\epsilon)^n\right\},$$
$$\Omega_\epsilon^+ = \left\{(e^{-2\pi R_1},\dots,e^{-2\pi R_n}) : R\in\mathcal{P}_\epsilon^+\right\}.$$

Lemma 4.13. *There is a constant $\kappa_0>0$ that depends only on $\operatorname{Im}\tau_1,\dots,\operatorname{Im}\tau_n$ such that if $P\in\mathbb{R}^n$ and $\epsilon>\epsilon'$, there exists $R\in\mathcal{P}_\epsilon^+$ such that for all $R'\in\mathcal{P}_{\epsilon'}^+$ we have*

$$(R'-R)\cdot P\le -\kappa_0(\epsilon-\epsilon')|P|. \tag{4.15}$$

Proof. Without loss of generality, we may assume that P is a unit vector. Let $\pi(x)P$ be the orthogonal projection from $x\in\mathbb{R}^n$ onto the line spanned by P. Choose $R\in\overline{\mathcal{P}}_\epsilon^+$ so that $\pi(R)$ has the largest value for $R\in\overline{\mathcal{P}_\epsilon^+}$. Note that R must be on the boundary of $\mathcal{P}_\epsilon^+$ and latter is contained in the half-space H defined by $\pi(y)\le\pi(R)$ for $y\in\mathbb{R}^n$. Hence, ∂H is orthogonal to P. Then for any $R'\in\mathcal{P}_{\epsilon'}^+$,

$$\pi(R)-\pi(R')=\operatorname{dist}(R',\partial H)\ge\operatorname{dist}(\partial\mathcal{P}_\epsilon^+,\mathcal{P}_{\epsilon'}^+)\ge(\epsilon-\epsilon')/C.$$

Therefore, we obtain $(R'-R)\cdot P=-(\pi(R)-\pi(R'))\le-(\epsilon-\epsilon')/C$. □

Remark 4.14. *Since $\mathcal{P}_\epsilon^+$ is a parallelotope, we can choose R to be a vertex of $\mathcal{P}_\epsilon^+$.*

In what follows, we denote the fixed constant :

$$\kappa := 2\pi\kappa_0. \tag{4.16}$$

Lemma 4.15 (Cauchy estimates)**.** *If $f \in \mathcal{A}_{\epsilon,r}$, then for all $(P,Q) \in \mathbb{Z}^n \times \mathbb{N}^d$,*

$$|f_{Q,P}| \le \frac{M}{r^{|Q|}\max_{s\in\Omega_\epsilon^+} s^P}. \tag{4.17}$$

Furthermore, if $f = \sum_{Q\in\mathbf{N}^d, P\in\mathbb{Z}^n} f_{QP}h^Pv^Q \in \mathcal{A}_{\epsilon,r}$ and $0<\delta<\kappa\epsilon$, then

$$\|f\|_{\epsilon-\delta/\kappa, re^{-\delta}} \le \frac{C\sup_{\Omega_{\epsilon,r}}|f|}{\delta^\nu},$$

where C and ν depends only on n and d.

Proof. According to Lemma 4.12 and Cauchy estimates on polydiscs, we have for any $s \in \Omega_\epsilon^+$,

$$|f_{Q,P}| \le \frac{\sup_{\Omega_\epsilon}|f_Q(h)|}{s^P} \le \frac{\sup_{\Omega_{\epsilon,r}}|f|}{s^Pr^{|Q|}}.$$

According to Lemma 4.12 and Cauchy estimates for polydiscs, we have if $(h,v) \in \Omega_{\epsilon e^{-\delta'}, re^{-\delta}}$, then for all $s \in \Omega_\epsilon^+$,

$$|f_Q(h)| \le \frac{\sup_{v\in\Delta_r^d}|f(h,v)|}{r^{|Q|}},$$
$$|f_{Q,P}h^P| \le \left|\frac{1}{(2\pi i)^n}\int_{|\zeta_1|=s_1,\dots,|\zeta_n|=s_n} f_Q(\zeta)\frac{h^P}{\zeta^P}\frac{d\zeta_1\wedge\cdots\wedge d\zeta_n}{\zeta_1\cdots\zeta_n}\right|.$$

Set $s_j = e^{-2\pi R_j}$, $|h_j| = e^{-2\pi R_j'}$, $R=(R_1,\dots,R_n)$ and $R'=(R_1',\dots,R_n')$. By Lemma 4.13,

$$\inf_{(|\zeta_1|,\dots,|\zeta_n|)=s\in\Omega_\epsilon^+}\ \sup_{h\in\Omega_{\epsilon-\delta'}}\left|\frac{h^P}{\zeta^P}\right| = \inf_{R\in\mathcal{P}_\epsilon^+}\ \sup_{R'\in\mathcal{P}^+_{\epsilon-\delta'}} e^{-2\pi<R-R',P>} \le e^{-\kappa\delta'|P|}, \tag{4.18}$$

where the positive constant κ, defined by Lemma 4.13 and (4.16), is independent of P,ζ_1,ζ_2. Thus

$$|f_{Q,P}h^Pv^Q| \le \frac{\sup_{\Omega_{\epsilon,r}}|f|e^{-\kappa\delta'|P|}r^{|Q|}e^{-\delta|Q|}}{r^{|Q|}} \le \sup_{\Omega_{\epsilon,r}}|f|e^{-\delta'\kappa|P|}e^{-\delta|Q|}. \tag{4.19}$$

Hence, setting $\delta' := \delta/\kappa$, we have

$$\|f\|_{\epsilon-\delta/\kappa, re^{-\delta}} \le \sum_{Q\in\mathbf{N}^d, P\in\mathbb{Z}^n}|f_{Q,P}h^Pv^Q| \le \frac{C\sup_{\Omega_{\epsilon,r}}|f|}{\delta^\nu},$$

where C and ν depends only on n and d. □

4.2. **Conjugacy of the deck transformations.** Let us show that there is a biholomorphism $\Phi = (h, v) + \phi(h, v)$ of some neighborhood $\Omega_{\tilde{\epsilon},\tilde{r}}$, fixing $\widetilde{C}$ pointwise (i.e. $\Phi(h, 0) = (h, 0)$) such that

$$\Phi \circ \hat{\tau}_i = \tau_i \circ \Phi, \quad i = 1, \dots, n.$$

This reads $\hat{\tau}_i + \phi(\hat{\tau}_i) = \hat{\tau}_i(Id + \phi) + \tau_i^{\bullet}(Id + \phi)$, that is, for all $i = 1, \dots n$,

$$\mathcal{L}_i(\phi) = \tau_i^{\bullet}(Id + \phi) + (\hat{\tau}_i(Id + \phi) - \hat{\tau}_i - D\hat{\tau}_i.\phi) = \tau_i^{\bullet}(Id + \phi). \tag{4.20}$$

Here we define $\mathcal{L}_i(\phi) := \phi(\hat{\tau}_i) - D\hat{\tau}_i.\phi$ and $(\mathcal{L}_i^h(\phi^h), \mathcal{L}_i^v(\phi^v)) := \mathcal{L}_i(\phi)$. We have

$$(\mathcal{L}_i^h(\phi^h), \mathcal{L}_i^v(\phi^v)) = \left(\phi^h(T_i h, M_i v) - T_i\phi^h(h, v), \phi^v(T_i h, M_i v) - M_i\phi^v(h, v)\right).$$

Since $\hat{\tau}_j$ are linear, then $D\hat{\tau}_j = \hat{\tau}_j$. Expand the latter in Taylor-Laurent expansions as

$$\begin{aligned}
\mathcal{L}_i^h(\phi^h) &= \sum_{Q\in\mathbb{N}_2^d}\left(\sum_{P\in\mathbb{Z}^n}\left(\lambda_i^P\mu_i^Q \times Id_n - T_i\right)\phi_{Q,P}^h h^P\right)v^Q, \quad \phi_{Q,P}^h \in \mathbb{C}^n,\\
\mathcal{L}_i^v(\phi^v) &= \sum_{Q\in\mathbb{N}_2^d}\left(\sum_{P\in\mathbb{Z}^n}\left(\lambda_i^P\mu_i^Q \times Id_d - M_i\right)\phi_{Q,P}^v h^P\right)v^Q, \quad \phi_{Q,P}^v \in \mathbb{C}^d.
\end{aligned}$$

Recall the notations $\lambda_\ell = (\lambda_{\ell,1}, \dots, \lambda_{\ell,n})$ and $\mu_\ell = (\mu_{\ell,1}, \dots, \mu_{\ell,d})$. With $P = (p_1, \dots, p_n) \in \mathbb{Z}^n$ and $Q = (q_1, \dots, q_d) \in \mathbb{N}^d$, we have

$$\lambda_\ell^P\mu_\ell^Q := \prod_{i=1}^n \lambda_{\ell,i}^{p_i}\prod_{j=1}^n \mu_{\ell,j}^{q_j}.$$

Lemma 4.16. *Let $\tau_j \in \mathcal{A}_{\epsilon,r}^{n+d}$ with $\tau_j = \hat{\tau}_j - F_j$ and $F_j(h, v) = O(|v|^{q+1})$ with $q \geq 1$. Suppose that $\tau_i\tau_j = \tau_j\tau_i$ in a neighborhood of $\Omega_0^{1\cdots n} \times \{0\}$ in $\mathbb{C}^{n+d}$. Then $\mathcal{L}_iF_j - \mathcal{L}_iF_i = O(|v|^{2q+1})$.*

Proof. Recall that $\hat{\tau}_i(h, v)$ are linear maps in h, v. Also, $\hat{\tau}_i\hat{\tau}_j$ sends $\Omega_\epsilon^j \times \Delta_\epsilon^d$ into $\mathbb{C}^{n+d}$. Since $\tau_j \in \mathcal{A}_{\epsilon,r}^{n+d}$ and $\tau_j = \hat{\tau}_j + O(|v|^2)$, the continuity implies that $\tau_i\tau_j$ is well-defined on the product domain $\Omega_{\epsilon'}^j \times \Delta_{r'}^d$ when ϵ', r' are sufficiently small. Fix $h \in \Omega_{\epsilon'}^j$. By Taylor expansions in v, we obtain

$$\tau_i\tau_j(h, v) = \hat{\tau}_i\hat{\tau}_j(h, v) + \hat{\tau}_jF_j(h, v) + F_i \circ \hat{\tau}_j(h, v) + O(|v|^{2q+1}).$$

Since $\hat{\tau}_i\hat{\tau}_j = \hat{\tau}_j\hat{\tau}_i$ and $\tau_i\tau_j = \tau_j\tau_i$ in a neighborhood of $\Omega_0^{1\cdots n} \times \{0\}$, we get $\mathcal{L}_iF_j = \mathcal{L}_jF_i = O(|v|^{2q+1})$ in a possibly smaller neighborhood of $\Omega_0^{1\cdots n} \times \{0\}$. $\square$

We will apply the following result to $F_j = \hat{\tau}_j - J^{2q}(\tau_j)$, where $J^{2q}(\tau_j)$ denotes the $2q$-jet at 0 in the variable v. These are holomorphic on $\Omega_{\epsilon,r}$. Recall that

$$\Omega_{\epsilon',r'}^{ij} := \Omega_{\epsilon',r'} \cap \hat{\tau}_i^{-1}(\Omega_{\epsilon',r'}) \cap \hat{\tau}_j^{-1}(\Omega_{\epsilon',r'})$$

Proposition 4.17. *Assume N_C is Diophantine. Fix $\epsilon_0, r_0, \delta_0$ in $(0, 1)$. Let $0 < \epsilon' < \epsilon < \epsilon_0$, $0 < r' < r < r_0$, $0 < \delta < \delta_0$, and $\frac{\delta}{\kappa} < \epsilon$. Suppose that $F_i \in \mathcal{A}_{\epsilon,r}$, $i = 1, \dots, n$, satisfy*

$$\mathcal{L}_i(F_j) - \mathcal{L}_j(F_i) = 0 \quad \text{on } \Omega_{\epsilon',r'}^{ij}. \tag{4.21}$$

There exist functions $G \in \tilde{\mathcal{A}}_{\epsilon-\delta/\kappa,re^{-\delta}}$ such that

$$\mathcal{L}_i(G) = F_i \quad \text{on } \Omega_{\epsilon-\delta/\kappa,re^{-\delta}}. \tag{4.22}$$

Furthermore, G satisfies

$$\|G\|_{\epsilon-\delta/\kappa, re^{-\delta}} \leq \max_i \|F_i\|_{\epsilon,r}\frac{C'}{\delta^{\tau+\nu}}, \tag{4.23}$$

$$\|G\circ\hat{\tau}_i\|_{\epsilon-\delta/\kappa, re^{-\delta}} \leq \max_i \|F_i\|_{\epsilon,r}\frac{C'}{\delta^{\tau+\nu}}. \tag{4.24}$$

for some constant κ, C' that are independent of $F, q, \delta, r, \epsilon$ and ν that depends only on n and d. Furthermore, if $F_j(h,v) = O(|v|^{q+1})$ for all j, then

$$G(h,v) = O(|v|^{q+1}), \quad G(h,v) = J^{2q}G(h,v). \tag{4.25}$$

Proof. Since $F_i \in \mathcal{A}_{\epsilon,r}$, we can write

$$F_i(h,v) = \sum_{Q\in\mathbb{N}^d, |Q|\geq 2}\sum_{P\in\mathbb{Z}^n} F_{i,Q,P}h^Pv^Q,$$

which converges normally for $(h,v)\in\Omega_{\epsilon,r}$. We emphasize that $F_{i,Q,P}$ are vectors, and its kth component is denoted by $F_{i,k,Q,P}$. For each $(Q,P)\in\mathbb{N}^d\times\mathbb{Z}^n$, each $i = 1,\ldots,n$, and each $j = 1,\ldots,d$, let $i_h := i_h(Q,P,i), i_v := i_v(Q,P,j)$ be in $\{1,\ldots,n\}$ as in Definition 4.4. Let us set

$$G_i^h := \sum_{Q\in\mathbb{N}^d, 2\leq|Q|\leq 2q}\sum_{P\in\mathbb{Z}} \frac{F^h_{i_h,i,Q,P}}{\lambda^P_{i_h}\mu^Q_{i_h} - \lambda_{i_h,i}}h^Pv^Q, \quad i = 1,\ldots,n \tag{4.26}$$

$$G_j^v := \sum_{Q\in\mathbb{N}^d, 2\leq|Q|\leq 2q}\sum_{P\in\mathbb{Z}} \frac{F^v_{i_v,j,Q,P}}{\lambda^P_{i_v}\lambda^Q_{i_v} - \mu_{i_v,j}}h^Pv^Q, \quad j = 1,\ldots.d. \tag{4.27}$$

According to (4.21), we have

$$(\lambda^P_{i_h}\mu^Q_{i_h} - \lambda_{i_h,i})F^h_{m,i,Q,P} = (\lambda^P_m\mu^Q_m - \lambda_{m,i})F^h_{i_h,i,Q,P}, \quad 2\leq|Q|\leq 2q. \tag{4.28}$$

Therefore, using (4.28), the ith-component of $\mathcal{L}_m(G)$ reads

$$\begin{aligned}\mathcal{L}_m(G_i^h) &= \sum_{Q\in\mathbb{N}^d, 2\leq|Q|\leq 2q}\sum_{P\in\mathbb{Z}^n}(\lambda^P_m\mu^Q_m - \lambda_{m,i})\frac{F^h_{i_h,i,Q,P}}{(\lambda^P_{i_h}\mu^Q_{i_h} - \lambda_{i_h,i})}h^Pv^Q\\ &= \sum_{Q\in\mathbb{N}^d, 2\leq|Q|\leq 2q}\sum_{P\in\mathbb{Z}^n} F^h_{m,i,Q,P}h^Pv^Q.\end{aligned}$$

Proceeding similarly for the vertical component, we have obtained, the formal equality :

$$\mathcal{L}_m(G) = F_m, \quad m = 1,\ldots,n. \tag{4.29}$$

Let us estimate these solutions. According to Definition 4.5 and formulas (4.26)-(4.27), we have

$$\max_{i,j}\left(|G^h_{i,Q,P}|, |G^v_{j,Q,P}|\right) \leq \max_i |F_{i,Q,P}|\frac{(|P|+|Q|)^\tau}{D}. \tag{4.30}$$

Let $(h,v)\in\Omega_{\epsilon-\delta/\kappa, re^{-\delta}}$. According to (4.19), we have

$$\begin{aligned}\|G^h_{Q,P}h^Pv^Q\| &\leq \max_i \|F_i\|_{\epsilon,r}e^{-\delta(|P|+|Q|)}\frac{(|P|+|Q|)^\tau}{D}\\ &\leq \max_i \|F_i\|_{\epsilon,r}e^{-\delta/2(|P|+|Q|)}\frac{(2\tau e)^\tau}{D\delta^\tau}.\end{aligned}$$

Summing over P and Q, we obtain

$$\|G\|_{\epsilon-\delta/\kappa,re^{-\delta}} \leq \max_i \|F_i\|_{\epsilon,r}\frac{C'}{\delta^{\tau+\nu}},$$

for some constants C',ν that are independent of F,ϵ,δ. Hence, $G\in\mathcal{A}_{\epsilon-\delta/\kappa,re^{-\delta}}$.

Let us prove (4.24). Let $B:=2\max_{k,i,j}(|\lambda_{k,i}|,|\mu_{k,j}|)$. Then, there is a constant D' such that

$$\max_{\ell\in\{1,\ldots,n\}}\left|\lambda_\ell^P\mu_\ell^Q-\lambda_{\ell,i}\right|\geq\frac{D'\max_k|\lambda_k^P\mu_k^Q|}{(|P|+[Q|)^\tau},\tag{4.31}$$

$$\max_{\ell\in\{1,\ldots,n\}}\left|\lambda_\ell^P\mu_\ell^Q-\mu_{\ell,j}\right|\geq\frac{D'\max_k|\lambda_k^P\mu_k^Q|}{(|P|+[Q|)^\tau}.\tag{4.32}$$

Indeed, if $\max_k|\lambda_k^P\mu_k^Q|<B$, then Definition 4.5 gives (4.31) with $D':=\frac{D}{B}$. Otherwise, if

$$|\lambda_{k_0}^P\mu_{k_0}^Q|:=\max_k|\lambda_k^P\mu_k^Q|\geq B,$$

then $|\lambda_{k_0,i}|\leq\frac{B}{2}\leq\frac{|\lambda_{k_0}^P\mu_{k_0}^Q|}{2}$. Hence, we have

$$\left|\lambda_{k_0}^P\mu_{k_0}^Q-\lambda_{k_0,i}\right|\geq\left||\lambda_{k_0}^P\mu_{k_0}^Q|-|\lambda_{k_0,i}|\right|\geq\frac{|\lambda_{k_0}^P\mu_{k_0}^Q|}{2}.$$

We have verified (4.31). Similarly, we can verified (4.32). Finally, combining all cases gives us, for $m=1,\ldots,n$,

$$\begin{aligned}|[G\circ\hat\tau_m]_{QP}|=\left|G_{Q,P}\lambda_m^P\mu_m^Q\right|&\leq\max_\ell|F_{\ell,Q,P}|\frac{|\lambda_m^P\mu_m^Q|}{|\lambda_{i_h}^P\mu_{i_h}^Q-\lambda_{i_h,i}|}\\&\leq\max_\ell|F_{\ell,Q,P}|\frac{|\lambda_m^P\mu_m^Q|(|P|+|Q|)^\tau}{D'\max_k|\lambda_k^P\mu_k^Q|}\\&\leq\max_\ell|F_{\ell,Q,P}|\frac{(|P|+|Q|)^\tau}{D'}.\end{aligned}$$

Hence, $\tilde G_m:=G\circ\hat\tau_m\in\mathcal{A}_{\epsilon-\delta/\kappa,re^{-\delta}}$. We can define $\tilde G\in\tilde{\mathcal{A}}_{\epsilon-\delta/\kappa,re^{-\delta}}$ such that $\tilde G=\tilde G_m\hat\tau_m^{-1}$ on $\hat\tau_m\Omega_{\epsilon,r}$. We verify that $\tilde G$ extends to a single-valued holomorphic function of class $\tilde{\mathcal{A}}_{\epsilon,r}$. Indeed, $\tilde G_i\hat\tau_i^{-1}=\tilde G_j\hat\tau_j^{-1}$ on $\hat\tau_i\Omega_{\epsilon,r}\cap\hat\tau_j\Omega_{\epsilon,r}$, since the latter is connected by Lemma 4.11 and the two functions agree with G on $\hat\tau_i\Omega_{\epsilon,r}\cap\hat\tau_j\Omega_{\epsilon,r}\cap\Omega_{\epsilon,r}$ that contains a neighborhood of $\tilde\Omega_\epsilon\times\{0\}$ in $\mathbb{C}^{n+d}$. □

In what follows, we shall set $\hat\tau_0=Id$ and for any $f\in(\tilde A_{\epsilon,r})^{n+d}$ and $F\in(\tilde{\mathcal{A}}_{\epsilon,r})^{n+d}$, we set

$$|||f|||_{\epsilon,r}:=\|f\|_{\epsilon,r}+\sum_{i=1}^n\|\hat\tau_i^{-1}f\circ\hat\tau_i\|_{\epsilon,r}=\sum_{i=0}^n\|\hat\tau_i^{-1}f\circ\hat\tau_i\|_{\epsilon,r},$$

$$F_{(i)}:=\hat\tau_i^{-1}F\circ\hat\tau_i\in\mathcal{A}_{\epsilon,r},\quad F_{(0)}:=F,\quad i=1,\ldots,n.$$

4.3. **Iteration scheme.** We shall prove the main result through a Newton scheme. Let us define sequences of positive real numbers

$$\delta_k:=\frac{\delta_0}{(k+1)^2},\quad r_{k+1}:=r_ke^{-5\delta_k},\quad \epsilon_{k+1}:=\epsilon_k-\frac{5\delta_k}{\kappa}$$

such that

$$\sigma := \sum_{k\geq 0} \delta_k < 2\delta_0 \tag{4.33}$$

We assume the following conditions hold :

$$\delta_0 < \frac{\kappa}{20}\epsilon_0, \tag{4.34}$$

$$\delta_0 < \frac{\ln 2}{10}. \tag{4.35}$$

Condition (4.34) ensures that

$$\frac{5}{\kappa}\sigma < \frac{10}{\kappa}\delta_0 < \frac{\epsilon_0}{2}, \quad \text{so that} \quad \epsilon_k > \frac{\epsilon_0}{2},\ k \geq 0.$$

Condition (4.35) ensures that

$$e^{-5\sigma} > e^{-10\delta_0} > \frac{1}{2}, \quad \text{so that} \quad r_k > \frac{r_0}{2},\ k \geq 0.$$

Let $m = 5$ be fixed. We define $\epsilon_{k+1} < \epsilon_k^{(\ell)} < \epsilon_k$ and $r_{k+1} < r_k^{(\ell)} < r_k$, $\ell = 1, \ldots m$ as follows :

$$\epsilon_k^{(\ell)} = \epsilon_k - \frac{\ell\delta_k}{\kappa}, \quad \epsilon_{k+1} := \epsilon_k^{(m)},$$
$$r_k^{(\ell)} = r_k e^{-\ell\delta_k}, \quad r_{k+1} := r_k^{(m)}.$$

We emphasize that condition (4.34) ensures $\epsilon_k^{(\ell)} > 0$. Let us assume that for each $i = 1, \ldots, n$, $\tau_i^{(k)} = \hat{\tau}_i + \tau_i^{\bullet(k)}$, is holomorphic and on Ω_{ϵ_k, r_k} it satisfies

$$||\tau^{\bullet(k)}||_{\epsilon_k, r_k} < \delta_k^{\mu}, \quad \tau^{\bullet(k)}(h, v) = O(|v|^{q_k+1}). \tag{4.36}$$

We further assume that $\tau_i^{(k)}, \tau_j^{(k)}$ commute on $\Omega_{\epsilon', r'}^{ij} \subset \Omega_{\epsilon_k, r_k}$ for some positive $\epsilon' < \epsilon_k, r' < r_k$. We take

$$q_{k+1} = 2q_k + 1 \geq q_0 2^k$$

for $q_0 \geq 1$ to be determined.

We will define a sequence $\Phi^{(k)}$ with $\Phi^{(k)}(h, 0) = (h, 0)$. Let us write on appropriate domains

$$\Phi^{(k)} = Id + \phi^{(k)}, \quad (\Phi^{(k)})^{-1} := Id - \psi^{(k)},$$
$$\tau_i^{(k+1)} = \Phi^{(k)} \circ \tau_i^{(k)} \circ (\Phi^{(k)})^{-1}, \quad i = 1, \ldots, n.$$

Note that there is a constant $C > 0$ (depending only on the $\mu_{i,j}$'s) such that if $\epsilon'' < \epsilon'$, $Cr'' < r'$ then

$$\Omega_{\epsilon'', r''}^{ij} \subset \Omega_{\epsilon'} \times \Delta_{r'}^d, \quad \Omega_{\epsilon''} \times \Delta_{r''}^d \subset \Omega_{\epsilon', r'}^{ij}.$$

Since the $\tau_i^{(k)}, \tau_j^{(k)}$ commute on $\Omega_{\epsilon'} \times \Delta_{\epsilon'}^d$ for some $\epsilon' > 0$ and $\Phi^{(k)}(h, v) = (h, v) + O(|v|^2)$, then $\tau_i^{(k+1)}, \tau_j^{(k+1)}$ still commute on the same kind of domains for a possibly smaller ϵ'. By Lemma 4.16, we obtain :

$$\mathcal{L}_j(\tau_i^{\bullet(k)}) - \mathcal{L}_i(\tau_j^{\bullet(k)}) = O(|v|^{2q_k+1}). \tag{4.37}$$

We want to find $\phi^{(k)}$ so that

$$\tau_i^{(k+1)} := \hat{\tau}_i + \tau_i^{\bullet(k+1)} = \Phi^{(k)} \circ \tau_i^{(k)} \circ (\Phi^{(k)})^{-1} \tag{4.38}$$
$$= \hat{\tau}_i(Id - \psi^{(k)}) + \tau_i^{\bullet(k)}(Id - \psi^{(k)})$$
$$+ \phi^{(k)}\left(\hat{\tau}_i(Id - \psi^{(k)}) + \tau_i^{\bullet(k)}(Id - \psi^{(k)})\right)$$

is defined and bounded on $\Omega_{\epsilon_{k+1},r_{k+1}}$.

We now define $\Phi^{(k)}$ by applying Proposition 4.17 with these $F_i := -J^{2q_k}(\tau_i^{\bullet(k)})$. Let $\phi^{(k)}$ stands for G. Therefore, given $0 < \delta < \kappa\epsilon_k^{(1)}$, $\phi^{(k)}$ is holomorphic and bounded on $\tilde{\Omega}_{\epsilon_k - \frac{\delta}{\kappa}, r_k e^{-\delta}}$ and it satisfies on $\Omega_{\epsilon_k^{(1)} - \frac{\delta}{\kappa}, r_k^{(1)} e^{-\delta}}$,

$$\mathcal{L}_i(\phi^{(k)}) := \phi^{(k)}(\hat{\tau}_i) - D\hat{\tau}_i.\phi^{(k)} = -J^{2q_k}(\tau_i^{\bullet(k)}), \quad i = 1, \ldots, n. \tag{4.39}$$

Writing formally $\Phi^{(k)} \circ \Psi^{(k)} = Id$ and using linearity of $\hat{\tau}_i$, we obtain

$$\psi_{(i)}^{(k)} = \phi_{(i)}^{(k)}(Id - \psi_{(i)}^{(k)}), \quad i = 0, \ldots, n, \tag{4.40}$$

recalling the notation $\phi_{(i)}^{(k)} = \hat{\tau}_i^{-1}\phi^{(k)} \circ \hat{\tau}_i$. According to Proposition 4.17, we have

$$|||\phi^{(k)}|||_{\epsilon_k - \frac{\delta}{\kappa}, r_k e^{-\delta}} \leq ||\tau^{\bullet(k)}||_{\epsilon_k, r_k} \frac{C'}{\delta^{\tau+\nu}} \leq C'\delta_k^{\mu}\delta^{-\tau-\nu}. \tag{4.41}$$

We recall that the constatn C' does not depend on k, nor on $\tau^{\bullet(k)}$.

Lemma 4.18. *There is constant $\tilde{D} > 0$ (independent of k) such that, given a positive number μ, so that $\delta < \min\{\tilde{D}, \ln 2, \frac{r_0}{2}\}$ satisfies*

$$2^{m+2}C'\delta_k^{\mu}\delta^{-\tau-\nu-3} < 1, \tag{4.42}$$

where $m = 5$. Then, for all $0 \leq \ell \leq m$ and $i = 0, \ldots, n$, the maps $\Phi_{(i)}^{(k)} := Id + \phi_{(i)}^{(k)}$ are biholomorphisms

$$\Phi_{(i)}^{(k)} : \Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}, r_k e^{-(\ell+1)\delta}} \to \Omega_{\epsilon_k - \frac{\ell\delta}{2\kappa}, r_k e^{-\frac{\ell\delta}{2}}}$$

with inverse $\Psi_{(i)}^{(k)} := Id - \psi_{(i)}^{(k)} : \Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}} \to \Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}, r_k e^{-(\ell+1)\delta}}$ satisfying

$$\Phi_{(i)}^{(k)} \circ \Psi_{(i)}^{(k)} = \hat{\tau}_i^{-1}(\Phi^{(k)} \circ \Psi^{(k)}) \circ \hat{\tau}_i = Id$$

on $\Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}}$, provided $q_0 > C(\delta_0, \mu)$.

Proof. We have $1 - e^{-\delta} > \delta/2$ and $\frac{1}{2} < e^{-\delta}$ for $0 < \delta < \ln 2$. Assuming $\delta < \frac{r_0}{2}$, we have $\delta < r_k$ so that

$$\mathrm{dist}(\Delta^d_{r_k e^{-(1+\ell)\delta}}, \partial\Delta^d_{r_k e^{-\ell\delta}}) = r_k e^{-\ell\delta}(1 - e^{-\delta}) > \frac{\delta^2}{2^{\ell+1}}.$$

On the other hand, by (4.6), we have

$$\mathrm{dist}(\Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}}, \partial\Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}}) = \mathrm{dist}(\Omega^+_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}}, \partial\Omega^+_{\epsilon_k - \frac{\ell\delta}{\kappa}}).$$

Let $(e^{-2\pi R}, e^{-2\pi R'}) \in \Omega^+_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}} \times \partial\Omega^+_{\epsilon_k - \frac{\ell\delta}{\kappa}}$. Since the matrix $(\mathrm{Im}\,\tau_i^j)_{1 \leq i,j \leq n}$ is invertible, there is a constant $\tilde{C}$ (independent of k) such that

$$|e^{-2\pi R} - e^{-2\pi R'}| \geq \tilde{C}^{-1}\left|1 - e^{-2\pi(R-R')}\right| > 2\pi\tilde{C}^{-1}|R - R'| \geq 2\pi\tilde{C}^{-1}\frac{\delta}{\kappa}.$$

Let us set $\tilde{D} := 2\frac{2\pi}{\tilde{C}\kappa}$. Assuming $\delta < \tilde{D}$, we have $\delta < \tilde{D} < 2^{\ell+2}\tilde{D} \leq 2^{m+2}\tilde{D}$. Assume $2^{m+2}C'\delta_k^{\mu}\delta^{-\tau-\nu-3} < 1$. Then according to (4.41), we have, for $(h,v) \in \Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}, r_k e^{-\ell\delta}}$,

$$
\begin{aligned}
(4.43) \qquad |\phi_{(i)}^{(k)}(h,v)| &\leq C'\delta_k^{\mu}\delta^{-\tau-\nu} < \frac{\delta^3}{2^{m+2}} < \frac{\delta}{2}\frac{\delta^2}{2^{\ell+1}} \\
&< \frac{\delta}{2}\operatorname{dist}(\Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}, r_k e^{-(\ell+1)\delta}}, \partial\Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}, r_k e^{-\ell\delta}}).
\end{aligned}
$$

By the Cauchy inequality, we have, for $(h,v) \in \Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}}$

(4.44)

$$
|D\phi_{(i)}^{(k)}(h,v)| \leq \frac{C'\delta_k^{\mu}\delta^{-\tau-\nu}}{\operatorname{dist}(\Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}}, \partial\Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}, r_k e^{-(\ell+1)\delta}})} \leq \frac{\delta}{2} < 1/2.
$$

We can apply the contraction mapping theorem to (4.40) together with the last inequality of (4.43). We find a holomorphic solution $\psi_{(i)}^{(k)}$ such that for $(h,v) \in \Omega_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}}$

$$
\begin{aligned}
(4.45) \qquad |\psi_{(i)}^{(k)}(h,v)| &\leq ||\phi_{(i)}^{(k)}||_{\epsilon_k - \frac{\delta(\textbf{Corrected})(\ell+2)}{\kappa}, r_k e^{-(\ell+2)\delta}} \leq C'\delta_k^{\mu}\delta^{-\tau-\nu} \leq \frac{\delta^3}{2^{m+2}} \\
&< \frac{\delta}{2^{m-\ell-1}}\operatorname{dist}(\Omega_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}}, \partial\Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}}).
\end{aligned}
$$

Hence, we have found a mapping $\Psi_{(i)}^{(k)} := Id - \psi_{(i)}^{(k)}$ such that

$$
\hat{\Omega}_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}} := \Psi_{(i)}^{(k)}(\Omega_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}}) \subset \Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}}.
$$

Also, $\Phi_{(i)}^{(k)} \circ \Psi_{(i)}^{(k)} = Id$ on $\Omega_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}}$. Therefore,

$$
\Phi_{(i)}^{(k)} : \hat{\Omega}_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}} \to \Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}, r_k e^{-(\ell+1)\delta}}
$$

is an (onto) biholomorphism such that $(\Phi_{(i)}^{(k)})^{-1} = \Psi_{(i)}^{(k)}$. □

Proposition 4.19. *Keep conditions on δ, μ in Lemma 4.18 as well as conditions (4.34),(4.35). If δ_0 small enough there is possibly larger $\mu > 0$ such that if for all $i = 1, \ldots, n$, $\tau_i^{(0)} \in (\mathcal{A}_{\epsilon_0, r_0})^{n+d}$ with $|||\tau_i^{\bullet(0)}|||_{\epsilon_0, r_0} \leq \delta_0^{\mu}$, then for all $k \geq 0$ we have the following :*

$$
\begin{aligned}
&\tau_i^{\bullet(k+1)} \in (\mathcal{A}_{\epsilon_{k+1}, r_{k+1}})^{n+d}, \quad \tau_i^{\bullet(k+1)} = O(|v|^{2q_k+1}), \\
(4.46) \qquad &||\tau_i^{\bullet(k+1)}||_{\epsilon_{k+1}, r_{k+1}} \leq \delta_{k+1}^{\mu}.
\end{aligned}
$$

Proof. Let us first show that $\tau_i^{(k+1)} := \Phi^{(k)} \circ \tau_i^{(k)} \circ (\Phi^{(k)})^{-1}$ is well defined on $\Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}, r_k e^{-\ell\delta}}$ for $m \geq \ell \geq 5$ and all $i = 1, \ldots, n$. Here m is fixed from Lemma 4.18. Indeed, we have

$$
\tau_i^{(k+1)} = \hat{\tau}_i(I + \phi_{(i)}^{(k)}) \circ (I + \hat{\tau}_i^{-1}\tau_i^{\bullet(k)}) \circ (\Phi^{(k)})^{-1}.
$$

Since $\tau_i^{\bullet(k)}$ is of order $\geq 2q_{k-1}+1$, Schwarz inequality gives

$$
\|\hat{\tau}_i^{-1}\tau_i^{\bullet(k)}\|_{\epsilon_k - \frac{(\ell-1)\delta}{\kappa}, r_k e^{-(\ell-1)\delta}} \leq \max_i \|\hat{\tau}_i^{-1}\|_{\epsilon_0, r_0} C e^{-(2q_{k-1}+1)\delta} \|\tau_i^{\bullet(k)}\|_{\epsilon_k - \frac{(\ell-2)\delta}{\kappa}, r_k e^{-(\ell-2)\delta}}.
$$

We recall that δ_0 satisfies (4.34) and (4.35). Setting $\delta := \delta_k$ and if q_0 large enough, we have

$$\max_i \|\hat{\tau}_i^{-1}\|_{\epsilon_0,r_0} Ce^{-(2q_{k-1}+1)\delta} \leq \max_i \|\hat{\tau}_i^{-1}\|_{\epsilon_0,r_0} Ce^{-\frac{2^{k-1}}{(k+1)^2}q_0\delta_0} < 1.$$

According to Lemma 4.18, (4.42) and the distance estimate in (4.43), we have

$$(I+\hat{\tau}_i^{-1}\tau_i^{\bullet(k)})\circ(\Phi^{(k)})^{-1}(\Omega_{\epsilon_k-\frac{\ell\delta}{\kappa},r_ke^{-\ell\delta}}) \subset \Omega_{\epsilon_k-\frac{(\ell-3)\delta}{\kappa},r_ke^{-(\ell-3)\delta}}$$

so that $\tau_i^{(k+1)}$ is defined on $\Omega_{\epsilon_k-\frac{\ell\delta}{\kappa},r_ke^{-\ell\delta}}$ for $\ell = 4 \leq m$ since $\phi_{(i)}^{(k)}$ is defined on $\Omega_{\epsilon_k-\delta/\kappa,r_ke^{-\delta/\kappa}}$. For the rest of the proof, we fix $\ell = 4$. From the argument above, we have

$$\tau_i^{(k+1)}(\Omega_{\epsilon_k-4\delta/\kappa,r_ke^{-4\delta/\kappa}}) \subset \hat{\tau}_i(\Phi_{(i)}^k(\Omega_{\epsilon_k-\delta/\kappa,r_ke^{-\delta/\kappa}})) \subset \hat{\tau}_i(\Omega_{\epsilon_k,r_k}) \subset \hat{\tau}_i(\Omega_{\epsilon_0,r_0}).$$

Hence, $\tau_i^{\bullet(k+1)}$ is uniformly bounded on $\Omega_{\epsilon_k-4\delta/\kappa,r_ke^{-4\delta/\kappa}}$ w.r.t k :

$$\|\tau_i^{\bullet(k+1)}\|_{\epsilon_k-4\delta/\kappa,r_ke^{-4\delta}} \leq C.$$

On the other hand, on $\Omega_{\epsilon_k-4\delta/\kappa,r_ke^{-4\delta/\kappa}}$, we have

$$\begin{aligned}\tau_i^{\bullet(k+1)} &= \hat{\tau}_i(\phi^{(k)}-\psi^{(k)}) + \left(\tau_i^{\bullet(k)}(Id-\psi^{(k)}) - \tau_i^{\bullet(k)}\right)\\ &+ \left(\phi^{(k)}\left(\hat{\tau}_i - \hat{\tau}_i\psi^{(k)} + \tau_i^{\bullet(k)}(Id-\psi^{(k)})\right) - \phi^{(k)}(\hat{\tau}_i)\right)\\ &+ (\phi^{(k)}(\hat{\tau}_i) - \hat{\tau}_i\phi^{(k)} + \tau_i^{\bullet(k)}).\end{aligned} \tag{4.47}$$

The last term is equal to $\tau_i^{\bullet(k)} - J^{2q_k}(\tau_i^{\bullet(k)}) = O(|v|^{2q_k+1})$. We also have $\phi^{(k)} - \psi^{(k)} = O(|v|^{2q_k+1})$, $\phi^{(k)} = O(|v|^{q_k+1})$. Thus

$$\tau_i^{\bullet(k+1)} = O(|v|^{2q_k+1}). \tag{4.48}$$

Improving the estimate by the Schwarz inequality, we obtain

$$\|\tau_i^{\bullet(k+1)}\|_{\epsilon_k-\frac{5\delta}{\kappa},r_ke^{-5\delta}} \leq Ce^{-q_{k+1}\delta}.$$

We have $e^{-x} < 1/x$ for $x > 0$. For $\delta := \delta_k < \min\{\tilde{D}, \ln 2, \frac{r_0}{2}\}$, let μ satisfy $2^{m+2}C'\delta_k^{\mu-\tau-\nu-3} < 1$, in which case assumptions of Lemma 4.18 are satisfied. We then obtain

$$C\delta_{k+1}^{-\mu}e^{-q_{k+1}\delta} \leq C\left(\frac{(k+2)^2}{\delta_0}\right)^{\mu+1}\frac{1}{q_0 2^k} < 1$$

provided $q_0 > C(\delta_0, \mu)$. □

Finally, quite classically, since (4.44) and (4.33), the sequence of diffeomorphisms $\{\Phi_k \circ \Phi_{k-1} \circ \cdots \circ \Phi_1\}_k$ converges uniformly on the open set $\Omega_{\frac{\epsilon_0}{2},r_0e^{-\sigma}}$ to a diffeomorphism Φ which satisfies

$$\Phi \circ \hat{\tau}_i = \tau_i \circ \Phi, \quad i = 1, \ldots, n$$

provided $q_0 \geq C(\delta_0, \mu)$ and $||\tau_i^{\bullet(0)}||_{\epsilon_0,r_0} \leq \delta_0^\mu$ for some δ_0, μ are fixed. The condition $q_0 > C(\delta_0, \mu, \tau, \nu)$ can be achieved by using finitely many $\Phi_0, \ldots, \Phi_m$. Then initial condition $||\tau_i^{\bullet(0)}||_{\epsilon_0,r_0} \leq \delta_0^\mu$ can be achieved easily by a dilation in the v variable. Indeed, we apply the dilation in v to the original $\tau_1, \ldots, \tau_n$ with $q_0 = 1$. This allows us to construct Φ_0 in Proposition 4.19 with $k = 0$ and define $\Phi_0\tau_j\Phi_0^{-1}$ to achieve $q_1 \geq 2$. Then $\Phi_0\tau_j\Phi_0^{-1}$, $j = 1, \ldots, n$ still commute pairwise on $\Omega_{\epsilon'} \times \Delta_{\epsilon'}^d$ for some $\epsilon' > 0$. Applying the procedure again, this allows us to find $\Phi_1, \ldots, \Phi_{k-1}$ to achieve

$q_k \geq 2^k > C(\delta_0, \mu)$. Finally using dilation we can apply the full version of Proposition 4.19 for all k to construction a new sequence of desired mapping $\Phi_0, \Phi_1, \ldots$. Hence, the torus C has a neighborhood in M biholomorphic to a neighborhood of the zero section in its normal bundle N_C since there is a biholomorphism fixing $\widetilde{C}$ that conjugate the deck transformations of the covering of the latter to those of the former.

Remark 4.20. *The assumption "$T_C M$ splits" can be replaced by the condition that for all $P \in \mathbb{Z}^n$, for all $Q \in \mathbb{N}^d$, $|Q| = 1$ and for all $i = 1, \ldots, n$*

$$\max_{\ell \in \{1,\ldots,n\}} \left| \lambda_\ell^P \mu_\ell^Q - \lambda_{\ell,i} \right| > \frac{D}{(|P|+1)^\tau}.$$

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