MULTIPLE SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS

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ABSTRACT. Let (M, g) be a smooth, compact Riemannian n-manifold, and h be a Hölder continuous function on M. We prove the existence of multiple changing sign solutions for equations like $\Delta_g u + hu = |u|^{2^*-2} u$, where Δ_g is the Laplace–Beltrami operator and the exponent $2^* = 2n/(n-2)$ is critical from the Sobolev viewpoint.

1. INTRODUCTION

1.1. Statement of the results

Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ and h be a Hölder continuous function on M, namely a function which belongs to $C^{0,\theta}(M)$ for some real number θ in $(0, 1)$. We consider equations like

$$\Delta_g u + hu = |u|^{2^*-2} u\,, \tag{1.1}$$

where $\Delta_g = -\operatorname{div}_g \nabla$ is the Laplace–Beltrami operator, and $2^* = 2n/(n-2)$. If $H_1^2(M)$ stands for the Sobolev space of all functions in $L^2(M)$ with one derivative in $L^2(M)$, then 2^* is the critical exponent for the embeddings of $H_1^2(M)$ into Lebesgue spaces. We provide $H_1^2(M)$ with the scalar product

$$\langle u, v\rangle_{H_1^2(M)} = \int_M \langle \nabla u, \nabla v\rangle_g \, dv_g + \Lambda \int_M uv dv_g\,, \tag{1.2}$$

where Λ is a positive constant to be chosen large later on. The Hölder continuity of h provides the regularity of weak solutions of equation (1.1). In case there holds $h \equiv \frac{n-2}{4(n-1)} \operatorname{Scal}_g$, where Scal_g is the scalar curvature of the manifold (M, g), equation (1.1) is the intensively studied Yamabe equation whose positive solutions u are such that the scalar curvature of the conformal metric $u^{2^*-2} g$ is constant (see Aubin [3], Schoen [49], Trudinger [58], and Yamabe [59]). In this paper, we deal with multiplicity of solutions for equation (1.1) when the function h is locally less than $\frac{n-2}{4(n-1)} \operatorname{Scal}_g$ in Theorem 1.1, and globally less than $\frac{n-2}{4(n-1)} \operatorname{Scal}_g$ in Theorems 1.2 and 1.3. We define the energy of a solution u of equation (1.1) to be the real number $E(u)$ given by

$$E(u) = \int_M |u|^{2^*} \, dv_g\,, \tag{1.3}$$

where dv_g is the volume element of the manifold (M, g). We say that an operator like $\Delta_g + h$ is coercive on $H_1^2(M)$ if the energy associated to this operator controls the H_1^2-norm. We

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let $D^{1,2}(\mathbb{R}^n)$ be the homogeneous Sobolev space defined as the completion of the space of all smooth functions on $\mathbb{R}^n$ with compact support with respect to the scalar product

$$\langle u, v\rangle_{D^{1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \langle \nabla u, \nabla v\rangle \, dx\,.$$

We let also K_n be the sharp constant for the embedding of $D^{1,2}(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$, namely

$$K_n = \sqrt{\frac{4}{n(n-2)\,\omega_n^{2/n}}}\,,$$

where ω_n is the volume of the unit n-sphere. We associate each solution of equation (1.1) with its opposite one, and call that a pair of solutions. We state our first result as follows.

Theorem 1.1. *Let (M,g) be a smooth, compact Riemannian manifold of dimension $n \geq 4$ and h be a Hölder continuous function on M such that the operator $\Delta_g + h$ is coercive on $H_1^2(M)$. If there exists a point x_0 in M such that $h(x_0) < \frac{n-2}{4(n-1)} \operatorname{Scal}_g(x_0)$, then equation* (1.1) *admits at least $(n+2)/2$ pairs of nontrivial solutions with energy less than $2K_n^{-n}$.*

More precisely, we prove that either we do have infinitely many solutions of equation (1.1) or the $(n+2)/2$ pairs of nontrivial solutions we get in Theorem 1.1 have distinct energies. In the particular case where the manifold is locally conformally flat, $n \geq 7$, and h is a C^1-function less than $\frac{n-2}{4(n-1)} \operatorname{Scal}_g$ on the whole manifold, the above result can be improved. In such a setting, we establish two results. We first consider families of equations like

$$\Delta_g u + hu = |u|^{p_\alpha - 2}\, u\,, \tag{1.4}$$

where $(p_\alpha)_\alpha$ is a sequence in $[2, 2^*]$ converging to 2^*. A sequence $(u_\alpha)_\alpha$ is said to be a sequence of solutions for the family (1.4) if for any α, u_α is a solution of equation (1.4). First, we prove a compactness result for the family of equations (1.4) similar to the one proved by Devillanova–Solimini [17] in the case of smooth, bounded domains of the Euclidean space. Our compactness result is as follows.

Theorem 1.2. *Let (M,g) be a smooth, compact, locally conformally flat Riemannian manifold of dimension $n \geq 7$ and h be a C^1-function on M. We let $(p_\alpha)_\alpha$ be a sequence in $[2, 2^*]$ converging to 2^*, and we consider the family of equations* (1.4)*. If there holds $h < \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ in M, then any bounded sequence in $H_1^2(M)$ of solutions for this family of equations remains bounded in $C^0(M)$.*

An equivalent conclusion of Theorem 1.2 is that any bounded sequence in $H_1^2(M)$ of solutions for the family of equations (1.4) is compact in $H_1^2(M)$. In particular, such a sequence converges up to a subsequence in $H_1^2(M)$ to a solution of the critical equation (1.1). This easily follows from standard elliptic estimates (see, for instance, Gilbarg–Trudinger [28] Theorem 9.11) and the compactness of the embedding of $H_2^p(M)$ into $H_1^2(M)$ for all real numbers $p > 2n/(n-2)$. As a remark, note that $p_\alpha \to 2^*$ is the only interesting difficult case for compactness since the embeddings of $H_1^2(M)$ into $L^p(M)$ are compact for $p < 2^*$. Theorem 1.2 is the key argument in the proof of our last result which states as follows.

Theorem 1.3. *Let (M,g) be a smooth, compact, locally conformally flat Riemannian manifold of dimension $n \geq 7$ and h be a C^1-function on M. If there holds $h < \frac{n-2}{4(n-1)} \operatorname{Scal}_g$ in M, then equation* (1.1) *admits infinitely many solutions with unbounded energies.*

There are several situations where we do know that the solutions we get in Theorems 1.1 and 1.3 truly change sign. Such changing sign solutions are referred to as nodal solutions. Let us assume, for instance, that the Ricci curvature Ric_g of the manifold (M, g) satisfies

$$\mathrm{Ric}_g \geq \frac{4(n-1)}{n(n-2)} \lambda g \tag{1.5}$$

for some positive real number λ, in the sense of bilinear forms, the inequality being strict when the manifold is conformally diffeomorphic to the sphere. Then, as proved by Bidaut-Véron–Véron [6], equation (1.1) with $h \equiv \lambda$ has a unique positive solution, which turns out to be $u = \lambda^{(n-2)/4}$. In particular, in such a situation, all but one pairs of solutions we get in Theorem 1.1 are nodal. Concerning Theorem 1.3, it has been proved by Druet [21] that there is an *a priori* bound on the energy of positive solutions of equation (1.1) when $h < \frac{n-2}{4(n-1)} \mathrm{Scal}_g$ in M. More precisely, for any smooth, compact Riemannian manifold (M, g) of dimension $n \geq 3$, there exists a real number E_0 such that if u is a positive solution of equation (1.1), then $E(u) \leq E_0$ where $E(u)$ is as in (1.3). In particular, as a direct consequence of the existence of this *a priori* bound for positive solutions, Theorem 1.3 provides infinitely many nodal solutions for equation (1.1). Summarizing, the following corollary holds true.

Corollary 1.4. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and h be a C^1-function on M such that $h < \frac{n-2}{4(n-1)} Scal_g$ in M. If $n \geq 7$ and the manifold is locally conformally flat, then equation* (1.1) *admits infinitely many nodal solutions. If $n \geq 4$, the manifold is arbitrary, $h \equiv \lambda$ for some $\lambda > 0$, and* (1.5) *holds true, the inequality being strict when the manifold is conformally diffeomorphic to the sphere, then equation* (1.1) *admits at least $n/2$ pairs of nodal solutions.*

Compactness of positive solutions of equations like (1.1) have been intensively studied in recent years. Possible references on this topic, in the case of manifolds, are Druet [21,22], Li–Zhang [40–42], Li–Zhu [43], Marques [45], and Schoen [50–52]. A survey reference on the subject is Druet–Hebey [23]. We refer also to Hebey [31,32] for compactness of positive solutions of critical elliptic systems in potential form and to Hebey–Robert–Wen [33] for compactness of positive solutions of critical fourth order equations. Compactness of changing sign solutions of equations like (1.1), in the case of smooth, bounded domains of the Euclidean space, have been studied in Devillanova–Solimini [17]. We follow this reference by Devillanova–Solimini [17] in several places in Section 3, as well as we follow the reference Clapp–Weth [15] in several places in Section 2. Possible other references on the existence of multiple nodal solutions for equations like (1.1) are Atkinson–Brezis–Peletier [2], Bahri–Lions [4], Capozzi–Fortunato–Palmieri [8], Castro–Cossio–Neuberger [9], Cerami–Fortunato–Struwe [10], Cerami–Solimini–Struwe [11], Devillanova–Solimini [18], Ding [19], Djadli–Jourdain [20], Fortunato–Jannelli, [26], Hebey–Vaugon [34], Holcman [35], Jourdain [36], Solimini [53], Tarantello [57], and Zhang [60]. Needless to say, the above list does not pretend to exhaustivity. We refer also to the recent very nice paper by Ammann–Humbert [1] where the question of the existence of at least one changing sign solution to the Yamabe equation is addressed.

A final remark in this introduction concerns the condition $h < \frac{n-2}{4(n-1)} \mathrm{Scal}_g$ in Theorem 1.2. Let $(\mathbb{S}^n, \mathrm{std})$ be the unit n-sphere. There holds $\mathrm{Scal}_{\mathrm{std}} \equiv n(n-1)$, and the Yamabe equation on the unit n-sphere reads as

$$\Delta_{\mathrm{std}} u + \frac{n(n-2)}{4} u = u^{2^*-1}. \tag{1.6}$$

For any $\beta > 1$ and any point x_0 in $\mathbb{S}^n$, we define the function u_{β,x_0} on $\mathbb{S}^n$ by

$$u_{\beta,x_0}(x) = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}} \left(\frac{\sqrt{\beta^2-1}}{\beta - \cos(d_{\mathrm{std}}(x_0,x))}\right)^{\frac{n-2}{2}} .$$

All these functions are solutions of equation (1.6), and have the same energy, namely K_n^{-n}. They are uniformly bounded in $H_1^2(M)$ but there holds $u_{\beta,x_0}(x_0) \to +\infty$ as $\beta \to 1^+$. In particular, when dealing with the unit n-sphere and equation (1.6), for which $h \equiv \frac{n-2}{4(n-1)} \mathrm{Scal}_{\mathrm{std}}$, there are no uniform bounds in $C^0(M)$. More sophisticated examples can be found in Druet–Hebey [23] for positive solutions of equations like (1.6), and in Ding [19] for changing sign solutions of equations like (1.6). We lose Theorem 1.2 when we do not assume something like $h < \frac{n-2}{4(n-1)} \mathrm{Scal}_g$.

1.2. **Preliminary material**

In the following, we let (M,g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ and h be a Hölder continuous function on M. We define the functional I_g by

$$I_g(u) = \frac{1}{2}\int_M |\nabla u|_g^2 \, dv_g + \frac{1}{2}\int_M hu^2 dv_g - \frac{1}{2^*}\int_M |u|^{2^*} dv_g .$$

Its critical points are the solutions of equation (1.1). Let us recall the basics about the H_1^2-theory of blow-up that we need in the proof of Theorems 1.1, 1.2, and 1.3. A sequence $(u_\alpha)_\alpha$ in $H_1^2(M)$ is said to be Palais–Smale for the functional I_g if the sequence $(I_g(u_\alpha))_\alpha$ is bounded and if there holds $DI_g(u_\alpha) \to 0$ in $H_1^2(M)'$ as $\alpha \to +\infty$. If moreover $I_g(u_\alpha)$ converges to a real number c as $\alpha \to +\infty$, then the sequence $(u_\alpha)_\alpha$ is said to be Palais–Smale for the functional I_g at level c. In particular, bounded sequences in $H_1^2(M)$ of solutions of equation (1.1) are Palais–Smale for the functional I_g. The H_1^2-theory of blow-up deals with the asymptotic behaviour in $H_1^2(M)$ of Palais–Smale sequences for the functional I_g.

Let η be a smooth cutoff function such that $0 \leq \eta \leq 1$ in $\mathbb{R}^n$, $\eta \equiv 1$ in $B_0(i_g/3)$, and $\eta \equiv 0$ out of $B_0(2i_g/3)$, where i_g is the injectivity radius of the manifold (M,g). Given a converging sequence $(x_\alpha)_\alpha$ of points in M, a sequence $(\mu_\alpha)_\alpha$ of positive real numbers converging to 0, and a function u in $D^{1,2}(\mathbb{R}^n)$, we shall call rescaling of u on M of centers $(x_\alpha)_\alpha$ and weights $(\mu_\alpha)_\alpha$, a sequence $(\varrho_\alpha(u))_\alpha$ of functions defined on M by

$$\varrho_\alpha(u)(x) = \mu_\alpha^{\frac{2-n}{2}} \eta_\alpha(x)\, u\left(\mu_\alpha^{-1} \exp_{x_\alpha}^{-1}(x)\right),$$

where $\eta_\alpha = \eta \circ \exp_{x_\alpha}^{-1}$. One can easily see that $(\varrho_\alpha(u))_\alpha$ converges to 0 weakly in $H_1^2(M)$, strongly in $L^2(M)$ but that the H_1^2-norm of the functions $\varrho_\alpha(u)$ converges to $\|u\|_{D^{1,2}(\mathbb{R}^n)}$ as $\alpha \to +\infty$. An important and usefull remark is that the H_1^2-range of interaction of a rescaling is of the order of its weights, namely that there hold

$$\lim_{\alpha\to+\infty} \int_{B_{x_\alpha}(R\mu_\alpha)} |\nabla \varrho_\alpha(u)|_g^2 \, dv_g = \int_{B_0(R)} |\nabla u|^2 \, dx$$

and

$$\limsup_{\alpha\to+\infty} \int_{M\setminus B_{x_\alpha}(R\mu_\alpha)} |\nabla \varrho_\alpha(u)|_g^2 \, dv_g = \varepsilon_R$$

for all positive real numbers R, where $\varepsilon_R \to 0$ as $R \to +\infty$.

We shall call bubble a rescaling on M of a nontrivial solution in $D^{1,2}(\mathbb{R}^n)$ of the equation

$$\Delta_\delta u = |u|^{2^*-2} u , \tag{1.7}$$

where δ is the Euclidean metric on $\mathbb{R}^n$. Given a bubble $(B_\alpha)_\alpha$, we define its energy $E(B_\alpha)$ by

$$E(B_\alpha) = \int_{\mathbb{R}^n} |u|^{2^*} dx\,.$$

Nonnegative solutions in $D^{1,2}(\mathbb{R}^n)$ of equation (1.7) are all of the form

$$u_{\mu,x_0}(x) = \left(\frac{\mu}{\mu^2 + \frac{|x-x_0|^2}{n(n-2)}}\right)^{\frac{n-2}{2}}, \tag{1.8}$$

where μ is a nonnegative real number and x_0 is a point in the Euclidean space (see Caffarelli–Gidas–Spruck [7] and Obata [47]). They are the extremal functions for the sharp Euclidean Sobolev inequality, and one can easily compute

$$\int_{\mathbb{R}^n} |u_{\mu,x_0}|^{2^*} dx = \int_{\mathbb{R}^n} |\nabla u_{\mu,x_0}|^2 dx = K_n^{-n}.$$

As for nodal solutions u of equation (1.7), there holds

$$\int_{\mathbb{R}^n} |u|^{2^*} dx = \int_{\mathbb{R}^n} |\nabla u|^2 dx > 2K_n^{-n}. \tag{1.9}$$

In other words, the energy of a constant sign bubble is K_n^{-n} while the one of a nodal bubble is greater than $2K_n^{-n}$. In order to prove (1.9), we decompose the function u into its positive part $u^+ = \max(u,0)$ and its negative part $u^- = \max(-u,0)$, and we write

$$\int_{\mathbb{R}^n} \left|\nabla u^\pm\right|^2 dx = \int_{\mathbb{R}^n} \Delta_\delta u u^\pm dx = \int_{\mathbb{R}^n} |u|^{2^*-2} u u^\pm dx = \int_{\mathbb{R}^n} \left|u^\pm\right|^{2^*} dx\,.$$

By taking into account that $u^\pm$ cannot be of the form (1.8), it follows that

$$\int_{\mathbb{R}^n} \left|\nabla u^\pm\right|^2 dx > K_n^{-n},$$

and we sum to get (1.9).

We recall the following result proved by Struwe [56] for equation (1.7) in smooth, bounded domains of the Euclidean space. We also refer to Druet–Hebey–Robert [24] for a complete exposition in book form in the Riemannian case.

Lemma 1.5. *Let (M,g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ and h be a Hölder continuous function on M. For any Palais–Smale sequence $(u_\alpha)_\alpha$ for the functional I_g, there exist a solution u_∞ of equation (1.1), a natural number k, and bubbles $(B_\alpha^1)_\alpha, \dots, (B_\alpha^k)_\alpha$ such that up to a subsequence,*

$$u_\alpha = u_\infty + \sum_{i=1}^k B_\alpha^i + R_\alpha$$

for all α, where $R_\alpha \to 0$ in $H_1^2(M)$ as $\alpha \to +\infty$ and moreover, there holds

$$I_g(u_\alpha) = I_g(u_\infty) + \frac{1}{n}\sum_{i=1}^k E\left(B_\alpha^i\right) + \mathrm{o}(1)$$

as $\alpha \to +\infty$.

We prove Theorem 1.1 in Section 2 by using a negative gradient flow, the H_1^2-theory of blow-up, and the relative equivariant Lusternik–Schnirelmann categories. We prove Theorems 1.2 and 1.3 in Section 3 thanks to a fine analysis of blow-up and topological arguments involving the Krasnosel′skiĭ genus.

2. Multiple solutions with bounded energies

We purpose to prove Theorem 1.1 in this section. We first set some notations. We let $\mathcal{P}$ be the set of all nonnegative functions in $H_1^2(M)$. Given a positive real number δ and a subset C of $H_1^2(M)$, we let $\mathcal{B}_\delta(C)$ stand for the neighbourhood of C formed by all functions in $H_1^2(M)$ at a distance from C less than or equal to δ. Given a real number c, we set $I_g^c = I_g^{-1}((-\infty, c])$.

2.1. **The negative gradient flow**

In the following, we let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ and h be a continuous function on M. For the moment, we do not need to assume that the operator $\Delta_g + h$ is coercive on $H_1^2(M)$ nor to restrict the dimension of M. We let ∇I_g stand for the operator acting on $H_1^2(M)$ satisfying

$$\langle \nabla I_g(u), v \rangle_{H_1^2(M)} = DI_g(u).v$$

for all functions u and v in $H_1^2(M)$, where $\langle ., . \rangle_{H_1^2(M)}$ is as in (1.2). We also let φ_g stand for the flow defined by

$$\begin{cases} \dfrac{\partial \varphi_g}{\partial t}(t, u) = -\nabla I_g(\varphi_g(t, u)) & \text{if } 0 \leq t < T(u), \\ \varphi_g(0, u) = u, \end{cases}$$

where for any function u in $H_1^2(M)$, $T(u)$ is the maximal existence time for the trajectory $t \to \varphi_g(t, u)$. As a remark, for any function u in $H_1^2(M)$ and for any positive time t, we get

$$\frac{\partial (I_g \circ \varphi_g)}{\partial t}(t, u) = -\left\| \nabla I_g(\varphi_g(t, u)) \right\|_{H_1^2(M)}^2. \tag{2.1}$$

A subset D of $H_1^2(M)$ is said to be strictly positively invariant for the flow φ_g if for any u in D and any time t in $(0, T(u))$, the function $\varphi_g(t, u)$ belongs to the interior of D. As an example, since by (2.1), the function $I_g \circ \varphi_g(\cdot, u)$ is decreasing for all non-critical points u in $H_1^2(M)$, the set I_g^c is strictly positively invariant for the flow φ_g for all non-critical values c. The following lemma provides some other examples of subsets of $H_1^2(M)$ which are strictly positively invariant for the flow φ_g and that we use in the proof of Theorem 1.1.

Lemma 2.1. *Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ and h be a continuous function on M. Let Λ be the positive constant appearing in the definition of the scalar product* (1.2)*. If Λ is large enough, then for small positive real numbers δ, the sets $\mathcal{B}_\delta(\mathcal{P})$ and $\mathcal{B}_\delta(-\mathcal{P})$ are strictly positively invariant for the flow φ_g.*

Proof. Since the operator ∇I_g is odd, it suffices to state the proof for the sets $\mathcal{B}_\delta(\mathcal{P})$. We write $\nabla I_g(u) = u - L_1(u) - L_2(u)$, where L_1 (resp. L_2) is the operator acting from $L^2(M)$ (resp. $L^{2^*}(M)$) into $H_1^2(M)$ which satisfy for any function u in $L^2(M)$ (resp. $L^{2^*}(M)$), the equation

$$\Delta_g L_1(u) + \Lambda L_1(u) = (\Lambda - h) u, \tag{2.2}$$

$$\text{resp. } \Delta_g L_2(u) + \Lambda L_2(u) = |u|^{2^*-2} u. \tag{2.3}$$

As a first step, we show that if Λ is large enough, then for small positive real numbers δ, there exists a real number ν in $(0,1)$ such that for any function u in $\mathcal{B}_\delta(\mathcal{P})$, there holds

$$d\left(L_1(u)+L_2(u),\mathcal{P}\right)\le \nu d(u,\mathcal{P}), \tag{2.4}$$

where d is the distance on the Sobolev space $H_1^2(M)$. We begin with estimating $d(L_1(u),\mathcal{P})$ for all functions u in $H_1^2(M)$. We assume that the constant Λ is greater than or equal to h so that by the weak maximum principle, the operator L_1 sends the set $\mathcal{P}$ into itself. By (2.2) and by Hölder's inequality, we get

$$\begin{aligned}\|L_1(u)\|^2_{H_1^2(M)}&=\int_M(\Lambda-h)\,uL_1(u)\,dv_g\\&\le\|\Lambda-h\|_{C^0(M)}\|u\|_{L^2(M)}\|L_1(u)\|_{L^2(M)}\\&\le\Lambda^{-2}\|\Lambda-h\|_{C^0(M)}\|u\|_{H_1^2(M)}\|L_1(u)\|_{H_1^2(M)}\,.\end{aligned}$$

It follows that

$$\|L_1(u)\|_{H_1^2(M)}\le\Lambda^{-2}\|\Lambda-h\|_{C^0(M)}\|u\|_{H_1^2(M)}\,. \tag{2.5}$$

We let v be the orthogonal projection of u on the closed convex set $\mathcal{P}$. By applying (2.5) to the function $u-v$ and since L_1 is a linear operator, we get

$$d(L_1(u),\mathcal{P})\le\|L_1(u)-L_1(v)\|_{H_1^2(M)}\le\Lambda^{-2}\|\Lambda-h\|_{C^0(M)}\,d(u,\mathcal{P})\,. \tag{2.6}$$

We then estimate $d(L_2(u),\mathcal{P})$. By the weak maximum principle, the operator L_2 also sends the set $\mathcal{P}$ into itself. Multiplying (2.3) by the function $-L_2(u)^-$ and integrating by parts on M yield

$$\left\|L_2(u)^-\right\|^2_{H_1^2(M)}=-\int_M|u|^{2^*-2}uL_2(u)^-\,dv_g\le\int_M\left|u^-\right|^{2^*-2}u^-L_2(u)^-\,dv_g\,.$$

By Hölder's inequality, it follows that

$$\left\|L_2(u)^-\right\|^2_{H_1^2(M)}\le\left\|u^-\right\|^{2^*-1}_{L^{2^*}(M)}\left\|L_2(u)^-\right\|_{L^{2^*}(M)}\,. \tag{2.7}$$

Note that there holds

$$\left\|u^-\right\|_{L^{2^*}(M)}=\min_{v\in\mathcal{P}}\|u-v\|_{L^{2^*}(M)}\;.$$

By (2.7) and the continuity of the embedding of $H_1^2(M)$ into $L^{2^*}(M)$, it follows that there exists a positive constant C independent of u such that there holds

$$\left\|L_2(u)^-\right\|_{H_1^2(M)}\le Cd(u,\mathcal{P})^{2^*-1}\,. \tag{2.8}$$

Summing (2.6) with (2.8) yields

$$d\left(L_1(u)+L_2(u),\mathcal{P}\right)\le\Lambda^{-2}\|\Lambda-h\|_{C^0(M)}\,d(u,\mathcal{P})+Cd(u,\mathcal{P})^{2^*-1}\,.$$

It follows that if Λ is large enough so that $\|\Lambda-h\|_{C^0(M)}<\Lambda^2$, then for small positive real numbers δ, there exists ν in $(0,1)$ such that (2.4) is satisfied for all functions u in $\mathcal{B}_\delta(\mathcal{P})$. In particular, for any positive real number λ in $(0,1]$ and any function u in $\mathcal{B}_\delta(\mathcal{P})$, we get

$$d\left(u-\lambda\nabla I_g(u),\mathcal{P}\right)\le d\left((1-\lambda)u,\mathcal{P}\right)+d\left(\lambda\left(L_1(u)+L_2(u)\right),\mathcal{P}\right)<d(u,\mathcal{P})\,.$$

It follows that there holds $d\left(u-\lambda\nabla I_g(u),\mathcal{B}_\delta(\mathcal{P})\right)=0$ for all positive real numbers λ in $(0,1]$ and all functions u in $\mathcal{B}_\delta(\mathcal{P})$. Moreover, the set $\mathcal{B}_\delta(\mathcal{P})$ is closed, convex, and its interior is nonempty. Therefore, by Deimling [16, Theorem 5.2], $\mathcal{B}_\delta(\mathcal{P})$ is positively invariant, that is to say for any function u in $\mathcal{B}_\delta(\mathcal{P})$, the trajectory $t\to\varphi_g(t,u)$ stays in the set $\mathcal{B}_\delta(\mathcal{P})$ for all positive times. It remains to exhibit a contradiction in case such a trajectory intersects $\partial\mathcal{B}_\delta(\mathcal{P})$ for some time $t_0>0$. In that case, by Mazur's separation theorem (see, for instance, Megginson [46]), there exists a continuous linear form ℓ on $H_1^2(M)$ such that there holds $\ell(\varphi_g(t_0,u)) < \ell(\text{interior}(\mathcal{B}_\delta(\mathcal{P})))$, where interior$(\mathcal{B}_\delta(\mathcal{P}))$ is the interior of the set $\mathcal{B}_\delta(\mathcal{P})$. By (2.4), the operator L_1+L_2 sends the set $\mathcal{B}_\delta(\mathcal{P})$ into its interior, thus we get

$$\frac{\partial(\ell\circ\varphi_g)}{\partial t}(t_0,u) = \ell(L_1(\varphi_g(t_0,u)) + L_2(\varphi_g(t_0,u))) - \ell(\varphi_g(t_0,u)) > 0\,.$$

It follows that for small $\varepsilon>0$ there holds $\ell(\varphi_g(t_0-\varepsilon,u)) < \ell(\varphi_g(t_0,u))$ and thus by continuity, $\varphi_g(t_0-\varepsilon,u)$ does not belong to $\mathcal{B}_\delta(\mathcal{P})$. This contradicts the positive invariance of the set $\mathcal{B}_\delta(\mathcal{P})$, and ends the proof of Lemma 2.1. □

Henceforth, we assume that Λ is large enough so that for δ small, the sets $\mathcal{B}_\delta(\mathcal{P})$ and $\mathcal{B}_\delta(-\mathcal{P})$ are strictly positively invariant for the flow φ_g. We shall say that a subset D of $H_1^2(M)$ is symmetric if there holds $D=-D$. Another essential ingredient for the proof of Theorem 1.1 is the following deformation lemma.

Lemma 2.2. *Let (M,g) be a smooth, compact Riemannian manifold, h be a continuous function on M, and D be a symmetric, closed subset of $H_1^2(M)$ strictly positively invariant for the flow φ_g. Let $c\in\mathbb{R}$, $\delta,\varepsilon\in\mathbb{R}^+$, and a symmetric subset C of $H_1^2(M)$ be such that for any u in $I_g^{-1}([c-\varepsilon,c+\varepsilon])\cap\mathcal{B}_\delta(C)$, there holds*

$$\|\nabla I_g(u)\|_{H_1^2(M)} \ge \frac{2\varepsilon}{\delta}\,. \tag{2.9}$$

Then there exists an odd, continuous map $\nu : \left(I_g^{c+\varepsilon}\cap C\right)\cup D \to I_g^{c-\varepsilon}\cup D$ such that $\nu\equiv\text{id}$ in the set D.

Proof. As a first step, we show that for any function u in $I_g^{c+\varepsilon}\cap C$, the trajectory $t\to\varphi_g(t,u)$ cannot stay in the set $I_g^{-1}((c-\varepsilon,c+\varepsilon])$ for all positive times, and thus belongs to $I_g^{c-\varepsilon}$ for large times since the function $I_g\circ\varphi_g(\cdot,u)$ is nonincreasing. We proceed by contradiction, and assume that the function $\varphi_g(t,u)$ belongs to the set $I_g^{-1}((c-\varepsilon,c+\varepsilon])$ for all positive times t. As long as $\varphi_g(t,u)$ belongs to $\mathcal{B}_\delta(C)$, by assumption (2.9), there holds

$$\begin{aligned}\|\varphi_g(t,u)-u\|_{H_1^2(M)} &\le \int_0^t \left\|\frac{\partial\varphi_g}{\partial t}(t,u)\right\|_{H_1^2(M)} dt\\ &\le \frac{\delta}{2\varepsilon}\int_0^t \|\nabla I_g(\varphi_g(t,u))\|^2_{H_1^2(M)}\,dt\\ &= -\frac{\delta}{2\varepsilon}\int_0^t \frac{\partial(I_g\circ\varphi_g)}{\partial t}(t,u)\,dt\\ &= \frac{\delta}{2\varepsilon}\left(I_g(u)-I_g(\varphi_g(t,u))\right).\end{aligned} \tag{2.10}$$

In particular, the trajectory stays in the ball $\mathcal{B}_\delta(u)$. Moreover, the above computations yield

$$t \le \left(\frac{\delta}{2\varepsilon}\right)^2\left(I_g(u)-I_g(\varphi_g(t,u))\right) \le \frac{\delta^2}{2\varepsilon}\,.$$

By the standard extension theorem for solutions of ordinary differential equations, it follows that the trajectory cannot stay in the set $\mathcal{B}_\delta(C)$ for all positive times t. We then let t_0 be the first positive time that the trajectory intersects $\partial\mathcal{B}_\delta(C)$. By (2.10) with $t=t_0$, we get

$$I_g(\varphi_g(t_0,u)) \le I_g(u) - 2\varepsilon \le c-\varepsilon\,,$$

and this leads to a contradiction. In particular, we have proved that for any function u in $I_g^{c+\varepsilon}\cap C$, the trajectory $t\to\varphi_g(t,u)$ belongs to $I_g^{c-\varepsilon}$ for large times. By the positive invariance of the set D, it follows that for any function u in $\left(I_g^{c+\varepsilon}\cap C\right)\cup D$, there exists a nonnegative time $\tau(u)$ from which the trajectory $t\to\varphi_g(t,u)$ belongs to $I_g^{c-\varepsilon}\cup D$. The function $\tau:\left(I_g^{c+\varepsilon}\cap C\right)\cup D\to\mathbb{R}_+$ is even. In order to get the continuity of τ, the only non-obvious thing we have to prove is its upper continuity. Let u be a function in $\left(I_g^{c+\varepsilon}\cap C\right)\cup D$. In case $\varphi_g(\tau(u),u)$ belongs to the boundary of the set D, the upper continuity of the function τ at u follows from the strict positive invariance of D. In case $I_g(\varphi_g(\tau(u),u))=c-\varepsilon$, by (2.10) with $t=\tau(u)$, we get that the function $\varphi_g(\tau(u),u)$ belongs to the set $\mathcal{B}_\delta(C)$ and assumption (2.9) together with (2.1) then leads to the upper continuity of the function τ at u. Now that we have proved the continuity of τ, we get the expected odd, continuous map $\nu:\left(I_g^{c+\varepsilon}\cap C\right)\cup D\to I_g^{c-\varepsilon}\cup D$ by setting $\nu(u)=\varphi_g(\tau(u),u)$. □

2.2. **Proof of Theorem 1.1**

In the following, we let (M,g) be a smooth, compact Riemannian manifold of dimension $n\geq 4$ and h be a Hölder continuous function on M such that the operator Δ_g+h is coercive on $H_1^2(M)$. We use the same notations as in the previous section. We assume that Λ is large enough and that δ is small enough so that the sets $\mathcal{B}_\delta(\mathcal{P})$ and $\mathcal{B}_\delta(-\mathcal{P})$ are strictly positively invariant for the flow φ_g.

We introduce the notion of relative equivariant Lusternik–Schnirelmann category. Let A and D be two symmetric, closed subsets of a Banach space E such that D is included in A. The equivariant Lusternik–Schnirelmann category of A relatively to D, denoted $\gamma_D(A)$, is the smallest natural number k such that there exist symmetric, open subsets $U_0,\dots,U_k$ of E which cover A and such that $D\subset U_0$ and odd, continuous maps $\chi_i:U_i\to\{-1,1\}$, $i=1,\dots,k$ and $\chi_0:U_0\to D$ such that $\chi_0\equiv\mathrm{id}$ in the set D. If no such natural number exist, then we set $\gamma_D(A)=+\infty$. If D is empty, then the equivariant Lusternik–Schnirelmann category of A relatively to D is called the Krasnosel′skiĭ genus of A, and it is denoted $\gamma(A)$. As is easily seen, the Krasnosel′skiĭ genus of a symmetric, closed subset A of E can also be defined as the smallest natural number k such that there exists an odd, continuous map $\chi:A\to\mathbb{R}^{k+1}\backslash\{0\}$. We now state some properties that we repeatedly use in the proof of Theorem 1.1. We let A, B, and D be three symmetric, closed subsets of E. A first easy estimate states that if D is included in $A\cap B$ and if there exists an odd, continuous map $\nu:A\to B$ such that $\nu\equiv\mathrm{id}$ in the set D, then there holds $\gamma_D(A)\leq\gamma_D(B)$. In particular, this estimate is satisfied when $D\subset A\subset B$. Another easy property states that if D is included in A, then there holds $\gamma_D(A\cup B)\leq\gamma_D(A)+\gamma(B)$. We refer to Bartsch–Clapp [5] and Clapp–Puppe [13, 14] for more material about the relative equivariant Lusternik–Schnirelmann category.

We set $\mathcal{D}_\delta=\mathcal{B}_\delta(\mathcal{P}\cup(-\mathcal{P}))$. For any real number c, we let K_c be the set of all critical points of the functional I_g at level c. One can easily check that there holds $\cup_{c\leq 0}K_c=\{0\}$. By Lemma 2.1 and by (2.1), it follows that the set $I_g^0\cup\mathcal{D}_\delta$ is strictly positively invariant for the flow φ_g. As an easy consequence of this strict positive invariance, there holds $\gamma_{I_g^0\cup\mathcal{D}_\delta}\left(I_g^0\cup\mathcal{D}_\delta\right)=0$. We then define

$$c_\beta=\inf\left\{c>0;\ \gamma_{I_g^0\cup\mathcal{D}_\delta}\left(I_g^c\cup\mathcal{D}_\delta\right)\geq\beta\right\}$$

for all natural numbers $\beta\geq 1$ with the convention that $\inf\emptyset=+\infty$. A preliminary remark is that the sequence $(c_\beta)_\beta$ is nondecreasing. We now claim that for any $\beta\geq 1$, if c_β is finite, then there exists a Palais–Smale sequence $(u_\alpha^{(\beta)})_\alpha$ for the functional I_g at level c_β with the additional property that the function $u_\alpha^{(\beta)}$ belongs to $\overline{H_1^2(M)\backslash\mathcal{D}_{\delta/2}}$ for all α. It suffices to prove that

$$\forall\varepsilon>0\ \exists u\in I_g^{-1}\left([c_\beta-\varepsilon\,,c_\beta+\varepsilon]\right)\cap\overline{H_1^2(M)\backslash\mathcal{D}_{\delta/2}}\text{ s.t. }\|\nabla I_g(u)\|_{H_1^2(M)}<\frac{4\varepsilon}{\delta}\tag{2.11}$$

By contradiction, if (2.11) is false, then there exists a positive real number ε_0 such that for any function u in $I_g^{-1}([c_\beta-\varepsilon_0,c_\beta+\varepsilon_0])\cap\overline{H_1^2(M)\backslash\mathcal{D}_{\delta/2}}$, there holds $\|\nabla I_g(u)\|_{H_1^2(M)}\geq 4\varepsilon_0/\delta$. We clearly get $B_{\delta/2}(\overline{H_1^2(M)\backslash\mathcal{D}_\delta})\subset\overline{H_1^2(M)\backslash\mathcal{D}_{\delta/2}}$. Letting $C=\overline{H_1^2(M)\backslash\mathcal{D}_\delta}$, $D=I_g^0\cup\mathcal{D}_\delta$ and applying Lemma 2.2 with $\delta/2$ instead of δ give an odd, continuous map

$$\nu:(I_g^{c_\beta+\varepsilon_0}\cap\overline{H_1^2(M)\backslash\mathcal{D}_\delta})\cup I_g^0\cup\mathcal{D}_\delta\to I_g^{c_\beta-\varepsilon_0}\cup I_g^0\cup\mathcal{D}_\delta$$

such that $\nu\equiv\text{id}$ in the set $I_g^0\cup\mathcal{D}_\delta$. Since $I_g^{c_\beta-\varepsilon_0}\cup I_g^0=I_g^{\max\left(c_\beta-\varepsilon_0,0\right)}$ and $(I_g^{c_\beta+\varepsilon_0}\cap\overline{H_1^2(M)\backslash\mathcal{D}_\delta})\cup I_g^0\cup\mathcal{D}_\delta=I_g^{c_\beta+\varepsilon_0}\cup\mathcal{D}_\delta$, by the above listed properties of the relative equivariant Lusternik–Schnirelmann category, it follows that

$$\gamma_{I_g^0\cup\mathcal{D}_\delta}\left(I_g^{c_\beta+\varepsilon_0}\cup\mathcal{D}_\delta\right)\leq\gamma_{I_g^0\cup\mathcal{D}_\delta}\left(I_g^{\max\left(c_\beta-\varepsilon_0,0\right)}\cup\mathcal{D}_\delta\right).$$

Whenever c_β is equal to 0 or not, this contradicts the definition of c_β. This proves the above claim, namely that if c_β is finite, then there exists a Palais–Smale sequence $(u_\alpha^{(\beta)})_\alpha$ for the functional I_g at level c_β such that the function $u_\alpha^{(\beta)}$ belongs to $\overline{H_1^2(M)\backslash\mathcal{D}_{\delta/2}}$ for all α. By Lemma 1.5, since there holds $d(u_\alpha^{(\beta)},\mathcal{P}\cup(-\mathcal{P}))\geq\delta/2$ for all α and since 0 is the only critical point of the functional I_g at level 0, we get that c_β cannot be equal to 0. We also get that in case $0<c_\beta\leq K_n^{-n}/n$, there exists a subsequence of $(u_\alpha^{(\beta)})_\alpha$ converging to a nontrivial nodal critical point of the functional I_g. Similarly, in case $K_n^{-n}/n<c_\beta<2K_n^{-n}/n$, there is at most one constant sign bubble in the decomposition of the sequence $(u_\alpha^{(\beta)})_\alpha$, thus either c_β or $c_\beta-K_n^{-n}/n$ is a critical level of the functional I_g.

Aiming to prove Theorem 1.1, we shall state four preliminary steps. The first one states as follows.

Step 2.3. *If there exists β such that $c_\beta=c_{\beta+1}<2K_n^{-n}/n$, then the functional I_g has infinitely many critical points at level c_β.*

Proof. We proceed by contradiction, and assume that the set K_{c_β} is finite. When K_{c_β} is not empty, there holds $\gamma(K_{c_\beta})=1$ and there exists a small positive real number θ such that there holds $\gamma(\mathcal{B}_{2\theta}\left(K_{c_\beta}\right))=\gamma(K_{c_\beta})=1$. We first consider the case $c_\beta\leq K_n^{-n}/n$. In that case, by the above discussion, the set K_{c_β} is not empty, and Palais–Smale sequences for the functional I_g at level c_β are compact in $H_1^2(M)$. In particular, there exists a real number ε in $(0,c_\beta)$ such that for any function u in $\overline{I_g^{-1}([c_\beta-\varepsilon,c_\beta+\varepsilon])\backslash\mathcal{B}_\theta\left(K_{c_\beta}\right)}$, there holds

$$\|\nabla I_g(u)\|_{H_1^2(M)}\geq\frac{2\varepsilon}{\theta}.$$

Applying Lemma 2.2 with $C=\overline{H_1^2(M)\backslash\mathcal{B}_{2\theta}\left(K_{c_\beta}\right)}$ and $D=I_g^0\cup\mathcal{D}_\delta$ then yields an odd, continuous map $\nu:\overline{I_g^{c_\beta+\varepsilon}\backslash\mathcal{B}_{2\theta}\left(K_{c_\beta}\right)}\cup I_g^0\cup\mathcal{D}_\delta\to I_g^{c_\beta-\varepsilon}\cup\mathcal{D}_\delta$ such that $\nu\equiv\text{id}$ in the set $I_g^0\cup\mathcal{D}_\delta$. By the definition of c_β and by the properties of the relative equivariant Lusternik–Schnirelmann category, it follows that

$$\begin{aligned}\beta+1 &\le \gamma_{I_g^0\cup\mathcal{D}_\delta}\left(I_g^{c_\beta+\varepsilon}\cup\mathcal{D}_\delta\right)\\ &\le \gamma_{I_g^0\cup\mathcal{D}_\delta}\left(\overline{I_g^{c_\beta+\varepsilon}\backslash\mathcal{B}_{2\theta}\left(K_{c_\beta}\right)}\cup I_g^0\cup\mathcal{D}_\delta\right)+\gamma\left(\mathcal{B}_{2\theta}\left(K_{c_\beta}\right)\right)\\ &\le \gamma_{I_g^0\cup\mathcal{D}_\delta}\left(I_g^{c_\beta-\varepsilon}\cup\mathcal{D}_\delta\right)+\gamma\left(\mathcal{B}_{2\theta}\left(K_{c_\beta}\right)\right)\\ &< \beta+\gamma\left(\mathcal{B}_{2\theta}\left(K_{c_\beta}\right)\right).\end{aligned}$$

This is in contradition with $\gamma(\mathcal{B}_{2\theta}\left(K_{c_\beta}\right))=1$. We now consider the case $c_\beta > K_n^{-n}/n$. We set

$$U_\theta=\mathcal{B}_{2\theta}\left(K_{c_\beta-K_n^{-n}/n}+P_\theta\right),$$

where

$$P_\theta=\left\{(\eta u_{\mu,0})\circ\exp_{x_0}^{-1};\ 0<\mu\le\theta\quad\text{and}\quad x_0\in M\right\},$$

where η is a smooth cutoff function as in Section 1.2 and where the functions $u_{\mu,0}$ are as in (1.8). We claim that if θ is small enough, then the sets U_θ and $-U_\theta$ are disjoint. In order to prove this claim, we proceed by contradiction, and assume that there exist sequences of functions u_α^1, u_α^2 in $K_{c_\beta-K_n^{-n}/n}$ and B_α^1, B_α^2 in $P_{1/\alpha}$ such that there holds

$$\left(u_\alpha^1+B_\alpha^1\right)-\left(u_\alpha^2-B_\alpha^2\right)\longrightarrow 0 \tag{2.12}$$

in $H_1^2(M)$ as $\alpha\to+\infty$. Passing if necessary to a subsequence, $(B_\alpha^1)_\alpha$ and $(B_\alpha^2)_\alpha$ are two positive bubbles. By taking into account that bubbles converge weakly to 0 and that sequences in $K_{c_\beta-K_n^{-n}/n}$ are compact in $H_1^2(M)$ since by assumption there holds $c_\beta-K_n^{-n}/n<K_n^{-n}/n$, it follows that up to a subsequence, $(u_\alpha^1)_\alpha$ and $(u_\alpha^2)_\alpha$ converge to the same limit in $H_1^2(M)$. This leads to a contradiction since by (2.12), the bubbles $(B_\alpha^1)_\alpha$ and $(B_\alpha^2)_\alpha$ would converge up to a subsequence to 0 in $H_1^2(M)$. We assumed here that the set $K_{c_\beta-K_n^{-n}/n}$ is not empty but the proof goes similarly otherwise. The above claim is proved, and we may now assume that θ is small enough so that the sets $\mathcal{B}_{2\theta}(K_{c_\beta})$, U_θ and $-U_\theta$ are mutually disjoint. For any positive real number θ', we adopt here the convention that $\mathcal{B}_{\theta'}(K_{c_\beta})=\emptyset$ when the set K_{c_β} is empty, and we set

$$Z_{\theta'}=\mathcal{B}_{\theta'}(K_{c_\beta})\cup U_{\theta'/2}\cup(-U_{\theta'/2}).$$

We proceed in the same way as in the first case. Since Palais–Smale sequences for the functional I_g at level c_β have at most one constant sign bubble in their decomposition, there exists a real number ε in $(0,c_\beta)$ such that for any function u in $\overline{I_g^{-1}\left([c_\beta-\varepsilon,c_\beta+\varepsilon]\right)\backslash Z_\theta}$, there holds

$$\|\nabla I_g(u)\|_{H_1^2(M)}\ge\frac{2\varepsilon}{\theta}.$$

Applying Lemma 2.2 with $C=\overline{H_1^2(M)\backslash Z_{2\theta}}$ and $D=I_g^0\cup\mathcal{D}_\delta$ then yields an odd, continuous map $\nu:\overline{I_g^{c_\beta+\varepsilon}\backslash Z_{2\theta}}\cup I_g^0\cup\mathcal{D}_\delta\to I_g^{c_\beta-\varepsilon}\cup\mathcal{D}_\delta$ such that $\nu\equiv\mathrm{id}$ in the set $I_g^0\cup\mathcal{D}_\delta$. By the definition of c_β and by the properties of the relative equivariant Lusternik–Schnirelmann category, it follows that

$$\begin{aligned}\beta+1 &\le \gamma_{I_g^0\cup\mathcal{D}_\delta}\left(I_g^{c_\beta+\varepsilon}\cup\mathcal{D}_\delta\right)\\ &\le \gamma_{I_g^0\cup\mathcal{D}_\delta}\left(\overline{I_g^{c_\beta+\varepsilon}\backslash Z_{2\theta}}\cup I_g^0\cup\mathcal{D}_\delta\right)+\gamma\left(Z_{2\theta}\right)\\ &\le \gamma_{I_g^0\cup\mathcal{D}_\delta}\left(I_g^{c_\beta-\varepsilon}\cup\mathcal{D}_\delta\right)+\gamma\left(Z_{2\theta}\right)\\ &< \beta+\gamma\left(Z_{2\theta}\right).\end{aligned}$$

Whenever the set K_{c_β} is empty or not, there holds $\gamma(Z_{2\theta}) = 1$, and the contradiction follows. This ends the proof of Step 2.3. □

We introduce the Nehari manifold $\mathcal{N}$ of the functional I_g defined by

$$\mathcal{N} = \left\{u \in H_1^2(M) \setminus \{0\}\,;\ DI_g(u).u = 0\right\}$$

and the radial projection $\varrho : H_1^2(M)\setminus\{0\} \to \mathcal{N}$ defined by

$$\varrho(u) = \left(\frac{\int_M |\nabla u|^2 dv_g + \int_M h u^2 dv_g}{\int_M |u|^{2^*} dv_g}\right)^{\frac{n-2}{4}} u\,.$$

For any function u in $\mathcal{N}$, there holds

$$I_g(\varrho(u)) = \max_{t \geq 0} I_g(tu)\,. \tag{2.13}$$

We also clearly get that $\varrho(tu) = \varrho(u)$ for all positive real numbers t and all functions u in $H_1^2(M)\setminus\{0\}$ and that $\varrho(u) = u$ for all functions u in $\mathcal{N}$. Moreover, by the coercivity of the operator $\Delta_g + h$ on $H_1^2(M)$ and by the continuity of the embedding of $H_1^2(M)$ into $L^{2^*}(M)$, there exists a positive constant E_0 such that for any function u in $\mathcal{N}$, there holds

$$\int_M |u|^{2^*} dv_g \geq E_0\,. \tag{2.14}$$

The second step in the proof of Theorem 1.1 is as follows. An example of positive real numbers μ_ε which satisfy $\mu_\varepsilon^{\varepsilon^2} = \mathrm{O}(\varepsilon)$ as $\varepsilon \to 0$ is $\mu_\varepsilon = e^{-\varepsilon^{-\theta}}$ for some $\theta > 2$.

Step 2.4. *For any point x in M and any real number ε in $(0, i_g)$, let $\psi(x,\varepsilon)$ be the function defined on M by*

$$\psi(x,\varepsilon)(y) = \frac{\eta\left(\varepsilon^{-1} d_g(x,y)\right)}{\left(\mu_\varepsilon + d_g(x,y)^2\right)^{\frac{n-2}{2}}}$$

where d_g is the geodesic distance on M with respect to the metric g, where η is a smooth cutoff function on $\mathbb{R}$ such that $\eta \equiv 1$ in $[-1/2, 1/2]$ and $\eta \equiv 0$ out of $[-1,1]$, and where μ_ε is a positive real number. In case $n = 4$, if there holds $\mu_\varepsilon^{\varepsilon^2} = \mathrm{O}(\varepsilon)$ as $\varepsilon \to 0$, then

$$I_g(\varrho(\psi(x,\varepsilon))) = \frac{1}{4} K_4^{-4} + \frac{K_4^{-4}}{16}\left(\mathrm{Scal}_g(x) - 6h(x)\right)\mu_\varepsilon \ln \mu_\varepsilon + \mathrm{o}\left(\mu_\varepsilon \ln \mu_\varepsilon\right) \tag{2.15}$$

as $\varepsilon \to 0$, uniformly with respect to x. In case $n > 4$, if there holds $\mu_\varepsilon = \mathrm{O}(\varepsilon^\theta)$ as $\varepsilon \to 0$ for some $\theta > 2\frac{n-2}{n-4}$, then

$$I_g(\varrho(\psi(x,\varepsilon))) = \frac{1}{n} K_n^{-n} - \frac{K_n^{-n}}{2n(n-4)}\left(\mathrm{Scal}_g(x) - \frac{4(n-1)}{n-2} h(x)\right)\mu_\varepsilon + \mathrm{o}(\mu_\varepsilon) \tag{2.16}$$

as $\varepsilon \to 0$, uniformly with respect to x.

Proof. We proceed as in Aubin [3] but with the tricky difference here that the supports of the functions $\psi(x,\varepsilon)$ have diameters of the order of ε instead of 1. For any point x in M and for positive real numbers r close to 0, there holds

$$\frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_x(r)} \sqrt{|g|}\, d\sigma = 1 - \frac{1}{6n} \mathrm{Scal}_g(x)\, r^2 + \mathrm{O}\left(r^4\right),$$

where $|g|$ is the determinant of the components of the metric g in geodesic normal coordinates. By standard properties of the exponential map, the remainder $\mathrm{O}(r^4)$ can be made uniform with respect to x. We set $I_p^q = \int_0^{+\infty} (1+r)^{-p} r^q dr$ for all positive real numbers p and q such that $p - q > 1$. In case $n = 4$, if there holds $\mu_\varepsilon^{\varepsilon^2} = \mathrm{O}(\varepsilon)$ as $\varepsilon \to 0$, then we compute

$$\int_M |\nabla\psi(x,\varepsilon)|_g^2 dv_g = 2\omega_3\mu_\varepsilon^{-1}\left(I_4^2 + \frac{1}{24}\operatorname{Scal}_g(x)\mu_\varepsilon\ln\mu_\varepsilon + \mathrm{o}(\mu_\varepsilon\ln\mu_\varepsilon)\right)$$

and

$$\int_M h\psi(x,\varepsilon)^2 dv_g = -\frac{\omega_3}{2}h(x)\ln\mu_\varepsilon + \mathrm{o}(\ln\mu_\varepsilon)$$

as $\varepsilon \to 0$, uniformly with respect to x. In case $n > 4$, if there holds $\mu_\varepsilon = \mathrm{O}(\varepsilon^\theta)$ as $\varepsilon \to 0$ for some $\theta > 2\frac{n-2}{n-4}$, then we compute

$$\int_M |\nabla\psi(x,\varepsilon)|_g^2 dv_g = \frac{(n-2)^2}{2}\omega_{n-1}I_n^{n/2}\mu_\varepsilon^{1-n/2}\left(1 - \frac{n+2}{6n(n-4)}\operatorname{Scal}_g(x)\mu_\varepsilon + \mathrm{o}(\mu_\varepsilon)\right)$$

and

$$\int_M h\psi(x,\varepsilon)^2 dv_g = \frac{2(n-2)(n-1)}{n(n-4)}\omega_{n-1}I_n^{n/2}h(x)\mu_\varepsilon^{2-n/2} + \mathrm{o}\left(\mu_\varepsilon^{2-n/2}\right)$$

as $\varepsilon \to 0$, uniformly with respect to x. In both cases, we also compute

$$\int_M \psi(x,\varepsilon)^{2^*} dv_g = \frac{\omega_{n-1}}{2}I_n^{n/2-1}\mu_\varepsilon^{-n/2}\left(1 - \frac{1}{6(n-2)}\operatorname{Scal}_g(x)\mu_\varepsilon + \mathrm{o}(\mu_\varepsilon)\right)$$

as $\varepsilon \to 0$, uniformly with respect to x. By noting that there hold

$$\frac{n-2}{n}I_n^{n/2} = I_n^{n/2-1} = \frac{\omega_n}{2^{n-1}\omega_{n-1}}$$

and

$$\frac{(n-2)^2}{2}\omega_{n-1}I_n^{n/2} = K_n^{-2}\left(\frac{n-2}{2n}\omega_{n-1}I_n^{n/2}\right)^{2/2^*},$$

and by writing

$$I_g(\varrho(\psi(x,\varepsilon))) = \frac{1}{n}\left(\frac{\int_M |\nabla\psi(x,\varepsilon)|_g^2 dv_g + \int_M h\psi(x,\varepsilon)^2 dv_g}{\left(\int_M \psi(x,\varepsilon)^{2^*} dv_g\right)^{2/2^*}}\right)^{n/2},$$

we then get (2.15) and (2.16). □

We let x_0 be as in Theorem 1.1, namely such that $h(x_0) < \frac{n-2}{4(n-1)}\operatorname{Scal}_g(x_0)$. The next step in the proof of Theorem 1.1 states as follows.

Step 2.5. *There exists an odd, continuous map $\Phi : \mathbb{R}^{n+2} \to H_1^2(M)$ such that*

$$\text{(i) } I_g \circ \Phi < \frac{2}{n}K_n^{-n},$$
$$\text{(ii) } \lim_{|y|\to+\infty} I_g \circ \Phi(y) = -\infty.$$

Proof. By Step 2.4, there exist ε_0 in $(0, i_g)$ and r_0 in $(0, i_g/3)$ such that for any real number ε in $(0, \varepsilon_0]$ and any point x in $\overline{B_{x_0}(2r_0)}$, there holds

$$I_g(\varrho(\psi(x,\varepsilon))) < \frac{1}{n}K_n^{-n}, \tag{2.17}$$

where the functions $\psi(x,\varepsilon)$ are as in Step 2.4. We then claim that there exist a real number ε_1 in $(0,\varepsilon_0)$ and a smooth cutoff function v such that $v \equiv 1$ in $B_{x_0}(\varepsilon_1)$, $v \equiv 0$ out of $B_{x_0}(\varepsilon_0)$, and such that there holds

$$I_g\left(\varrho\left((1-v)\,\psi(x,\varepsilon_0)\right)\right) < \frac{1}{n}K_n^{-n} \tag{2.18}$$

for all points x in the ball $\overline{B_{x_0}(2r_0)}$. In order to prove this claim, by standard properties of the capacities of balls, we first note that

$$\inf_{u\in\mathcal{H}_{\varepsilon,\varepsilon_0}}\left(\int_{B_{x_0}(\varepsilon_0)\backslash B_{x_0}(\varepsilon)} |\nabla u|_g^2\, dv_g\right) \longrightarrow 0 \tag{2.19}$$

as $\varepsilon \to 0$, where $\mathcal{H}_{\varepsilon,\varepsilon_0}$ is the set of all functions u in $H_1^2(M)$ such that $u \equiv 1$ in $B_{x_0}(\varepsilon)$ and $u \equiv 0$ out of $B_{x_0}(\varepsilon_0)$. We refer to Grigor′yan [29] for more details about (2.19). Moreover, the Poincaré inequality holds in $\mathcal{H}_{\varepsilon,\varepsilon_0}$ (see, for instance, Hebey [30]). In other words, there exists a positive constant C such that for any function u in $\mathcal{H}_{\varepsilon,\varepsilon_0}$, there holds $\|u\|_{L^2(M)} \le C\,\|\nabla u\|_{L^2(M)}$. The existence of a real number ε_1 and a smooth cutoff function v such that $v \equiv 1$ in $B_{x_0}(\varepsilon_1)$, $v \equiv 0$ out of $B_{x_0}(\varepsilon_0)$, and such that (2.18) holds true for all points x in the ball $\overline{B_{x_0}(r_0)}$ then follows from (2.17) and (2.19) by an easy density argument and by the continuity of the functionals in I_g. Without loss of generality, we may assume that r_0 is small enough so that there exists a constant $C_0 > 1$ such that there holds

$$|x-y| \le C_0 d_g\left(\exp_{x_0}(x), \exp_{x_0}(y)\right) \tag{2.20}$$

for all points x and y in the ball $B_0(r_0)$. We may assume moreover that ε_0 is small enough so that $2C_0\varepsilon_0 < r_0$. For any natural number $k > 0$, we let $\mathbb{B}^k$ be the unit ball and $\mathbb{S}^k$ be the unit sphere in $\mathbb{R}^{k+1}$. We define two maps $\varPhi_1, \varPhi_2 : \overline{\mathbb{B}^n} \to \mathcal{N}$ by

$$\varPhi_1(y) = \begin{cases} \varrho\left(\psi(x_0,\varepsilon_1)\right) & \text{if } |y| \le \dfrac{1}{2} \\ \varrho\left(\psi(x_1(y), \varepsilon(y))\right) & \text{otherwise,} \end{cases}$$

and

$$\varPhi_2(y) = \begin{cases} \varrho\left((1-v)\,\psi(x_2(y), \varepsilon_0)\right) & \text{if } |y| \le \dfrac{1}{2} \\ \varrho\left(\psi(x_3(y), \varepsilon_0)\right) & \text{otherwise,} \end{cases}$$

where $\varepsilon(y) = 2(\varepsilon_0 - \varepsilon_1)|y| + 2\varepsilon_1 - \varepsilon_0$ and where

$$x_1(y) = \exp_{x_0}\left(-2C_0\varepsilon_0\left(2 - \frac{1}{|y|}\right)y\right),\ x_2(y) = \exp_{x_0}(4C_0\varepsilon_0 y),\ x_3(y) = \exp_{x_0}\left(2C_0\varepsilon_0\frac{y}{|y|}\right).$$

In particular, for any point y such that $1/2 \le |y| \le 1$, the real number $\varepsilon(y)$ belongs to $[\varepsilon_1,\varepsilon_0]$ and the point $x_i(y)$ belongs to the ball $\overline{B_{x_0}(2r_0)}$ for $i = 1,2,3$. For any point y such that $|y| = 1/2$, there hold $x_1(y) = x_0$ and $\varepsilon(y) = \varepsilon_1$. It follows that the map $\varPhi_1$ is continuous. Similarly, for any point y such that $|y| = 1/2$, there hold $x_2(y) = x_3(y)$ and $v \equiv 0$ in $B_{x_2(y)}(\varepsilon_0)$ since $d_g(x_0, x_2(y)) = 2C_0\varepsilon_0 > \varepsilon(y) + \varepsilon_0$. It follows that the map $\varPhi_2$ is continuous. We then show that for any point y in $\overline{\mathbb{B}^n}$, there holds

$$\operatorname{Supp}\varPhi_1(y) \cap \operatorname{Supp}\varPhi_2(y) = \emptyset\,. \tag{2.21}$$

If $|y| \le 1/2$, then $\operatorname{Supp}\varPhi_1(y) = \overline{B_{x_0}(\varepsilon_1)}$ and $\operatorname{Supp}\varPhi_2(y) \subset \overline{B_{x_2(y)}(\varepsilon_0)}\backslash B_{x_0}(\varepsilon_1)$, and thus (2.21) holds true. If $|y| \ge 1/2$, then $\operatorname{Supp}\varPhi_1(y) = \overline{B_{x_1(y)}(\varepsilon(y))}$ and $\operatorname{Supp}\varPhi_2(y) = \overline{B_{x_3(y)}(\varepsilon_0)}$ while by (2.20), we get $d_g(x_1(y), x_3(y)) \ge 4\varepsilon_0|y|$, and it follows that here again (2.21) holds true. As a last remark on the maps $\varPhi_1$ and $\varPhi_2$, there holds $\varPhi_1(y) = \varPhi_2(-y)$ for all points y in $\mathbb{S}^{n-1}$. We now define the map $\varPhi_0 : \mathbb{S}^n \to \mathcal{N}$ by

$$\varPhi_0(y_1, \dots, y_{n+1}) = \begin{cases} \varPhi_1(y_1, \dots, y_n) & \text{if } y_{n+1} \geq 0 \\ \varPhi_2(-y_1, \dots, -y_n) & \text{otherwise.} \end{cases}$$

It is easily checked that $\varPhi_0$ is also continuous. We then introduce the map $\widetilde{\varPhi} : (\mathbb{S}^n \times (-1,1)) \cup (\mathbb{B}^{n+1} \times \{-1,1\}) \to H_1^2(M) \setminus \{0\}$ defined by

$$\widetilde{\varPhi}(y,t) = \begin{cases} (1+t)\varPhi_0(y) - (1-t)\varPhi_0(-y) & \text{if } y \in \mathbb{S}^n, \\ 2|y|\varPhi_0\left(\dfrac{y}{|y|}\right) + (1-|y|)\varrho(\psi(y_0, \varepsilon_0)) & \text{if } t = 1, \\ -2|y|\varPhi_0\left(-\dfrac{y}{|y|}\right) - (1-|y|)\varrho(\psi(y_0, \varepsilon_0)) & \text{if } t = -1, \end{cases}$$

where $y_0 = \exp_{x_0}(2r_0\theta_0)$ for some point θ_0 in $\mathbb{S}^n$. By noting that for any point y in $\overline{\mathbb{B}^n}$, the supports of the functions $\varPhi_1(y)$ and $\varPhi_2(y)$ are included in the ball $B_{x_0}(2C_0\varepsilon_0 + \varepsilon_0)$, and since $d_g(x_0, y_0) = 2r_0 > 2C_0\varepsilon_0 + 2\varepsilon_0$, we get

$$\operatorname{Supp}\varrho(\psi(y_0, \varepsilon_0)) \cap \operatorname{Supp}\varPhi_1(y) = \emptyset \tag{2.22}$$

and

$$\operatorname{Supp}\varrho(\psi(y_0, \varepsilon_0)) \cap \operatorname{Supp}\varPhi_2(y) = \emptyset. \tag{2.23}$$

By (2.21), (2.22), and (2.23), the supports of the functions $\varPhi_1(y)$, $\varPhi_2(y)$ and $\varrho(\psi(y_0, \varepsilon_0))$ are mutually disjoint for all points y in $\overline{\mathbb{B}^n}$, thus $\widetilde{\varPhi}$ takes its values in $H_1^2(M) \setminus \{0\}$. It is easily checked that $\widetilde{\varPhi}$ is odd and continuous. By taking into account that the domain of definition of the map $\widetilde{\varPhi}$ is precisely the boundary of the set $\mathbb{B}^{n+1} \times (-1,1)$, we may define the radial extension of $\widetilde{\varPhi}$ as the map $\varPhi : \mathbb{R}^{n+2} \to H_1^2(M) \setminus \{0\}$ given by $\varPhi(ty) = t\widetilde{\varPhi}(y)$ for all positive real numbers t and for all points y in $\partial(\mathbb{B}^{n+1} \times (-1,1))$. The map $\varPhi$ is then odd and continuous. By (2.13), (2.21), (2.22), and (2.23), property (i) follows from (2.17) and (2.18). By (2.14), we get

$$\max_{u \in \mathcal{N}} I_g(tu) \to -\infty$$

as $t \to +\infty$. By (2.21) and (2.22), and (2.23), we then also get property (ii). This ends the proof of Step 2.5. ☐

The last ingredient we need in the proof of Theorem 1.1 is as follows.

Step 2.6. *There holds $c_{n+1} < 2K_n^{-n}/n$.*

Proof. We set

$$k = \gamma_{I_g^0 \cup \mathcal{D}_\delta}\left(I_g^{\sup(I_g \circ \varPhi)} \cup \mathcal{D}_\delta\right),$$

where $\varPhi$ is the map we get in Step 2.5. We may assume that k is finite. We purpose to prove that k is greater than or equal to $n+1$. Step 2.6 then obviously follows from Step 2.5. By the definition of k, there exist $k+1$ symmetric, open subsets $U_0, \dots, U_k$ of $H_1^2(M)$ which cover $I_g^{\sup(I_g \circ \varPhi)} \cup \mathcal{D}_\delta$ and such that $\left(I_g^0 \cup \mathcal{D}_\delta\right) \subset U_0$ and $k+1$ odd, continuous maps $\chi_i : U_i \to \{-1,1\}$, $i = 1, \dots, k$ and $\chi_0 : U_0 \to I_g^0 \cup \mathcal{D}_\delta$ such that $\chi_0 \equiv \mathrm{id}$ in the set $I_g^0 \cup \mathcal{D}_\delta$. Up to a restriction of U_0, by using Dugundji's extension of Tietze's theorem (see Dugundji [25]), we may extend the map χ_0 into an odd, continuous map still denoted χ_0, defined from the whole Sobolev space $H_1^2(M)$ into itself. We show that there exists an odd, continuous map $\chi_{k+1} : \mathcal{N} \cap \mathcal{D}_\delta \to \{-1,1\}$. We let $\mathcal{E}$ be the set of all functions in the Nehari manifold $\mathcal{N}$ whose positive and negative parts also belong to $\mathcal{N}$. For any functions u and v in $\mathcal{P}$ with disjoint support, the function

$\varrho(u) - \varrho(v)$ belongs to the set $\mathcal{E}$. The distance between $\mathcal{E}$ and $\mathcal{P} \cup (-\mathcal{P})$ is positive. Indeed, by the continuity of the embedding of $H_1^2(M)$ into $L^{2^*}(M)$, we get that there exists a positive constant C such that for any u in $\mathcal{E}$ and v in $\mathcal{P}$, there holds

$$\|u \pm v\|_{H_1^2(M)} \geq C \|u \pm v\|_{L^{2^*}(M)} \geq C \left\|u^\pm\right\|_{L^{2^*}(M)} \geq C E_0^{1/2^*},$$

where E_0 is as in (2.14). Decreasing δ if necessary, we may now assume that the sets $\mathcal{E}$ and $\mathcal{D}_\delta$ are disjoint. In the same way as in Castro–Cossio–Neuberger [9, Lemma 2.5], we get that the set $\mathcal{N} \backslash \mathcal{E}$ consists in two connected components, namely $\{u \in \mathcal{N};\ u \geq 0 \text{ or } DI_g(u^+).u^+ < 0\}$ and its symmetric. Therefore, the set $\mathcal{N} \cap \mathcal{D}_\delta$ also consists in two connected components. It follows that there exists an odd, continuous map $\chi_{k+1} : \mathcal{N} \cap \mathcal{D}_\delta \to \{-1, 1\}$. We let $\mathcal{O}$ be the inverse image by the map $\chi_0 \circ \Phi$ of the connected component of $H_1^2(M) \backslash \mathcal{N}$ which contains 0. By Step 2.5 (ii), $\mathcal{O}$ is a symmetric, bounded, open neighbourhood of 0. The boundary of $\mathcal{O}$ is covered by the sets $\partial\mathcal{O} \cap \Phi^{-1}(U_i)$, $i = 0, \dots, k$. Taking a partition of unity consisting of even functions $\{\pi_0, \dots, \pi_k\}$ subordinated to this covering, we define the map $\chi : \partial\mathcal{O} \to \mathbb{R}^{k+1}$ by

$$\chi(y) = \pi_0(y)\, \chi_{k+1} \circ \chi_0 \circ \Phi(y)\, e_{k+1} + \sum_{i=1}^{k} \pi_i(y)\, \chi_i \circ \Phi(y)\, e_i\,,$$

where e_i is the i-th vector in the canonical basis of $\mathbb{R}^{k+1}$. This map is odd, continuous, and nowhere vanishing. By the Borsuk–Ulam theorem (see, for instance, Kavian [37]), it follows that $k + 1$ is greater than or equal to $n + 2$, and this ends the proof of Step 2.6. □

We let c_0 stand for the minimum of the functional I_g on its Nehari manifold. As a preliminary remark on c_0, by Step 2.4, we get $c_0 < K_n^{-n}/n$. By reasoning as in Aubin [3], we then get that c_0 is reached for a positive solution of equation (1.1). Moreover, c_0 can only be reached for constant sign solutions. Indeed, if c_0 was reached for a nodal solution u, then it is easily seen that the function $|u|$ would also be a solution of equation (1.1), and this would contradict the maximum principle.

We now prove Theorem 1.1 by using the above preliminary steps.

Proof of Theorem 1.1. By Steps 2.3 and 2.6, we may assume that the sequence $(c_1, \dots, c_{n+1})$ is increasing and strictly bounded from above by $2K_n^{-n}/n$. We let k be the greater index such that $c_k \leq K_n^{-n}$. If $k \geq 1$, then for $\beta = 1, \dots, k$, c_β is a critical level of the functional I_g for a nontrivial nodal solution of equation (1.1). In particular, we get $c_1 > c_0$. It follows that there exist at least $k + 1$ distinct critical levels of the functional I_g less than or equal to K_n^{-n}/n. Moreover, for $\beta = k + 1, \dots, n + 1$, either c_β or $c_\beta - K_n^{-n}/n$ is a critical level of the functional I_g, thus we also get the existence of $n + 1 - k$ distinct critical levels of I_g in $(0, 2K_n^{-n}/n)$. We finally conclude that there exist at least $\frac{(k+1)+(n+1-k)}{2} = \frac{n+2}{2}$ distinct critical levels of the functional I_g in $(0, 2K_n^{-n}/n)$. □

As a remark, the above proof yields the more precise following result. Namely that if we denote by p the number of pairs of positive solutions of equation (1.1) with energy less than $2K_n^{-n}$ and by q_1 (resp. q_2) the number of pairs of nontrivial nodal solutions of equation (1.1) with energy less than or equal to K_n^{-n} (resp. greater than or equal to K_n^{-n}), then there holds $p + 2q_1 + q_2 \geq n + 2$. In particular, if there does not exist any nontrivial nodal solution of equation (1.1) with energy less than or equal to K_n^{-n}, then there exist at least $n + 2$ pairs of nontrivial solutions of equation (1.1) with energy less than $2K_n^{-n}$.

As another remark, by slightly modifying the asymptotic expansions in (2.15) and (2.16), we could also have included the case of the geometric equation

$$\Delta_g u + \frac{n-2}{4(n-1)} \operatorname{Scal}_g u = |u|^{2^*-2} u$$

when $n \geq 6$ and the manifold (M, g) is non-locally conformally flat.

3. The case of locally conformally flat manifolds of high dimension

We prove Theorems 1.2 and 1.3 in this section. We start with the proof of Theorem 1.2. For this purpose, we let $(p_\alpha)_\alpha$ be a sequence in $[2, 2^*]$ such that $p_\alpha \to 2^*$ as $\alpha \to +\infty$, h be an Hölder continuous function on M, and we consider the family of equations

$$\Delta_g u + hu = |u|^{p_\alpha - 2} u\,. \tag{3.1}$$

A preliminary result we easily get by following the lines in Solimini [55] and Devillanova–Solimini [17] is Lemma 3.1 below. We refer also to Robert [48] and Struwe [56] for related references.

Lemma 3.1. *Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ and h be a Hölder continuous function on M.For any bounded sequence $(u_\alpha)_\alpha$ in $H_1^2(M)$ of solutions for the family of equations* (3.1)*, there exist a solution u_∞ of equation* (1.1)*, a natural number k, bubbles $(B_\alpha^1)_\alpha, \dots, (B_\alpha^k)_\alpha$, and real numbers $a_1, \dots, a_k$ greater than or equal to 1 such that up to a subsequence,*

$$u_\alpha = u_\infty + \sum_{i=1}^k a_i B_\alpha^i + R_\alpha$$

for all α, where $R_\alpha \to 0$ in $H_1^2(M)$ as $\alpha \to +\infty$.

Moreover, together with Lemma 3.1, we can also assume that there holds

$$\frac{\mu_\alpha^i}{\mu_\alpha^j} + \frac{\mu_\alpha^j}{\mu_\alpha^i} + \frac{d_g\left(x_\alpha^i, x_\alpha^j\right)^2}{\mu_\alpha^i \mu_\alpha^j} \longrightarrow +\infty \tag{3.2}$$

as $\alpha \to +\infty$, for all distinct $i, j = 1, \dots, k$, where $(x_\alpha^i)_\alpha$ and $(\mu_\alpha^i)_\alpha$ stand for the centers and the weights of the bubble $(B_\alpha^i)_\alpha$ in Lemma 3.1.

We prove integral estimates in what follows, then we prove local estimates, and at last we prove Theorems 1.2 and 1.3.

3.1. **Integral estimates**

In the following, we let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ and h be a Hölder continuous function on M. For the moment, we do not need to assume that (M, g) is locally conformally flat nor to restrict neither the dimension of M, nor the regularity of h. We let $(u_\alpha)_\alpha$ be a bounded sequence in $H_1^2(M)$ of solutions for the family of equations (3.1), and we assume that $(u_\alpha)_\alpha$ blows up as $\alpha \to +\infty$, that is to say the natural number k in Lemma 3.1 is not zero. For $i = 1, \dots, k$, we let $(x_\alpha^i)_\alpha$ and $(\mu_\alpha^i)_\alpha$ be the centers and the weights of the bubble $(B_\alpha^i)_\alpha$ in Lemma 3.1. Renumbering and passing if necessary to a subsequence, we may assume that

$$\mu_\alpha^1 = \max_{1 \leq i \leq k} \mu_\alpha^i$$

for all α. Then, we let μ_α stand for μ_α^1 and x_α stand for x_α^1 for all α.

For any real number p_1 and p_2 such that $1 \le p_2 < 2^* < p_1$ and for any positive real number σ, we define the norm $\|.\|_{p_1,p_2,\sigma}$ on $L^\infty(M)$ by

$$\|u\|_{p_1,p_2,\sigma} = \inf\{C > 0;\ \exists u_1, u_2 \in L^\infty(M) \text{ s.t. } |u| \le u_1 + u_2\,,$$
$$\|u_1\|_{L^{p_1}(M)} \le C\,, \quad \text{and} \quad \|u_2\|_{L^{p_2}(M)} \le C\sigma^{n/2^*-n/p_2}\}.$$

We now fix a positive real number a. For any α and any positive real number ε, we define the function u_α^ε on M by

$$u_\alpha^\varepsilon = \sqrt{\varepsilon^2 + u_\alpha^2}\,. \tag{3.3}$$

We then compute

$$\Delta_g u_\alpha^\varepsilon = \frac{u_\alpha \Delta_g u_\alpha}{\sqrt{\varepsilon^2 + u_\alpha^2}} - \frac{\varepsilon^2 |\nabla u_\alpha|^2}{(\varepsilon^2 + u_\alpha^2)^{3/2}} \le \frac{u_\alpha \Delta_g u_\alpha}{\sqrt{\varepsilon^2 + u_\alpha^2}}\,.$$

It follows that there exists two constants A and B such that for any α and ε, there holds

$$\Delta_g u_\alpha^\varepsilon + a u_\alpha^\varepsilon \le A |u_\alpha|^{2^*-1} + B\,. \tag{3.4}$$

We even get that for any real number $A > 1$, there exists $B > 0$ such that (3.4) holds true for all α and ε. For α fixed, there holds $u_\alpha^\varepsilon \to |u_\alpha|$ in $C^0(M)$ as $\varepsilon \to 0$.

For the sake of completeness, we also prove the following result. Namely that for any $p > 1$, if a function v in $H_2^p(M)$ and a function f in $L^p(M)$ satisfy the equation $\Delta_g v + av = f$, then there exists a positive constant C independent of v and f such that there holds

$$\|v\|_{H_2^p(M)} \le C \|f\|_{L^p(M)}\,. \tag{3.5}$$

By standard elliptic theory (see, for instance, Gilbarg–Trudinger [28, Theorem 9.11]), we get

$$\|v\|_{H_2^p(M)} \le C\left(\|f\|_{L^p(M)} + \|v\|_{L^p(M)}\right)\,. \tag{3.6}$$

Therefore, it suffices to prove that there holds $\|v\|_{L^p(M)} \le C\|f\|_{L^p(M)}$. We proceed by contradiction, and assume that there exists a sequence $(v_\alpha)_\alpha$ in $H_2^p(M)$ such that $\|v_\alpha\|_{L^p(M)} = 1$ for all α and $\|\Delta_g v_\alpha + a v_\alpha\|_{L^p(M)} \to 0$ as $\alpha \to +\infty$. By (3.6), the sequence $(v_\alpha)_\alpha$ is bounded in $H_2^p(M)$. By the compactness of the embedding of $H_2^p(M)$ into $H_1^p(M)$, it follows that $(v_\alpha)_\alpha$ converges up to a subsequence in $H_1^p(M)$ to a function v_∞. Passing to the limit as $\alpha \to +\infty$ yields $\|v_\infty\|_{L^p(M)} = 1$ and $\Delta_g v_\infty + a v_\infty = 0$, but this last equation implies that $v_\infty = 0$ since a is positive, and the contradiction follows. This proves that there exists a positive constant C such that (3.5) holds true for all functions v in $H_2^p(M)$ and all functions f in $L^p(M)$ which satisfy the equation $\Delta_g v + av = f$.

We purpose to prove the following integral estimates.

Lemma 3.2. *Let (M,g) be a smooth, compact Riemannian manifold of dimension $n \ge 3$, h be a Hölder continuous function on M, and $(u_\alpha)_\alpha$ be a bounded sequence in $H_1^2(M)$ of solutions for the family of equations* (3.1)*. Let p_1 and p_2 be two real numbers such that $2^*/2 < p_2 < 2^* < p_1$. If the sequence $(u_\alpha)_\alpha$ blows up as $\alpha \to +\infty$, then up to a subsequence, there holds*

$$\|u_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}} = \mathrm{O}(1) \tag{3.7}$$

as $\alpha \to +\infty$.

In the sequel, $p(n)$ denotes $\frac{n2^*}{n-2.2^*}$ in case $n \ge 7$ and $+\infty$ in case $n < 7$. Aiming to prove Lemma 3.2, we shall state two preliminary steps. The first one is as follows.

Step 3.3. (3.7) *holds true in case $p_1 < p(n)$ and $p_2 > \max(\frac{n2^*}{n+2.2^*}, \frac{2^*}{2})$.*

Proof. We fix a constant $C > 1$. For any α, there exist two functions u_α^1 and u_α^2 in $L^\infty(M)$ such that there holds $|u_\alpha| \le u_\alpha^1 + u_\alpha^2$ and such that

$$\left\|u_\alpha^1\right\|_{L^{p_1}(M)} \le C \left\|u_\alpha\right\|_{p_1,p_2,\mu_\alpha^{-1}} \tag{3.8}$$

and

$$\left\|u_\alpha^2\right\|_{L^{p_2}(M)} \le C \left\|u_\alpha\right\|_{p_1,p_2,\mu_\alpha^{-1}} \mu_\alpha^{n/p_2 - n/2^*}. \tag{3.9}$$

We let $\mathcal{G}$ be the Green's function of the operator $\Delta_g + a$. This function is positive. By (3.4), letting $\varepsilon \to 0$, we get

$$|u_\alpha(x)| \le \int_M \mathcal{G}(x,.) \left(A |u_\alpha|^{2^*-1} + B\right) dv_g$$

for all points x in M and all α. Writing $|u_\alpha|^{2^*-1} = |u_\alpha|^{2^*-2}|u_\alpha|$ and decomposing the functions u_α as in Lemma 3.1 yield

$$|u_\alpha| \le A'\left(v_\alpha^1 + v_\alpha^2 + w_\alpha\right) + B\int_M \mathcal{G}(\cdot,y)\, dv_g(y), \tag{3.10}$$

where $A' > A$ does not depend on α and where

$$v_\alpha^1(x) = \int_M \mathcal{G}(x,.) |u_\infty|^{2^*-2} |u_\alpha|\, dv_g,$$

$$v_\alpha^2(x) = \sum_{i=1}^k a_i^{2^*-2} \int_M \mathcal{G}(x,.) \left|B_\alpha^i\right|^{2^*-2} |u_\alpha|\, dv_g,$$

$$w_\alpha(x) = \int_M \mathcal{G}(x,.) |R_\alpha|^{2^*-2} |u_\alpha|\, dv_g.$$

We are led to estimate the norm $\|.\|_{p_1,p_2,\mu_\alpha^{-1}}$ of the terms in the right hand side of (3.10). We first consider the functions w_α. We let w_α^1 and w_α^2 be two functions in $H_1^2(M)$ which satisfy the equations

$$\Delta_g w_\alpha^1 + a w_\alpha^1 = |R_\alpha|^{2^*-2} u_\alpha^1 \tag{3.11}$$

and

$$\Delta_g w_\alpha^2 + a w_\alpha^2 = |R_\alpha|^{2^*-2} u_\alpha^2. \tag{3.12}$$

By standard elliptic regularity, we get that the functions w_α^1 and w_α^2 belong to $H_2^p(M)$ for all real numbers $p \ge 1$ and then to $L^\infty(M)$. We write

$$\Delta_g w_\alpha + a w_\alpha = |R_\alpha|^{2^*-2} |u_\alpha| \le (\Delta_g + a)\left(w_\alpha^1 + w_\alpha^2\right).$$

By the maximum principle, it follows that $w_\alpha \le w_\alpha^1 + w_\alpha^2$. Moreover, for both $i = 1, 2$, if we assume that $p_i > 2^*/2$, then the continuity of the embedding of $H_2^{np_i/(n+2p_i)}(M)$ into $L^{p_i}(M)$ and elliptic theory as in (3.5) for equations (3.11) and (3.12) yield

$$\left\|w_\alpha^i\right\|_{L^{p_i}(M)} = \mathrm{O}\left(\left\|w_\alpha^i\right\|_{H_2^{\frac{np_i}{n+2p_i}}(M)}\right) = \mathrm{O}\left(\left\||R_\alpha|^{2^*-2} u_\alpha^i\right\|_{L^{\frac{np_i}{n+2p_i}}(M)}\right) \tag{3.13}$$

as $\alpha \to +\infty$. Hölder's inequality gives

$$\left\||R_\alpha|^{2^*-2} u_\alpha^i\right\|_{L^{\frac{np_i}{n+2p_i}}(M)} \le \|R_\alpha\|_{L^{2^*}(M)}^{2^*-2} \left\|u_\alpha^i\right\|_{L^{p_i}(M)} = \mathrm{o}\left(\left\|u_\alpha^i\right\|_{L^{p_i}(M)}\right) \tag{3.14}$$

as $\alpha \to +\infty$. By (3.8), (3.9), (3.13), and (3.14), we get

$$\|w_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}} = \mathrm{o}\left(\|u_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}}\right) \tag{3.15}$$

as $\alpha \to +\infty$. We now consider the functions v_α^1 and v_α^2. They satisfy the equations

$$\Delta_g v_\alpha^1 + a v_\alpha^1 = |u_\infty|^{2^*-2} |u_\alpha|$$

and

$$\Delta_g v_\alpha^2 + a v_\alpha^2 = \sum_{i=1}^k a_i^{2^*-2} \left|B_\alpha^i\right|^{2^*-2} |u_\alpha| \ .$$

Here again, the Sobolev embeddings and elliptic theory yield

$$\left\|v_\alpha^1\right\|_{L^{p_1}(M)} = \mathrm{O}\left(\left\||u_\infty|^{2^*-2} u_\alpha\right\|_{L^{\frac{np_1}{n+2p_1}}(M)}\right) \tag{3.16}$$

and

$$\left\|v_\alpha^2\right\|_{L^{p_2}(M)} = \mathrm{O}\left(\left\|\sum_{i=1}^k a_i^{2^*-2} \left|B_\alpha^i\right|^{2^*-2} u_\alpha\right\|_{L^{\frac{np_2}{n+2p_2}}(M)}\right) \tag{3.17}$$

as $\alpha \to +\infty$. On the one hand, if we assume that $p_1 < p(n)$, then $H_1^2(M)$ embeds into $L^{np_1/(n+2p_1)}(M)$, and thus the sequence $(u_\alpha)_\alpha$ remains bounded in $L^{np_1/(n+2p_1)}(M)$. By (3.16), it follows that

$$\left\|v_\alpha^1\right\|_{L^{p_1}(M)} = \mathrm{O}(1) \tag{3.18}$$

as $\alpha \to +\infty$. On the other hand, assuming $p_2 > \frac{n2^*}{n+2.2^*}$ allows us to apply Hölder's inequality in order to get

$$\left\|\left|B_\alpha^i\right|^{2^*-2} u_\alpha\right\|_{L^{\frac{np_2}{n+2p_2}}(M)} \le \left\|B_\alpha^i\right\|_{L^{q(2^*-2)}(M)}^{2^*-2} \|u_\alpha\|_{L^{2^*}(M)} \le C \left\|B_\alpha^i\right\|_{L^{q(2^*-2)}(M)}^{2^*-2} \tag{3.19}$$

for all α and for $i = 1, \dots, k$, where q is such that $\frac{1}{q} + \frac{1}{2^*} = \frac{n+2p_2}{np_2}$ and where C is a positive constant independent of α and i which existence is ensured by the boundedness of the sequence $(u_\alpha)_\alpha$ in $H_1^2(M)$ and the continuity of the embedding of $H_1^2(M)$ into $L^{2^*}(M)$. For $i = 1, \dots, k$ and for any real number δ in $(0, i_g/3)$, one can easily check

$$\int_{M \backslash B_{x_\alpha^i}(\delta)} \left|B_\alpha^i\right|^{(2^*-2)q} dv_g = \mathrm{O}\left(\left(\mu_\alpha^i\right)^{2q}\right) \tag{3.20}$$

as $\alpha \to +\infty$. By taking into account that our assumption $p_2 > \frac{n2^*}{n+2.2^*}$ implies that $q > n/4$, another easy computation yields

$$\int_{B_{x_\alpha^i}(\delta)} \left|B_\alpha^i\right|^{(2^*-2)q} dv_g = \mathrm{O}\left(\left(\mu_\alpha^i\right)^{qn/p_2 - qn/2^*}\right) \tag{3.21}$$

as $\alpha \to +\infty$, and thus summing (3.20) with (3.21) gives

$$\int_M \left|B_\alpha^i\right|^{(2^*-2)q} dv_g = \mathrm{O}\left(\left(\mu_\alpha^i\right)^{qn/p_2 - qn/2^*}\right) \tag{3.22}$$

as $\alpha \to +\infty$. We note that for any α, there holds $(\mu_\alpha^i)^{n/p_2 - n/2^*} \le \mu_\alpha^{n/p_2 - n/2^*}$ since μ_α is the largest weight of the bubbles $(B_\alpha^1)_\alpha, \dots, \left(B_\alpha^k\right)_\alpha$ and since $p_2 < 2^*$. By (3.17), (3.19), and (3.22), it follows that

$$\left\|v_\alpha^2\right\|_{L^{p_2}(M)} = \mathrm{O}\left(\mu_\alpha^{n/p_2 - n/2^*}\right) \tag{3.23}$$

as $\alpha \to +\infty$. By (3.18) and (3.23), we get

$$\left\|v_\alpha^1 + v_\alpha^2\right\|_{p_1, p_2, \mu_\alpha^{-1}} = \mathrm{O}(1) \tag{3.24}$$

as $\alpha \to +\infty$. Finally, by (3.10), (3.15), and (3.24), we get that there holds $\|u_\alpha\|_{p_1, p_2, \mu_\alpha^{-1}} = \mathrm{O}(1)$ as $\alpha \to +\infty$. This ends the proof of Step 3.3. ☐

The second step in the proof of Lemma 3.2 states as follows.

Step 3.4. *If* (3.7) *holds true for some* $p_1 < n(2^*-1)/2$ *and* $p_2 > 2^*-1$, *then*

$$\|u_\alpha\|_{f(p_1),f(p_2),\mu_\alpha^{-1}} = \mathrm{O}(1) \tag{3.25}$$

as $\alpha \to +\infty$, *where* $f(p) = \frac{np}{n(2^*-1)-2p}$.

Proof. Let p_1 and p_2 satisfy $2^*-1 < p_2 < 2^* < p_1 < n(2^*-1)/2$. We fix a constant $C > 1$. For any α, there exist two functions u_α^1 and u_α^2 in $L^\infty(M)$ such that $|u_\alpha| \le u_\alpha^1 + u_\alpha^2$ and such that there hold

$$\left\|u_\alpha^1\right\|_{L^{p_1}(M)} \le C\|u_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}} \tag{3.26}$$

and

$$\left\|u_\alpha^2\right\|_{L^{p_2}(M)} \le C\|u_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}}\,\mu_\alpha^{n/p_2-n/2^*}. \tag{3.27}$$

We let v_α^1 and v_α^2 be two functions in $H_1^2(M)$ which satisfy the equations

$$\Delta_g v_\alpha^1 + a v_\alpha^1 = 2^{2^*-2}A\left(u_\alpha^1\right)^{2^*-1} + B \tag{3.28}$$

and

$$\Delta_g v_\alpha^2 + a v_\alpha^2 = 2^{2^*-2}A\left(u_\alpha^2\right)^{2^*-1}. \tag{3.29}$$

where A and B are as in (3.4). By standard elliptic regularity, we get that the functions v_α^1 and v_α^2 belong to $H_2^p(M)$ for all real numbers $p \ge 1$ and then to $L^\infty(M)$. By (3.4), letting $\varepsilon \to 0$, we also get

$$\Delta_g|u_\alpha| + a|u_\alpha| \le (\Delta_g + a)\left(v_\alpha^1 + v_\alpha^2\right).$$

By the maximum principle, it follows that $|u_\alpha| \le v_\alpha^1 + v_\alpha^2$. Moreover, for both $i = 1, 2$, since we assumed that $2^*-1 < p_i < n(2^*-1)/2$, the continuity of the embedding of $H_2^{p_i/(2^*-1)}(M)$ into $L^{f(p_i)}(M)$ and elliptic theory as in (3.5) give

$$\left\|v_\alpha^i\right\|_{L^{f(p_i)}(M)} = \mathrm{O}\left(\left\|v_\alpha^i\right\|_{H_2^{\frac{p_i}{(2^*-1)}}(M)}\right) = \mathrm{O}\left(\left\|\Delta_g v_\alpha^i + a v_\alpha^i\right\|_{L^{\frac{p_i}{(2^*-1)}}(M)}\right)$$

as $\alpha \to +\infty$. By (3.26) and (3.28), it follows that

$$\left\|v_\alpha^1\right\|_{L^{f(p_1)}(M)} = \mathrm{O}\left(\left\|u_\alpha^1\right\|_{L^{p_1}(M)}^{2^*-1} + 1\right) = \mathrm{O}\left(\|u_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}}^{2^*-1} + 1\right) \tag{3.30}$$

as $\alpha \to +\infty$. Analogously, by (3.27) and (3.29), we get

$$\left\|v_\alpha^2\right\|_{L^{f(p_1)}(M)} = \mathrm{O}\left(\left\|u_\alpha^2\right\|_{L^{p_2}(M)}^{2^*-1}\right) = \mathrm{O}\left(\|u_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}}^{2^*-1}\,\mu_\alpha^{n/f(p_2)-n/2^*}\right). \tag{3.31}$$

If we assume that (3.7) holds true for our fixed p_1 and p_2, then (3.25) finally follows from (3.30) and (3.31). ☐

We prove Lemma 3.2 by induction, by using the initialization Step 3.3 and the bootstrap Step 3.4.

Proof of Lemma 3.2. We define $f : (0, n(2^*-1)/2) \to (0, +\infty)$ by $f(p) = \frac{np}{n(2^*-1)-2p}$. This function is increasing, and realizes a bijection from $(0, n(2^*-1)/2)$ onto $(0, +\infty)$. We let $q_\beta^i = f^{-\beta}(p_i)$ for $i = 1, 2$ and for all β. By noting that there holds $f(p) > 2^*/2$ for all real numbers $p > 2^*-1$ and since $f(2^*) = 2^*$, we get that there holds $2^*-1 < q_\beta^2 < 2^* < q_\beta^1 < n(2^*-1)/2$ for all $\beta \ge 1$. It is easily seen that the sequence $(q_\beta^1)_\beta$ is decreasing while the sequence $(q_\beta^2)_\beta$ is increasing. Since there holds $f(q_{\beta+1}^i) = q_\beta^i$ for $i = 1, 2$ and for all β, it follows that both sequences converge to 2^*. By Step 3.3, for β large enough so that $q_\beta^1 < p(n)$ and $q_\beta^2 > \max(\frac{n2^*}{n+2.2^*}, \frac{2^*}{2})$, there holds

$$\|u_\alpha\|_{q_\beta^1, q_\beta^2, \mu_\alpha^{-1}} = \mathrm{O}(1)$$

as $\alpha \to +\infty$. We finally get (3.7) by β iterations of Step 3.4. □

3.2. **Local estimates**

In the following, we let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ and h be a Hölder continuous function on M. Here again, we do not need to assume that (M, g) is locally conformally flat nor to restrict neither the dimension of M, nor the regularity of h. As in the previous section, we let $(u_\alpha)_\alpha$ be a bounded sequence in $H_1^2(M)$ of solutions for the family of equations (3.1), and we assume that $(u_\alpha)_\alpha$ blows up as $\alpha \to +\infty$. For $i = 1, \ldots, k$, we let $(x_\alpha^i)_\alpha$ and $(\mu_\alpha^i)_\alpha$ be the centers and the weights of the bubble $(B_\alpha^i)_\alpha$ in Lemma 3.1, and we assume that

$$\mu_\alpha^1 = \max_{1 \leq i \leq k} \mu_\alpha^i .$$

Then, for any α, we let μ_α stand for μ_α^1, x_α stand for x_α^1, and for any positive real number a and b such that $a < b$, we define the open annulus

$$A_\alpha^{a,b} = \{x \in M;\ a\sqrt{\mu_\alpha} < d_g(x, x_\alpha) < b\sqrt{\mu_\alpha}\},$$

where d_g is the geodesic distance on M with respect to the metric g.

In what follows, we repeatedly have to estimate the functions ϕ_α defined on $M \times [0, i_g)$ by

$$\phi_\alpha(x, r) = \begin{cases} \dfrac{1}{r^{n-1}} \displaystyle\int_{\partial B_x(r)} |u_\alpha|\, d\sigma_g & \text{if } r > 0 \\ \omega_{n-1} |u_\alpha(x)| & \text{if } r = 0 \end{cases}$$

for all α, where $d\sigma_g$ is the volume element on $\partial B_x(r)$ induced by the metric g. We also introduce the functions ϕ_α^ε defined on $M \times [0, i_g)$ by

$$\phi_\alpha^\varepsilon(x, r) = \begin{cases} \dfrac{1}{r^{n-1}} \displaystyle\int_{\partial B_x(r)} u_\alpha^\varepsilon d\sigma_g & \text{if } r > 0 \\ \omega_{n-1} u_\alpha^\varepsilon(x) & \text{if } r = 0 \end{cases}$$

for all α and ε, where the functions u_α^ε are as in (3.3). It is easily checked that the functions ϕ_α and ϕ_α^ε are continuous. For any point x in M, there exists a smooth function β_x defined around x such that for any function u in $C^1(M)$ and for any r in $(0, i_g)$, there holds

$$\frac{\partial}{\partial r}\left(\frac{1}{r^{n-1}} \int_{\partial B_x(r)} u d\sigma_g\right) = \frac{1}{r^{n-1}} \int_{\partial B_x(r)} \frac{\partial u}{\partial \nu} d\sigma_g + \frac{1}{r^{n-1}} \int_{\partial B_x(r)} \beta_x u d\sigma_g , \tag{3.32}$$

where $\frac{\partial}{\partial \nu}$ is the normal derivative with respect to the outward unit normal vector ν. As is well known (see, for instance, Chavel [12]), there exists a positive constant Γ such that there holds $|\beta_x(y)| \leq \Gamma d_g(x, y)$ for all points x and y in M which satisfy $d_g(x, y) < i_g$. For any α, ε, x, and r, by (3.32) with $u = u_\alpha^\varepsilon$, it follows that

$$\begin{aligned} \frac{\partial \phi_\alpha^\varepsilon}{\partial r}(x, r) &\geq -A \frac{1}{r^{n-1}} \int_{B_x(r)} |u_\alpha|^{2^*-1} dv_g - Br - \Gamma r \phi_\alpha^\varepsilon(x, r) \\ &\geq -A \frac{1}{r^{n-1}} \int_{B_x(r)} |u_\alpha|^{2^*-1} dv_g - B_\varepsilon r - \Gamma r \phi_\alpha(x, r) , \end{aligned} \tag{3.33}$$

where A and B are two positive constants which do not depend on α and ε and where $B_\varepsilon = B + \varepsilon \Gamma$.

We purpose to prove the following local estimates.

Lemma 3.5. *Let (M,g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$, h be a Hölder continuous function on M, and $(u_\alpha)_\alpha$ be a bounded sequence in $H_1^2(M)$ of solutions for the family of equations (3.1). If the sequence $(u_\alpha)_\alpha$ blows up as $\alpha \to +\infty$, then there exists a real number $c > 2$ such that up to a subsequence, there holds*

$$\text{(i)}\quad \|u_\alpha\|_{C^0\left(A_\alpha^{c-2,c+2}\right)} = \mathrm{O}(1),$$

$$\text{(ii)}\quad \int_{A_\alpha^{c-1,c+1}} |\nabla u_\alpha|_g^2\, dv_g = \mathrm{O}\left(\mu_\alpha^{\frac{n-2}{2}}\right)$$

as $\alpha \to +\infty$.

In order to prove Lemma 3.5, for any α, we consider the $k+1$ mutually disjoint open annuli $A_\alpha^{8i-4,8i+4}$, $i = 1,\dots,k+1$. For α large, they are nonempty, thus at least one of them does not contain any center of the k bubbles $(B_\alpha^i)_\alpha$, $i = 1,\dots,k$. It follows that there exists a real number $c > 4$ such that up to a subsequence, for any α, the points x_α^i, $i = 1,\dots,k$ do not belong to the annulus $A_\alpha^{c-4,c+4}$. The first step in the proof of Lemma 3.5 consists of the following weaker estimate.

Step 3.6. *As $\alpha \to +\infty$, there holds*

$$\|u_\alpha\|_{C^0\left(A_\alpha^{c-3,c+3}\right)} = \mathrm{O}\left(\mu_\alpha^{\frac{2-n}{4}}\right).$$

Proof. We proceed by contradiction, and assume that there exists a sequence of points y_α^0 in $A_\alpha^{c-3,c+3}$ such that there holds $\mu_\alpha^{(n-2)/4} |u_\alpha(y_\alpha^0)| \to +\infty$ as $\alpha \to +\infty$. For any α, we set $r_\alpha^0 = |u_\alpha(y_\alpha^0)|^{-2/(n-2)}$, and we show that there exists a point y_α in $B_{y_\alpha^0}(2r_\alpha^0)$ such that there hold $|u_\alpha(y_\alpha)| \geq |u_\alpha(y_\alpha^0)|$ and $|u_\alpha(y)| \leq 2|u_\alpha(y_\alpha)|$ for all points y in $B_{y_\alpha}(r_\alpha)$, where $r_\alpha = |u_\alpha(y_\alpha)|^{-2/(n-2)}$. If the point y_α^0 does not satisfy this condition, then there exists a point y_α^1 in $B_{y_\alpha^0}(r_\alpha^0)$ such that there holds $|u_\alpha(y_\alpha^1)| > 2|u_\alpha(y_\alpha^0)|$. We may iterate this argument as long as we do not find a point in $B_{y_\alpha^0}(2r_\alpha^0)$ which satisfies the above conditions. We then get a sequence of points y_α^β in $B_{y_\alpha^0}(2r_\alpha^0)$ such that there holds $|u_\alpha(y_\alpha^\beta)| > 2|u_\alpha(y_\alpha^{\beta-1})|$ for all β. Indeed, for any natural number β, the point y_α^β is chosen in the ball $B_{y_\alpha^{\beta-1}}(r_\alpha^{\beta-1})$, thus we compute

$$d_g\left(y_\alpha^\beta, y_\alpha^0\right) \leq \sum_{k=0}^{\beta-1} r_\alpha^k \leq \left(\sum_{k=0}^{\beta-1} 2^{-\frac{n-2}{2}k}\right) r_\alpha^0 \leq 2r_\alpha^0.$$

If we never find a satisfying point by this way, then there holds $|u_\alpha(y_\alpha^\beta)| \to +\infty$ as $\beta \to +\infty$ which contradicts the continuity of u_α. This proves the existence of a point y_α in $B_{y_\alpha^0}(2r_\alpha^0)$ such that there hold $|u_\alpha(y_\alpha)| \geq |u_\alpha(y_\alpha^0)|$ and $|u_\alpha(y)| \leq 2|u_\alpha(y_\alpha)|$ for all y in $B_{y_\alpha}(r_\alpha)$, where $r_\alpha = |u_\alpha(y_\alpha)|^{-2/(n-2)}$. We then let $\widetilde{u}_\alpha$ be the function defined on $B_0(i_g/r_\alpha)$ by

$$\widetilde{u}_\alpha(x) = r_\alpha^{\frac{n-2}{2}} u_\alpha \circ \exp_{y_\alpha}(r_\alpha x).$$

By our primary assumption, r_α^0 is asymptotically negligible compared to $\sqrt{\mu_\alpha}$ as $\alpha \to +\infty$, and thus so do r_α. For any real number ε in $(0,1)$ and for α large enough so that $2r_\alpha^0 \leq \varepsilon$, the point y_α belongs to the annulus $A_\alpha^{c-3-\varepsilon,c+3+\varepsilon}$. Hence, r_α remains asymptotically negligible compared to the distance between the points y_α and x_α^i as $\alpha \to +\infty$. By decomposing the functions u_α as in Lemma 3.1, we can deduce that the sequence $(\widetilde{u}_\alpha)_\alpha$ converges weakly to 0 in $D^{1,2}(\mathbb{R}^n)$. It follows that $(\widetilde{u}_\alpha)_\alpha$ converges up to a subsequence to 0 in $L^1_{\text{loc}}(\mathbb{R}^n)$. We now estimate the L^1-norm of the functions $\widetilde{u}_\alpha$ over small balls centered at 0 in order to exhibit a contradiction. We note that there exists a positive constant C such that for any α and for small positive real numbers r, if $d\sigma$ and $d\sigma_{g_\alpha}$ denotes the volume elements on $\partial B_0(r)$ respectively induced by the Euclidean metric and the metric $g_\alpha = \exp_{y_\alpha}^* g$ then there holds $d\sigma \geq C d\sigma_{g_\alpha}$. For any real number r and for α large, it follows that

$$\int_{\partial B_0(r)} |\widetilde{u}_\alpha| \, d\sigma \geq C r_\alpha^{\frac{n-2}{2}} r^{n-1} \phi_\alpha(y_\alpha, r_\alpha r) . \tag{3.34}$$

We are led to estimate the functions $\phi_\alpha(y_\alpha, .)$. For any $\varepsilon > 0$ and any real number r in $(0, r_\alpha)$, by (3.33), we get

$$\begin{aligned}\frac{\partial \phi_\alpha^\varepsilon}{\partial r}(y_\alpha, r) &\geq -2^{2^*-1} A |u_\alpha(y_\alpha)|^{2^*-1} \frac{\operatorname{Vol}_g(B_{y_\alpha}(r))}{r^{n-1}} - (B_\varepsilon + 2\Gamma |u_\alpha(y_\alpha)|) r \\ &\geq -\left(A' |u_\alpha(y_\alpha)|^{2^*-1} + B_\varepsilon + 2\Gamma |u_\alpha(y_\alpha)|\right) r ,\end{aligned}$$

where A' is a constant independent of α and ε. Integrating on $(0, r)$ and letting $\varepsilon \to 0$ yield

$$\phi_\alpha(y_\alpha, r) \geq \omega_{n-1} |u_\alpha(y_\alpha)| - \frac{1}{2}\left(A' |u_\alpha(y_\alpha)|^{2^*-1} + B + 2\Gamma |u_\alpha(y_\alpha)|\right) r^2. \tag{3.35}$$

For any real number r in $(0, 1)$, by (3.34) and (3.35), we get

$$\int_{\partial B_0(r)} |\widetilde{u}_\alpha| \, d\sigma \geq C \omega_{n-1} r^{n-1} - \frac{C}{2}\left(A' + B r_\alpha^{\frac{n+2}{2}} + 2\Gamma r_\alpha^2\right) r^{n+1}.$$

For small positive real numbers r, by integrating on $(0, r)$ we get that the L^1-norm of the functions $\widetilde{u}_\alpha$ over the ball $B_0(r)$ is bounded from below by a positive constant independent of α, and thus does not converge up to a subsequence to 0 as $\alpha \to +\infty$. This contradiction ends the proof of Step 3.6. □

The second step in the proof of Lemma 3.5 states as follows.

Step 3.7. *As $\alpha \to +\infty$, there holds*

$$\int_{A_\alpha^{c-3, c+3}} |u_\alpha| \, dv_g = \mathrm{O}\left(\mu_\alpha^{n/2}\right) . \tag{3.36}$$

Proof. By the continuity of the embedding of $H_1^2(M)$ into $L^1(M)$, the sequence $(u_\alpha)_\alpha$ remains bounded in $L^1(M)$, and thus there holds

$$\int_{B_{x_\alpha}(2i_g/3) \setminus B_{x_\alpha}(i_g/3)} |u_\alpha| \, dv_g \leq C$$

for all α, where C is a positive constant independent of α. It follows that there exists $(r_\alpha)_\alpha$ in $[i_g/3, 2i_g/3]$ such that the sequence $(\phi_\alpha(x_\alpha, r_\alpha))_\alpha$ is bounded. For any α and any real number r in $(0, r_\alpha)$, integrating (3.33) on $[r, r_\alpha]$ and letting $\varepsilon \to 0$ yield

$$e^{\Gamma \frac{r^2}{2}} \phi_\alpha(x_\alpha, r) \leq e^{\Gamma \frac{r_\alpha^2}{2}} \phi_\alpha(x_\alpha, r_\alpha) + \int_r^{r_\alpha} e^{\Gamma \frac{s^2}{2}} \left(\frac{A}{s^{n-1}} \int_{B_{x_\alpha}(s)} |u_\alpha|^{2^*-1} dv_g + Bs\right) ds.$$

It follows that

$$\phi_\alpha(x_\alpha, r) \leq C_1 + C_2 \int_r^{i_g} \frac{1}{s^{n-1}} \int_{B_{x_\alpha}(s)} |u_\alpha|^{2^*-1} dv_g ds , \tag{3.37}$$

where C_1 and C_2 are two positive constants independent of α and r. We are then led to estimate $\int_{B_{x_\alpha}(r)} |u_\alpha|^{2^*-1} dv_g$ for all α and all r. Applying Lemma 3.2 with $p_1 = n(2^* - 1)$ and

$p_2 = 2^* - 1$ yields that for any α, there exist two functions u_α^1 and u_α^2 in $L^\infty(M)$ such that there hold $|u_\alpha| \le u_\alpha^1 + u_\alpha^2$, $\|u_\alpha^1\|_{L^{n(2^*-1)}(M)} \le C$ and $\|u_\alpha^2\|_{L^{2^*-1}(M)} \le C\mu_\alpha^{n/(2^*-1)-n/2^*}$, where C is a positive constant independent of α. By these three estimates and by Hölder's inequality, we get

$$\int_{B_{x_\alpha}(r)} |u_\alpha|^{2^*-1} dv_g \le 2^{2^*-2}\left(\int_{B_{x_\alpha}(r)} \left|u_\alpha^1\right|^{2^*-1} dv_g + \int_{B_{x_\alpha}(r)} \left|u_\alpha^2\right|^{2^*-1} dv_g\right)$$
$$\le C_1 \operatorname{Vol}_g\left(B_{x_\alpha}(r)\right)^{1-1/n} + C_2\mu_\alpha^{\frac{n-2}{2}}, \tag{3.38}$$

where C_1 and C_2 are two positive constants independent of α and r. Bishop's inequality (see, for instance, Chavel [12]) provides a positive constant C independent of α and r such that there holds $\operatorname{Vol}_g\left(B_{x_\alpha}(r)\right) \le Cr^n$. By (3.38), it follows that

$$\int_{B_{x_\alpha}(r)} |u_\alpha|^{2^*-1} dv_g \le C_1 r^{n-1} + C_2\mu_\alpha^{\frac{n-2}{2}},$$

where C_1 and C_2 are two positive constants independent of α and r. Plugging into (3.37) gives

$$\phi_\alpha(x_\alpha, r) \le C_1 + C_2\left(\frac{\sqrt{\mu_\alpha}}{r}\right)^{n-2},$$

where C_1 and C_2 are two positive constants independent of α and r. Finally, multiplying by r^{n-1} and integrating on $\left[(c-3)\sqrt{\mu_\alpha}, (c+3)\sqrt{\mu_\alpha}\right]$ yield (3.36). □

We prove the first estimate in Lemma 3.5 by using Steps 3.6 and 3.7.

Proof of Lemma 3.5 (i). We fix a sequence of points y_α in the annulus $A_\alpha^{c-2,c+2}$, and we purpose to prove that up to a subsequence, $|u_\alpha(y_\alpha)|$ is bounded from above by a positive constant independent of y_α. We first consider the case where there holds $\phi_\alpha(y_\alpha, r) > \phi_\alpha(y_\alpha, 0)/2$ up to a subsequence for all α and all real numbers r in $\left[0, \sqrt{\mu_\alpha}\right)$. In that case, by multiplying by r^{n-1}, by integrating on $\left[(c-3)\sqrt{\mu_\alpha}, (c+3)\sqrt{\mu_\alpha}\right]$, and by using Step 3.7, we get that up to a subsequence, $|u_\alpha(y_\alpha)|$ is bounded from above by a positive constant independent of y_α. Passing if necessary to a subsequence, we may assume from now on that for any α, there holds $|u_\alpha(y_\alpha)| \ge 1$ and there exists a real number r_α in $\left[0, \sqrt{\mu_\alpha}\right)$ such that there holds $\phi_\alpha(y_\alpha, r_\alpha) \le \phi_\alpha(y_\alpha, 0)/2$. For any α, we let r_α^1 be a real number where the function $\phi_\alpha(y_\alpha, .)$ attains its maximum on $[0, r_\alpha]$. For any ε and any real number r in $(0, r_\alpha]$, by (3.33) and by writing $|u_\alpha|^{2^*-1} \le |u_\alpha|^{2^*-2}|u_\alpha|$, we get

$$\frac{\partial\phi_\alpha^\varepsilon}{\partial r}(y_\alpha, r) \ge -\|u_\alpha\|_{C^0\left(A_\alpha^{c-3,c+3}\right)}^{2^*-2}\frac{A}{r^{n-1}}\int_0^r s^{n-1}\phi_\alpha(y_\alpha, s)\,ds - Br - \varGamma r\phi_\alpha(y_\alpha, r)$$
$$\ge -\left(\left(\|u_\alpha\|_{C^0\left(A_\alpha^{c-3,c+3}\right)}^{2^*-2}\frac{A}{n} + \varGamma\right)\phi_\alpha\left(y_\alpha, r_\alpha^1\right) + B\right)r\,.$$

Since we assumed that $|u_\alpha(y_\alpha)| \ge 1$ and thus that $\phi_\alpha\left(y_\alpha, r_\alpha^1\right) \ge \omega_{n-1}$, Step 3.6 gives a positive constant C independent of y_α and ε such that for any real number r in $(0, r_\alpha]$, there holds

$$\frac{\partial\phi_\alpha^\varepsilon}{\partial r}(y_\alpha, r) \ge -\frac{C}{\mu_\alpha}\phi_\alpha\left(y_\alpha, r_\alpha^1\right)r\,.$$

We then set

$$r_\alpha^2 = \max\left\{r \in \left[r_\alpha^1, r_\alpha\right];\ \phi_\alpha(y_\alpha, r) \ge \phi_\alpha\left(y_\alpha, r_\alpha^1\right)/2\right\}.$$

On the one hand, integrating on $[r_\alpha^1, r_\alpha^2]$ and letting $\varepsilon \to 0$ yield

$$\frac{1}{2}\phi_\alpha\left(y_\alpha, r_\alpha^1\right) \le \frac{C}{2\mu_\alpha}\phi_\alpha\left(y_\alpha, r_\alpha^1\right)\left(\left(r_\alpha^2\right)^2 - \left(r_\alpha^1\right)^2\right) \le \frac{C}{\sqrt{\mu_\alpha}}\phi_\alpha\left(y_\alpha, r_\alpha^1\right)\left(r_\alpha^2 - r_\alpha^1\right),$$

and thus one can easily compute

$$\mu_\alpha^{n/2} \le (2C)^n\left(\left(r_\alpha^2\right)^n - \left(r_\alpha^1\right)^n\right). \tag{3.39}$$

On the other hand, we get

$$\int_{B_{y_\alpha}(r_\alpha^2)\backslash B_{y_\alpha}(r_\alpha^1)} |u_\alpha|\, dv_g = \int_{r_\alpha^1}^{r_\alpha^2} r^{n-1}\phi_\alpha\left(y_\alpha, r\right) dr \ge \frac{\omega_{n-1}}{2n}\left|u_\alpha\left(y_\alpha\right)\right|\left(\left(r_\alpha^2\right)^n - \left(r_\alpha^1\right)^n\right). \tag{3.40}$$

By (3.39), (3.40), and Step 3.7, here again, we get that $|u_\alpha(y_\alpha)|$ is bounded from above by a positive constant independent of y_α. This ends the proof of property (i). ☐

We prove the second estimate in Lemma 3.5 by using the first one.

Proof of Lemma 3.5 (ii). We let ζ be a smooth cutoff function on the Euclidean space such that $\zeta \equiv 1$ in the annulus $\{x \in \mathbb{R}^n;\ c-1 < |x| < c+1\}$ and $\zeta \equiv 0$ out of the annulus $\{x \in \mathbb{R}^n;\ c-2 < |x| < c+2\}$. For any α, we set the function $\varphi_\alpha = \zeta\left(d_g\left(x_\alpha, .\right)/\sqrt{\mu_\alpha}\right)$. Multiplying the equation $\Delta_g u_\alpha + h u_\alpha = |u_\alpha|^{p_\alpha - 2} u_\alpha$ by the function $\varphi_\alpha u_\alpha$ and integrating by parts on M yield

$$\int_M \varphi_\alpha |\nabla u_\alpha|_g^2\, dv_g + \frac{1}{2}\int_M u_\alpha^2 \Delta_g \varphi_\alpha dv_g + \int_M h\varphi_\alpha u_\alpha^2 dv_g = \int_M \varphi_\alpha |u_\alpha|^{p_\alpha} dv_g.$$

By (i), it follows that there exist two constants C_1 and C_2 such that there holds

$$\int_{A_\alpha^{c-1,c+1}} |\nabla u_\alpha|_g^2\, dv_g \le \left(C_1 + C_2\left\|\Delta_g\varphi_\alpha\right\|_{C^0(M)}\right)\mathrm{Vol}_g\left(A_\alpha^{c-2,c+2}\right) \tag{3.41}$$

for all α. One can easily check that there holds $\|\Delta_g\varphi_\alpha\|_{C^0(M)} = \mathrm{O}(\mu_\alpha^{-1})$ as $\alpha \to +\infty$. Moreover, we get $\mathrm{Vol}_g\left(A_\alpha^{c-2,c+2}\right) = \mathrm{O}(\mu_\alpha^{n/2})$ as $\alpha \to +\infty$ by Bishop's inequality. Finally, property (ii) follows from (3.41). ☐

3.3. **Proof of Theorem 1.2**

In the following, we let (M,g) be a smooth, compact Riemannian manifold of dimension $n \ge 7$ and h be a C^1-function on M such that there holds $h < \frac{n-2}{4(n-1)}\mathrm{Scal}_g$ in M. The proof of Theorem 1.2 is based on conformal invariance of the conformal Laplacian and on the Euclidean Pohožaev identity. Given a smooth, bounded domain Ω in the Euclidean space, the Euclidean Pohožaev identity states

$$\int_\Omega \langle x, \nabla u\rangle \Delta_\delta u dx + \frac{n}{2^*}\int_\Omega u\Delta_\delta u dx$$
$$= -\int_{\partial\Omega}\langle x, \nabla u\rangle \frac{\partial u}{\partial\nu} d\sigma\left(x\right) + \frac{1}{2}\int_{\partial\Omega}\langle x, \nu\rangle |\nabla u|^2 d\sigma\left(x\right) - \frac{n}{2^*}\int_{\partial\Omega} u\frac{\partial u}{\partial\nu} d\sigma \tag{3.42}$$

for all smooth functions u on $\overline{\Omega}$, where $d\sigma$ is the Euclidean volume element on $\partial\Omega$ and $\partial/\partial\nu$ is the normal derivative with respect to the outward unit normal vector ν. We easily get (3.42) by integrating by parts the first term in the left hand side.

We proceed by contradiction, and let $(u_\alpha)_\alpha$ be a bounded sequence in $H_1^2(M)$ of solutions for the family of equations (3.1) which does not remain bounded in $C^0(M)$. It is easily seen that the sequence $(u_\alpha)_\alpha$ cannot be compact in $H_1^2(M)$, and thus blows up as $\alpha \to +\infty$. As in the previous sections, for $i = 1, \dots, k$, we let $(x_\alpha^i)_\alpha$ and $(\mu_\alpha^i)_\alpha$ be the centers and the weights of the bubble $(B_\alpha^i)_\alpha$ in Lemma 3.1, and we assume

$$\mu_\alpha^1 = \max_{1 \le i \le k} \mu_\alpha^i \, .$$

Then, for any α, we let μ_α stand for μ_α^1 and x_α stand for x_α^1.

Since g is locally conformally flat, there exists a conformal metric $\widetilde{g}$ to g which is flat around the geometrical blow-up point $\widetilde{x}$, limit of the centers $(x_\alpha)_\alpha$. We set $\widetilde{g} = \varphi^{2^*-2} g$, where φ is smooth and positive and $\widetilde{u}_\alpha = u_\alpha / \varphi$ for all α. By conformal invariance of the conformal Laplacian (see, for instance, Lee–Parker [39]), $\widetilde{u}_\alpha$ satisfies the equation

$$\Delta_{\widetilde{g}} \widetilde{u}_\alpha + \widetilde{h} \widetilde{u}_\alpha = \varphi^{p_\alpha - 2^*} \left| \widetilde{u}_\alpha \right|^{p_\alpha - 2} \widetilde{u}_\alpha \tag{3.43}$$

for all α, where

$$\widetilde{h} = \frac{n-2}{4(n-1)} \operatorname{Scal}_{\widetilde{g}} + \frac{1}{\varphi^{2^*-2}} \left(h - \frac{n-2}{4(n-1)} \operatorname{Scal}_g \right) .$$

In particular, around the point $\widetilde{x}$, the flatness of $\widetilde{g}$ implies that $\operatorname{Scal}_{\widetilde{g}} \equiv 0$, and thus $\widetilde{h}$ is negative. We let c be the real number we get in Lemma 3.5. It follows from the second assertion of this lemma that there exists a sequence $(r_\alpha)_\alpha$ in $(c-1, c+1)$ such that, up to a subsequence, there holds

$$\int_{\partial B_{x_\alpha}\left(r_\alpha \sqrt{\mu_\alpha}\right)} \left| \nabla u_\alpha \right|_{\widetilde{g}}^2 d\sigma_{\widetilde{g}} = \mathrm{O}\left(\mu_\alpha^{\frac{n-3}{2}} \right) \tag{3.44}$$

as $\alpha \to +\infty$, where $d\sigma_{\widetilde{g}}$ is the volume element on $\partial B_{x_\alpha}\left(r_\alpha \sqrt{\mu_\alpha}\right)$ induced by the metric $\widetilde{g}$. For α large, $\widetilde{g}$ is flat in the ball $B_{x_\alpha}\left((c+1)\sqrt{\mu_\alpha}\right)$. In particular, the Euclidean Pohožaev identity (3.42) holds for the function $\hat{u}_\alpha = \widetilde{u}_\alpha \circ \exp_{x_\alpha}$ and the domain $\mathcal{B}_\alpha = B_0\left(r_\alpha \sqrt{\mu_\alpha}\right)$. By integrating by parts the first term in the left hand side of (3.42) and by using (3.43), we get

$$\begin{aligned}
\int_{\mathcal{B}_\alpha} \langle x, \nabla \hat{u}_\alpha \rangle \Delta_\delta \hat{u}_\alpha dx = & -\frac{n}{p_\alpha} \int_{\mathcal{B}_\alpha} \varphi_\alpha^{p_\alpha - 2^*} \left| \hat{u}_\alpha \right|^{p_\alpha} dx - \frac{2^* - p_\alpha}{p_\alpha} \int_{\mathcal{B}_\alpha} \langle x, \nabla \varphi_\alpha \rangle \varphi_\alpha^{p_\alpha - 2^* - 1} \left| \hat{u}_\alpha \right|^{p_\alpha} dx \\
& + \frac{n}{2} \int_{\mathcal{B}_\alpha} \hat{h}_\alpha \hat{u}_\alpha^2 dx + \frac{1}{2} \int_{\mathcal{B}_\alpha} \left\langle x, \nabla \hat{h}_\alpha \right\rangle \hat{u}_\alpha^2 dx + \frac{1}{p_\alpha} \int_{\partial \mathcal{B}_\alpha} \langle x, \nu \rangle \varphi_\alpha^{p_\alpha - 2^*} \left| \hat{u}_\alpha \right|^{p_\alpha} d\sigma(x) \\
& - \frac{1}{2} \int_{\partial \mathcal{B}_\alpha} \langle x, \nu \rangle \hat{h}_\alpha \left| \hat{u}_\alpha \right|^2 d\sigma(x) ,
\end{aligned}$$

where $\varphi_\alpha = \varphi \circ \exp_{x_\alpha}$, $\hat{h}_\alpha = \widetilde{h} \circ \exp_{x_\alpha}$, $d\sigma$ is the Euclidean volume element on $\partial \mathcal{B}_\alpha$ and ν is the outward unit normal vector to $\partial \mathcal{B}_\alpha$. Plugging into the Euclidean Pohožaev identity gives

$$\begin{aligned}
& n \left(\frac{1}{p_\alpha} - \frac{1}{2^*} \right) \int_{\mathcal{B}_\alpha} \varphi_\alpha^{p_\alpha - 2^* - 1} \left| \hat{u}_\alpha \right|^{p_\alpha} dx + \frac{2^* - p_\alpha}{p_\alpha} \int_{\mathcal{B}_\alpha} \langle x, \nabla \varphi_\alpha \rangle \varphi_\alpha^{p_\alpha - 2^* - 1} \left| \hat{u}_\alpha \right|^{p_\alpha} dx - \int_{\mathcal{B}_\alpha} \hat{h}_\alpha \hat{u}_\alpha^2 dx \\
& - \frac{1}{2} \int_{\mathcal{B}_\alpha} \left\langle x, \nabla \hat{h}_\alpha \right\rangle \hat{u}_\alpha^2 dx = \frac{1}{p_\alpha} \int_{\partial \mathcal{B}_\alpha} \langle x, \nu \rangle \varphi_\alpha^{p_\alpha - 2^* - 1} \left| \hat{u}_\alpha \right|^{p_\alpha} d\sigma(x) - \frac{1}{2} \int_{\partial \mathcal{B}_\alpha} \langle x, \nu \rangle \hat{h}_\alpha \hat{u}_\alpha^2 d\sigma(x) \\
& + \int_{\partial \mathcal{B}_\alpha} \langle x, \nabla \hat{u}_\alpha \rangle \frac{\partial \hat{u}_\alpha}{\partial \nu} d\sigma(x) - \frac{1}{2} \int_{\partial \mathcal{B}_\alpha} \langle x, \nu \rangle \left| \nabla \hat{u}_\alpha \right|^2 d\sigma(x) + \frac{n}{2^*} \int_{\partial \mathcal{B}_\alpha} \hat{u}_\alpha \frac{\partial \hat{u}_\alpha}{\partial \nu} d\sigma .
\end{aligned} \tag{3.45}$$

We first give a lower bound of the left hand side of (3.45). As a remark, there exists a positive constant ε_0 such that for any α, there holds $\hat{h}_\alpha < -\varepsilon_0$ in $\mathcal{B}_\alpha$. For α large, we get

$< x, \nabla \hat{h}_\alpha > \leq -\hat{h}_\alpha$ and $< x, \nabla \varphi_\alpha > \geq -(n/2^*)\,\varphi_\alpha$ for all points x in $\mathcal{B}_\alpha$. It follows that

$$-\frac{1}{2}\sup_{\mathcal{B}_\alpha}\hat{h}_\alpha\int_{\mathcal{B}_\alpha}\hat{u}_\alpha^2dx \leq n\left(\frac{1}{p_\alpha}-\frac{1}{2^*}\right)\int_{\mathcal{B}_\alpha}\varphi_\alpha^{p_\alpha-2^*-1}\left|\hat{u}_\alpha\right|^{p_\alpha}dx$$
$$+\frac{2^*-p_\alpha}{p_\alpha}\int_{\mathcal{B}_\alpha}\left\langle x,\nabla\varphi_\alpha\right\rangle\varphi_\alpha^{p_\alpha-2^*-1}\left|\hat{u}_\alpha\right|^{p_\alpha}dx-\int_{\mathcal{B}_\alpha}\hat{h}_\alpha\hat{u}_\alpha^2dx-\frac{1}{2}\int_{\mathcal{B}_\alpha}\left\langle x,\nabla\hat{h}_\alpha\right\rangle\hat{u}_\alpha^2dx\,.$$

Then, we are concerned with estimating the right hand side of (3.45). For any point x in $\partial\mathcal{B}_\alpha$, by the Cauchy-Schwarz inequality, we get

$$\left|\left\langle x,\nu\right\rangle\right| \leq (c+1)\sqrt{\mu_\alpha}\,,\quad \left|\left\langle x,\nabla\hat{u}_\alpha\right\rangle\right| \leq (c+1)\sqrt{\mu_\alpha}\left|\nabla\hat{u}_\alpha\right|,\quad\text{and}\quad\left|\frac{\partial\hat{u}_\alpha}{\partial\nu}\right|\leq\left|\nabla\hat{u}_\alpha\right|\,.$$

By (3.45), it follows that

$$-\frac{1}{2}\sup_{\mathcal{B}_\alpha}\hat{h}_\alpha\int_{\mathcal{B}_\alpha}\hat{u}_\alpha^2dx \leq \frac{c+1}{p_\alpha}\sqrt{\mu_\alpha}\int_{\partial\mathcal{B}_\alpha}\varphi_\alpha^{p_\alpha-2^*-1}\left|\hat{u}_\alpha\right|^{p_\alpha}d\sigma+\frac{c+1}{2}\sqrt{\mu_\alpha}\int_{\partial\mathcal{B}_\alpha}\left|\hat{h}_\alpha\right|\hat{u}_\alpha^2d\sigma$$
$$+\frac{3(c+1)}{2}\sqrt{\mu_\alpha}\int_{\partial\mathcal{B}_\alpha}\left|\nabla\hat{u}_\alpha\right|^2d\sigma+\frac{n}{2^*}\int_{\partial\mathcal{B}_\alpha}\left|\hat{u}_\alpha\right|\left|\nabla\hat{u}_\alpha\right|d\sigma\,. \quad (3.46)$$

Lemma 3.5 (i) gives

$$\sqrt{\mu_\alpha}\int_{\partial\mathcal{B}_\alpha}\varphi_\alpha^{p_\alpha-2^*-1}\left|\hat{u}_\alpha\right|^{p_\alpha}d\sigma+\frac{c+1}{2}\sqrt{\mu_\alpha}\int_{\partial\mathcal{B}_\alpha}\left|\hat{h}_\alpha\right|\hat{u}_\alpha^2d\sigma = \mathrm{O}\left(\mu_\alpha^{\frac{n}{2}}\right) \quad (3.47)$$

as $\alpha\to+\infty$. By Hölder's inequality, Lemma 3.5 (i), and (3.44), we also get

$$\int_{\partial\mathcal{B}_\alpha}\left|\hat{u}_\alpha\right|\left|\nabla\hat{u}_\alpha\right|d\sigma \leq \sqrt{\int_{\partial\mathcal{B}_\alpha}\hat{u}_\alpha^2d\sigma}\sqrt{\int_{\partial\mathcal{B}_\alpha}\left|\nabla\hat{u}_\alpha\right|^2d\sigma} = \mathrm{O}\left(\mu_\alpha^{\frac{n-2}{2}}\right) \quad (3.48)$$

as $\alpha\to+\infty$. By (3.44), (3.46), (3.47), and (3.48), we finally get

$$-\sup_{\mathcal{B}_\alpha}\hat{h}_\alpha\int_{\mathcal{B}_\alpha}\hat{u}_\alpha^2dx = \mathrm{O}\left(\mu_\alpha^{\frac{n-2}{2}}\right) \quad (3.49)$$

as $\alpha\to+\infty$. It remains to estimate $\int_{\mathcal{B}_\alpha}\hat{u}_\alpha^2dx$ from below. Since the ball $B_0(\mu_\alpha)$ is included in $\mathcal{B}_\alpha$ for α large, it suffices to estimate $\int_{B_0(\mu_\alpha)}\hat{u}_\alpha^2dx$. By decomposing the functions u_α as in Lemma 3.1 and by (3.2), one can easily check that up to a subsequence, there holds

$$\int_{B_0(\mu_\alpha)}\hat{u}_\alpha^2dx \geq C\mu_\alpha^2 \quad (3.50)$$

for all α, where C is a positive constant independent of α. By (3.49) and (3.50), it follows that up to a subsequence, there holds

$$-\sup_{\mathcal{B}_\alpha}\hat{h}_\alpha = \mathrm{O}\left(\mu_\alpha^{\frac{n-6}{2}}\right)$$

as $\alpha\to+\infty$. Since $n\geq 7$, passing to the limit as $\alpha\to+\infty$ yields that $\widetilde{h}(\widetilde{x})$ cannot be negative. This contradiction ends the proof of Theorem 1.2.

3.4. **Proof of Theorem 1.3**

In the following, we let (M,g) be a smooth, compact Riemannian manifold of dimension $n\geq 7$ and h be a C^1-function on M such that there holds $h < \frac{n-2}{4(n-1)}\,\mathrm{Scal}_g$ in M. We let

$(p_\alpha)_\alpha$ be a sequence in $(2, 2^*)$ converging to 2^*, and we purpose to deduce Theorem 1.3 from Theorem 1.2. For any α, we define the functional I_g^α on $H_1^2(M)$ by

$$I_g^\alpha(u) = \frac{1}{2}\int_M |\nabla u|_g^2 \, dv_g + \frac{1}{2}\int_M hu^2 dv_g - \frac{1}{p_\alpha}\int_M |u|^{p_\alpha} dv_g \,.$$

Its critical points are the solutions of the equation

$$\Delta_g u + hu = |u|^{p_\alpha - 2} u \,.$$

Another way of regarding the solutions of this equation is to say that up to a renormalization, they are the critical points of the functional I_g^α on the constraint

$$\mathcal{H} = \left\{ u \in H_1^2(M)\,;\; F(u) = 1 \right\},$$

where F is the functional defined on $H_1^2(M)$ by

$$F(u) = \int_M |\nabla u|_g^2 \, dv_g + \int_M hu^2 dv_g \,.$$

More precisely, the critical points of the functional I_g^α on the constraint $\mathcal{H}$ at level c_α are solutions of the equation

$$\Delta_g u + hu = \frac{2}{p_\alpha (1 - 2c_\alpha)} |u|^{p_\alpha - 2} u \,. \tag{3.51}$$

In order to use later on the min-max principle, we need to check that for α fixed, the functional I_g^α satisfies the Palais–Smale condition on the constraint $\mathcal{H}$ at any critical level c_α, namely that for any sequence $(u_\beta, \mu_\beta)_\beta$ in $\mathcal{H} \times \mathbb{R}$, if there hold $I_g^\alpha(u_\beta) \to c_\alpha$ and $DI_g^\alpha(u_\beta) - \mu_\beta DF(u_\beta) \to 0$ in $H_1^2(M)'$ as $\beta \to +\infty$, then $(u_\beta, \mu_\beta)_\beta$ converges up to a subsequence in $\mathcal{H} \times \mathbb{R}$. We set a sequence $(u_\beta, \mu_\beta)_\beta$ in $\mathcal{H} \times \mathbb{R}$ satisfying the above conditions. We first note that there holds

$$\int_M |u_\beta|^{p_\alpha} dv_g = p_\alpha \left(\frac{1}{2} - I_g^\alpha(u_\beta) \right)$$

for all β. It follows that the sequence $(u_\beta)_\beta$ is bounded in $L^{p_\alpha}(M)$ and thus in $L^2(M)$. By the very definition of the set $\mathcal{H}$, the sequence $(u_\beta)_\beta$ remains bounded in $H_1^2(M)$. On the one hand, evaluating the functional $DI_g^\alpha(u_\beta) - \mu_\beta DF(u_\beta)$ at the function u_β for all β and passing to the limit as $\beta \to +\infty$ yield that the sequence of real numbers $(\mu_\beta)_\beta$ converges to $(1 + p_\alpha(c_\alpha - 1/2))/2$. On the other hand, since $H_1^2(M)$ is reflexive and by the compactness of the embeddings of $H_1^2(M)$ into $L^{p_\alpha}(M)$ and into $L^2(M)$, we may assume that there exists a function u in $H_1^2(M)$ such that up to a subsequence, $(u_\beta)_\beta$ converges to u weakly in $H_1^2(M)$ and strongly in $L^{p_\alpha}(M)$ and in $L^2(M)$. We clearly get that u is a solution of equation (3.51). For any function φ in $H_1^2(M)$, we then get

$$(1 - 2\mu_\beta)\int_M \langle \nabla(u_\beta - u), \nabla\varphi \rangle_g \, dv_g = \mathrm{o}\left(\|\varphi\|_{H_1^2(M)} \right)$$

as $\beta \to +\infty$. By taking $\varphi = u_\beta - u$, it follows that $(u_\beta)_\beta$ converges up to a subsequence to the function u in $H_1^2(M)$. This proves that the functional I_g^α satisfies the Palais–Smale condition on the constraint $\mathcal{H}$ at any critical level c_α.

We let $(\lambda_\beta)_\beta$ be the sequence of eigenvalues of the Laplace–Beltrami operator Δ_g and for any β, we let ϕ_β be an eigenfunction corresponding to λ_β and Γ_β be the set of all symmetric, compact subsets of $\mathcal{H}$ whose Krasnosel′skiĭ genus is greater than or equal to β. For β large enough so that $h > -\lambda_\beta$, one can easily check with the Borsuk–Ulam theorem (see, for instance, Kavian [37]) that the set $\mathcal{H} \cap \mathrm{Span}\,(\phi_\beta, \dots, \phi_{2\beta-1})$ belongs to Γ_β and thus that the set Γ_β is not empty. We then define

$$c_\alpha^{(\beta)} = \inf_{A \in \Gamma_\beta} \max_{u \in A} I_g^\alpha(u)$$

for all α and similarly

$$c_\beta = \inf_{A \in \Gamma_\beta} \max_{u \in A} I_g(u) .$$

For β large, we claim that all these lower bounds are finite. We prove this claim for c_β. We let $\mathcal{E}_k$ be the eigenspace associated with λ_k for all natural numbers k, and we set $\mathcal{S}_\beta = \bigoplus_{k=0}^{\beta} \mathcal{E}_k$. We fix a natural number β_0 such that $h > -\lambda_{\beta_0+1}$. If $\beta > \dim(\mathcal{S}_{\beta_0})$, then the intersection of any set A in Γ_β with the orthogonal complement of $\mathcal{S}_{\beta_0}$ is not empty. If not the case, then the projection of A onto $\mathcal{S}_{\beta_0}$ would be an odd, continuous map with nonzero values in a vector space of dimension less than β, and this would contradict the definition of the Krasnosel′skiĭ genus. It follows from this remark that it suffices to prove that the functional I_g is bounded from below on the intersection $\mathcal{G}$ of the set $\mathcal{H}$ with the orthogonal complement of $\mathcal{S}_{\beta_0}$. For any function u in $\mathcal{G}$, by the min-max characterization of the eigenvalues of the Laplace–Beltrami operator Δ_g, there holds

$$\int_M (\lambda_{\beta_0+1} + h)\, u^2 dv_g \le 1 .$$

Thanks to our choice of β_0, we get that the set $\mathcal{G}$ is bounded in $L^2(M)$. By the very definition of $\mathcal{H}$, the set $\mathcal{G}$ remains bounded in $H_1^2(M)$ and thus in $L^{2^*}(M)$. This proves that the functional I_g is bounded from below on the set $\mathcal{G}$, and as already said, it follows that c_β is finite. A similar argument gives that the lower bounds c_α^β are also finite.

By the properties of the Krasnosel′skiĭ genus, one can easily check that we are under the conditions of the min-max principle for $c_\alpha^{(\beta)}$ (see, for instance, Kavian [37]). In particular, $c_\alpha^{(\beta)}$ is a critical level of the functional I_g^α on the constraint $\mathcal{H}$. Therefore, we get a critical level $\widetilde{c}_\alpha^{(\beta)}$ of the functional I_g^α by setting

$$\widetilde{c}_\alpha^{(\beta)} = \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) \left(\frac{2}{p_\alpha \left(1 - 2c_\alpha^{(\beta)}\right)}\right)^{\frac{2}{p_\alpha - 2}} .$$

Similarly, we set

$$\widetilde{c}_\beta = \frac{1}{n} \left(\frac{2}{2^*(1 - 2c_\beta)}\right)^{\frac{2}{2^*-2}} .$$

The proof of Theorem 1.3 consists of three steps. The first one is as follows.

Step 3.8. *There holds $\widetilde{c}_\alpha^{(\beta)} \to \widetilde{c}_\beta$ as $\alpha \to +\infty$ for all β.*

Proof. It comes to the same thing to prove that there holds $c_\alpha^{(\beta)} \to c_\beta$ as $\alpha \to +\infty$. We begin with estimating the upper limit of $c_\alpha^{(\beta)}$ as $\alpha \to +\infty$. For any set A in Γ_β, since the functionals I_g^α are equicontinuous, there holds

$$\max_{u \in A} I_g^\alpha(u) \to \max_{u \in A} I_g(u)$$

as $\alpha \to +\infty$. It follows that

$$\limsup_{\alpha \to +\infty} c_\alpha^{(\beta)} \le \max_{u \in A} I_g(u) ,$$

and since this is satisfied for all sets A in Γ_β, we get

$$\limsup_{\alpha\to+\infty} c_\alpha^{(\beta)} \le c_\beta\,. \tag{3.52}$$

It remains to estimate the lower limit of $c_\alpha^{(\beta)}$ as $\alpha\to+\infty$. For any α, by taking into account that there holds $t^{p_\alpha}/p_\alpha - t^{2^*}/2^* \le 1/p_\alpha - 1/2^*$ for all nonnegative real numbers t, we get

$$I_g \le I_g^\alpha + \left(\frac{1}{p_\alpha} - \frac{1}{2^*}\right)\mathrm{Vol}_g(M)\,.$$

It follows that

$$c_\beta \le \liminf_{\alpha\to+\infty} c_\alpha^{(\beta)}\,. \tag{3.53}$$

By (3.52) and (3.53), we get that there holds $c_\alpha^{(\beta)} \to c_\beta$ as $\alpha\to+\infty$. □

The next step in the proof of Theorem 1.3 states as follows. We prove it by using Step 3.8 and Theorem 1.2.

Step 3.9. *$\widetilde{c}_\beta$ is a critical level of the functional I_g for all β.*

Proof. For any α, we let $\widetilde{u}_\alpha^{(\beta)}$ be a critical point of the functional I_g^α at level $\widetilde{c}_\alpha^{(\beta)}$. The function $\widetilde{u}_\alpha^{(\beta)}$ is a solution of equation (1.1). We then write

$$\left\|\widetilde{u}_\alpha^{(\beta)}\right\|_{H_1^2(M)}^2 = \int_M \left|\widetilde{u}_\alpha^{(\beta)}\right|^{p_\alpha} dv_g + \int_M (\Lambda - h)\left(\widetilde{u}_\alpha^{(\beta)}\right)^2 dv_g\,,$$

where Λ is as in (1.2). By Hölder's inequality, it follows

$$\begin{aligned}\left\|\widetilde{u}_\alpha^{(\beta)}\right\|_{H_1^2(M)}^2 &\le \int_M \left|\widetilde{u}_\alpha^{(\beta)}\right|^{p_\alpha} dv_g + \|\Lambda - h\|_{C^0(M)}\mathrm{Vol}_g(M)^{\frac{p_\alpha-2}{p_\alpha}}\left(\int_M \left|\widetilde{u}_\alpha^{(\beta)}\right|^{p_\alpha} dv_g\right)^{\frac{2}{p_\alpha}}\\ &= \frac{2p_\alpha}{p_\alpha-2}\widetilde{c}_\alpha^{(\beta)} + \|\Lambda - h\|_{C^0(M)}\mathrm{Vol}_g(M)^{\frac{p_\alpha-2}{p_\alpha}}\left(\frac{2p_\alpha}{p_\alpha-2}\widetilde{c}_\alpha^{(\beta)}\right)^{\frac{2}{p_\alpha}}.\end{aligned}$$

By Step 3.8, the right hand side in this equation is converging as $\alpha\to+\infty$. It follows that the sequence $(\widetilde{u}_\alpha^{(\beta)})_\alpha$ is bounded in $H_1^2(M)$. By Theorem 1.2, we get that $(\widetilde{u}_\alpha^{(\beta)})_\alpha$ converges up to a subsequence in $H_1^2(M)$ to a critical point $\widetilde{u}_\beta$ of the functional I_g. Finally, it follows from Step 3.8 that the level of $\widetilde{u}_\beta$ is $\widetilde{c}_\beta$. This ends the proof of Step 3.9. □

The third and last step in the proof of Theorem 1.3 is as follows.

Step 3.10. *There holds $\widetilde{c}_\beta\to+\infty$ as $\beta\to+\infty$.*

Proof. We proceed by contradiction, and assume that the sequence of real numbers $(\widetilde{c}_\beta)_\beta$ is bounded. By Step 3.8, we may construct an increasing sequence of natural numbers $(\alpha_\beta)_\beta$ such that there holds $|\widetilde{c}_{\alpha_\beta}^{(\beta)} - \widetilde{c}_\beta| < 1$ for all β and thus such that the sequence of real numbers $(\widetilde{c}_{\alpha_\beta}^{(\beta)})_\beta$ is bounded. By Ghoussoub [27, Corollary 10.5] (see also Bahri–Lions [4], Lazer–Solimini [38] and Solimini [54]), for any β, we may select a critical point u_β of the functional $I_g^{\alpha_\beta}$ on the constraint $\mathcal{H}$ at level $c_{\alpha_\beta}^{(\beta)}$ whose augmented Morse index is greater than or equal to α_β. Here, the augmented Morse index of u_β as a critical point of the functional $I_g^{\alpha_\beta}$ turns out to be the number of nonpositive eigenvalues, this time counted as many times as their multiplicity, of the linearized operator $\Delta_g + (h - (p_{\alpha_\beta}-1)|u_\beta|^{p_{\alpha_\beta}-2})$. The same computations as in Step 3.9 yield that the sequence $(u_\beta)_\beta$ is bounded in $H_1^2(M)$ and thus in $L^{2^*}(M)$. It follows that the functions $(h - (p_{\alpha_\beta}-1)|u_\beta|^{p_{\alpha_\beta}-2})$ are bounded in $L^{n/2}(M)$. With Lieb-type [44] arguments, we then get that the augmented Morse indices of the critical points u_β is bounded. This contradiction ends the proof of Step 3.10. □

By Steps 3.9 and 3.10, we get a sequence of solutions for the equation

$$\Delta_g u + hu = |u|^{2^*-2} u .$$

These solutions have unbounded energies. Up to a subsequence, we may assume that their energies are increasing. This ends the proof of Theorem 1.3.

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