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Magnetic field transport and kinematic dynamo effect: a Lagrangian interpretation

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The growth of magnetic fluctuations in the inertial range of turbulence is investigated in terms of fluid particle dynamics. The existence of dynamo effect is related to the time behaviour of the correlations between tangent vectors evolving along Lagrangian trajectories. In the presence of dynamo effect, the correlations between tangent vectors grow exponentially in time; in the absence of dynamo effect they decay as power laws. The above behaviours are intimately related to statistical conservation laws for the Lagrangian dynamics.

Keywords: dynamo effect; Lagrangian dynamics; statistical conservation laws

1. Introduction

The Eulerian and Lagrangian approaches are complementary ways of dealing with the evolution of fluid dynamical systems. In the former case, the motion of the fluid is described in a fixed reference frame in terms of fields depending on space and time. In the latter case, attention is directed to fluid particle dynamics. The study of turbulent transport of scalar and vector fields has benefited greatly from the Lagrangian approach; the scaling properties of the transported fields, indeed, turn out to be strictly related to the geometry of fluid particle configurations (see Falkovich *et al.* (2001) for a review of the subject).

We address the issue of the Lagrangian interpretation of magnetic dynamo effect in the turbulent kinematic model. The magnetic field transported by the motion of a conducting fluid is amplified by the velocity gradients and dissipated by the non-zero resistivity of the fluid. The aim of kinematic dynamo theory is to establish under what conditions a given flow can enhance a weak initially 'seeded' magnetic field. In this case, the amplification of the magnetic energy is referred to as dynamo effect. Dynamo theory has wide applications in

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astrophysics; the magnetic fields occurring in celestial bodies are indeed believed to be generated by dynamo processes (e.g. Moffatt 1978).

Most of the main concepts of the Lagrangian statistical formalism have been initially introduced to investigate the intermittency of passive scalar fields. Examples of scalar fields are provided by the concentration of an atmospheric pollutant and the density of a dye dispersed in a fluid. The field is transported passively when its back reaction on the advecting flow can be disregarded. Intermittency is related to the fact that the multipoint correlations of scalar increments are not scale invariant and violate dimensional predictions. With decreasing separation, the probability density function of scalar increments deviates more and more from the Gaussian distribution and weak and strong fluctuations become increasingly frequent. This leads to the formation of fronts and structures (e.g. Sreenivasan 1991; Shraiman & Siggia 2000; Warhaft 2000). The Lagrangian interpretation of passive scalar intermittency relies on the notion of statistical conservation laws. Even when the statistics of the passive field is not stationary, there exist functions of fluid particle positions that are preserved in mean by the flow, that is are constant in time when averaged over all realizations of the turbulent velocity field. These statistical integrals of motion control the small-scale behaviour of the transported field (see Falkovich *et al.* (2001) for a review).

The existence of statistically conserved quantities has been proved rigorously for the Kraichnan (1968) statistical model. In this case, the correlation time of the advecting flow vanishes and the scalar correlations obey closed partial differential equations. The homogeneous solutions of such equations, the so-called zero modes, are the dominant part of scalar correlations at the small scales; they exhibit anomalous scaling and are, therefore, responsible for the intermittency of the scalar field (Chertkov *et al.* 1995; Gawędzki & Kupiainen 1995; Shraiman & Siggia 1995; Frisch *et al.* 1998). Zero modes turn out to be statistically preserved by the Lagrangian dynamics thanks to a compensation mechanism in the evolution of fluid particle geometry. Multiparticle configurations can be described by size and shape variables: zero modes are special functions of particle positions such that the increase due to the size growth balances the decrease due to the shape deformation. This means that intermittency is due to subgroups of fluid particles remaining close to each other for extremely long times (Bernard *et al.* 1998). The Lagrangian mechanism leading to the generation of statistically preserved quantities persists also when the correlation time of the flow is finite. Numerical simulations indicate that statistically conserved quantities are still present in the two-dimensional Navier–Stokes flow and dominate the small-scale behaviour of the transported field; the statistical integrals of motion are once more the anomalous parts of scalar correlations and this confirms the Lagrangian interpretation of intermittency (Celani & Vergassola 2001). The study of statistical conservation laws for scalar fields relies on the analysis of particle dynamics. However, the numerical investigation of shell models proves that statistically preserved objects exist even if no Lagrangian structures can be identified (Arad *et al.* 2001). The notion of statistical conservation laws, therefore, turns out to be relevant for a general comprehension of turbulent transport.

Here, we investigate the existence of statistical integrals of motions in the turbulent transport of magnetic fields and their relevance for the kinematic dynamo effect. Contrary to scalar fields, the properties of the magnetic field are related not only to fluid particle configurations in physical space, but also to tangent-space dynamics along Lagrangian trajectories. We focus on magnetic field fluctuations within the inertial interval of turbulence, i.e. the range of scales much smaller than the correlation length of the flow and much larger than the dissipative scale. The growth of small-scale fluctuations was investigated in terms of Lagrangian dynamics by Chertkov *et al.* (1999). They considered time-scales smaller than the typical time needed for Lagrangian trajectories to attain separations comparable with the dissipative scale of the flow. Our analysis holds for longer times and until particle separations are below the correlation length of the velocity. The problem is tackled in the framework of the Kraichnan–Kazantsev model: the velocity field is Gaussian, short-correlated in time and scale invariant (Kazantsev 1967; Kraichnan 1968). This model is fully solvable and provides a simple connection between theory and more realistic flows.

The study of astrophysical magnetic fields would require taking into account the nonlinear coupling between the magnetic field and the flow. The kinematic dynamo theory describes the initial growth of magnetic energy under the assumption that the magnetic field is weak and does not affect the turbulent flow. The growth of magnetic energy, however, saturates when the magnetic field becomes intense enough to influence the transporting flow. At this point, the interaction between the magnetic field and the flow becomes strongly nonlinear. An exact analysis, like the one presented in this paper, can be carried out only for the kinematic stage of the dynamo, where the evolution equations are linear. An analytical study of the nonlinear turbulent dynamo is still an open issue and that regime has to be investigated numerically (e.g. Schekochihin *et al.* 2004; Ponty *et al.* 2005).

The remainder of the paper is organized as follows. In §2, we introduce the Kraichnan–Kazantsev model and investigate the dynamics of tangent vectors evolving along Lagrangian trajectories. In §3, we discuss the implications for the kinematic dynamo effect. In §4, we show that the results of §§2 and 3 are related to the existence of statistical conservation laws for the Lagrangian dynamics. Section 5 is dedicated to conclusions.

2. Lagrangian dynamics

In turbulent flows, fluid particles move along stochastic trajectories due to both molecular diffusion and the statistical nature of the velocity field. The equation of motion for a fluid particle is (Taylor 1921)

$$\dot{\boldsymbol{a}} = \boldsymbol{v}(\boldsymbol{a}(s), s) + \sqrt{2\kappa}\boldsymbol{\beta}(s), \tag{2.1}$$

where $\boldsymbol{a}=\boldsymbol{a}(s)$ denotes the position of the particle, κ is its diffusivity, and $\boldsymbol{\beta}$ is a Gaussian $\boldsymbol{\delta}$-correlated random process which models molecular fluctuations:

$$\begin{aligned} \langle \beta_i(t) \rangle &= 0, \\ \langle \beta_i(t)\beta_j(t') \rangle &= \delta_{ij}\delta(t-t'), \end{aligned} \qquad i,j = 1,2,3.$$

The turbulent flow $\boldsymbol{v}$ is assumed to be three-dimensional and incompressible ($\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$). Vectors belonging to the tangent space at the point $\boldsymbol{a}(s)$ evolve according to the equation

$$\dot{\boldsymbol{\tau}} = \sigma(\boldsymbol{a}(s), s)\boldsymbol{\tau}, \tag{2.2}$$

where $\boldsymbol{\sigma}$ denotes the gradient of the velocity, $\sigma^i_j = \partial_j v^i$. The latter equation can be recast in the form (Cocke 1969)

$$\boldsymbol{\tau}(t) = W(t, \boldsymbol{x}; 0, \boldsymbol{x}_0)\boldsymbol{\tau}(0),$$

with

$$W(t, \boldsymbol{x}; 0, \boldsymbol{x}_0) = \exp_{\leftarrow}\left[\int_0^t \sigma(\boldsymbol{a}(s), s)\mathrm{d}s\right].$$

The symbol $\exp_{\leftarrow}$ means that all the matrix factors σ in the series expansion of the exponential are arranged from right to left in order of increasing time argument; $\boldsymbol{a} = \boldsymbol{a}(s)$ is the point of application of $\boldsymbol{\tau}$ at time s and satisfies $\boldsymbol{a}(0) = \boldsymbol{x}_0$, $\boldsymbol{a}(t) = \boldsymbol{x}$.

The equal-time correlation between tangent vectors can be written in the form

$$\langle \boldsymbol{\tau}_{(1)}(t)\cdot\boldsymbol{\tau}_{(2)}(t)\rangle_{\mathcal{L}} = \tau^k_{(1)}(0)\tau^\ell_{(2)}(0)\int F^{ii}_{k\ell}(t, \boldsymbol{\rho}|0, \boldsymbol{r})\mathrm{d}\boldsymbol{\rho}, \tag{2.3}$$

where[1]

$$F^{ij}_{k\ell}(t, \boldsymbol{\rho}|0, \boldsymbol{r}) = \langle [W_{(1)}(t)]^i_k [W_{(2)}(t)]^j_\ell \delta(\boldsymbol{\rho}(t) - \boldsymbol{\rho})\rangle_{\mathcal{L}}.$$

The symbol $\langle\cdot\rangle_{\mathcal{L}}$ denotes the average over the stochastic trajectories that are at separation $\boldsymbol{r}$ at time $t=0$. The initial vectors $\boldsymbol{\tau}_{(1)}(0)$ and $\boldsymbol{\tau}_{(2)}(0)$ belong to two different tangent spaces coupled to two trajectories that start at separation $\boldsymbol{r}$. The linear operator $F(t, \boldsymbol{\rho}|0, \boldsymbol{r})$ propagates forward in time the two-particle configuration, $\boldsymbol{\rho}$ being the final separation between the points of application of $\boldsymbol{\tau}_{(1)}$ and $\boldsymbol{\tau}_{(2)}$. The form of the propagator F depends on the statistics of the flow.

In the Kraichnan–Kazantsev model, $\boldsymbol{v}(\boldsymbol{x}, t)$ is a Gaussian random field with zero mean and second-order correlation (Kazantsev 1967; Kraichnan 1968):

$$\langle v^i(\boldsymbol{x} + \boldsymbol{\rho}, t)v^j(\boldsymbol{x}, t')\rangle_v = [D_0\delta^{ij} - S^{ij}(\boldsymbol{\rho})]\delta(t - t').$$

The constant D_0 represents the eddy diffusivity and $S^{ij}(\boldsymbol{\rho})$defines the statistics of velocity differences at scale ρ. The trace of the tensor S^{ij} scales as ρ^2 below the dissipative scale η and tends to a constant beyond the correlation length of the flow, L. In the inertial range $\eta \ll \rho \ll L$, the flow is assumed to be scale invariant with exponent $0 \le \xi \le 2$:

$$S^{ij}(\boldsymbol{\rho}) = D_1\rho^\xi[(\xi + 2)\delta^{ij} - \xi\hat{\rho}^i\hat{\rho}^j], \quad \hat{\rho}^i \equiv \rho^i/\rho.$$

The latter expression holds under the supplementary assumption of statistical invariance with respect to rotations and reflections (Monin & Yaglom 1975). The scaling exponent ξ measures the spatial roughness of the flow: with decreasing ξ the flow changes gradually from smooth ($\xi = 2$) to purely diffusive ($\xi = 0$). Here, we focus on fluctuations within the inertial range of turbulence. Accordingly, throughout the paper spatial separations are assumed to lie within the interval

[1] For notational convenience we shall drop from now on the explicit dependence of W on the Lagrangian trajectory. Moreover, summation over repeated indices will be understood throughout the paper.

$\eta \ll \rho \ll L$. This means that we neglect the effects of the dissipative scale and the correlation length of the flow.

When the velocity field has the Kraichnan–Kazantsev statistics the propagator F obeys the equation

$$\partial_t F^{ij}_{k\ell}(t, \boldsymbol{\rho}|0, \boldsymbol{r}) = [\mathcal{M}(\boldsymbol{\rho})]^{ij}_{pq} F^{pq}_{k\ell}(t, \boldsymbol{\rho}|0, \boldsymbol{r}), \tag{2.4}$$

where $\mathcal{M}$ is the differential operator

$$[\mathcal{M}]^{ij}_{pq} = \delta^i_p \delta^j_q S^{\alpha\beta} \partial_\alpha \partial_\beta - \delta^i_p (\partial_q S^{\alpha j}) \partial_\alpha - \delta^j_q (\partial_p S^{i\beta}) \partial_\beta + (\partial_p \partial_q S^{ij}) + 2\kappa \delta^i_p \delta^j_q \Delta. \tag{2.5}$$

Equation (2.4) can be derived by Gaussian integration by parts (e.g. Klyatskin 2005). The notation $\mathcal{M}(\boldsymbol{\rho})$ means that the spatial derivatives in the definition of the operator are taken with respect to the components of $\boldsymbol{\rho}$. The behaviour of the two-vector correlation (2.3) is ruled by the operator:

$$Q_{k\ell}(t, \boldsymbol{r}) = \int F^{ii}_{k\ell}(t, \boldsymbol{\rho}|0, \boldsymbol{r}) \mathrm{d}\boldsymbol{\rho}.$$

The operator $Q_{k\ell}$ satisfies the adjoint of equation (2.4)

$$\partial_t Q_{k\ell}(t, \boldsymbol{r}) = [\mathcal{M}^\dagger(\boldsymbol{r})]^{ij}_{k\ell} Q_{ij}(t, \boldsymbol{r}), \tag{2.6}$$

where $\mathcal{M}^\dagger$ reads

$$[\mathcal{M}^\dagger]^{ij}_{pq} = \delta^i_p \delta^j_q S^{\alpha\beta} \partial_\alpha \partial_\beta + \delta^i_p (\partial_q S^{\alpha j}) \partial_\alpha + \delta^j_q (\partial_p S^{i\beta}) \partial_\beta + (\partial_p \partial_q S^{ij}) + 2\kappa \delta^i_p \delta^j_q \Delta. \tag{2.7}$$

Under the assumptions of statistical homogeneity, isotropy and parity invariance, $Q_{k\ell}(t, \boldsymbol{r})$ takes the form $Q_{k\ell}(t, \boldsymbol{r}) = G_1(t, r)\delta_{k\ell} + G_2(t, r)\hat{r}_k \hat{r}_\ell$ (Monin & Yaglom 1975). If diffusivity is disregarded, then equation (2.6) yields

$$\left. \begin{aligned} \partial_t G_1 &= 2D_1 r^\xi [\partial_{rr} G_1 + 2r^{-1}\partial_r G_1 + \xi(\xi+3) r^{-2} G_1 + 2(\xi+1) r^{-2} G_2], \\ \partial_t G_2 &= 2D_1 r^\xi [\partial_{rr} G_2 + 3\xi r^{-1} \partial_r G_1 + (3\xi+2) r^{-1} \partial_r G_2 + (\xi+3)(\xi-2) r^{-2} G_1 \\ &\qquad - (3\xi^2 - \xi - 6) r^{-2} G_2]. \end{aligned} \right\} \tag{2.8}$$

Hereafter, we drop the subscripts '1' and '2' since the analysis is the same for G_1 and G_2. We look for solutions of the kind $g_E(r)\mathrm{e}^{-\mathrm{Et}}$, where the set of values E is the spectrum of the operator $\mathcal{M}^\dagger$. The full solutions $G(t, r)$ is then recovered as a linear combination of the functions g_E. After equations (2.8), g_E should take the form

$$g_E(r) = \mathcal{G}((E/D_1)^{1/(2-\xi)} r),$$

where for small arguments $\mathcal{G}(z) \sim z^{\bar{\zeta}}$ with

$$\bar{\zeta} = \frac{1-\xi}{2} + \frac{1}{2}\sqrt{3(1-\xi)(3+\xi)}. \tag{2.9}$$

The exponent $\bar{\zeta}$ is non-negative, decreases with ξ, equals two for $\xi = 0$, and vanishes for $\xi = 1$. We consider the cases $0 \le \xi < 1$ and $1 < \xi \le 2$ separately.

In the range $0 \le \xi < 1$, the spectrum of $\mathcal{M}^\dagger$ is known to be continuous and composed of non-negative eigenvalues (Kazantsev 1967; Vergassola 1996); hence

$$G(t,r) = \int_0^\infty f(E)\mathcal{G}((E/D_1)^{1/(2-\xi)} r) \mathrm{e}^{-Et}\, \mathrm{d}E.$$

The spectral density $f(E)$ can be obtained by specifying the initial condition for Q_{kl}. Since $Q_{kl}(0, r) = \delta_{kl}$, we must impose $G_1(0, r) = 1$ and $G_2(0, r) = 0$, and this implies $f(E) \sim E^{-1}$. We now plug the spectral density back into the linear decomposition of G and look for its long-time asymptotics. Exploiting the expansion of $\mathcal{G}$ for small arguments, we end up with

$$\langle \boldsymbol{\tau}_{(1)}(t) \cdot \boldsymbol{\tau}_{(2)}(t) \rangle_{\mathcal{L}} \sim r^{\bar{\zeta}} t^{-\bar{\zeta}/(2-\xi)}, \quad 0 \le \xi < 1. \tag{2.10}$$

Therefore, the correlation between tangent vectors decays in time as a power law. The decay becomes increasingly slower as the scaling exponent of the flow, ξ, approaches one.

In the range $1 < \xi \le 2$, the spectrum of $\mathcal{M}^\dagger$ has a countable branch of negative eigenvalues (Kazantsev 1967; Vergassola 1996). The long-time behaviour of $Q_{k\ell}$ is then dominated by the 'ground state' of $\mathcal{M}^\dagger$ and the correlations between tangent vectors increase exponentially in time:

$$\langle \boldsymbol{\tau}_{(1)}(t) \cdot \boldsymbol{\tau}_{(2)}(t) \rangle_{\mathcal{L}} \sim \mathrm{e}^{|E_0|t}, \quad 1 < \xi \le 2, \tag{2.11}$$

$E_0 < 0$ being the lowest eigenvalue of $\mathcal{M}^\dagger$. The rate-of-growth $|E_0|$ is an increasing function of ξ (Kazantsev 1968; Vincenzi 2002; Horvai in preparation).

In conclusion, the time correlations between tangent vectors evolving along Lagrangian trajectories depend crucially on the scaling exponent ξ, that is on the spatial roughness of the turbulent flow. In the range $0 \le \xi < 1$, the correlations *decrease as power laws*, whereas in the range $1 < \xi \le 2$ they *grow exponentially with time*. In §3, we investigate how these results apply to the study of the dynamo effect.

3. Kinematic dynamo effect

Kinematic dynamo theory deals with the problem of magnetic field generation. In this case, it is natural to assume that the initial magnetic field is weak and does not affect the dynamics of the conducting medium. The study of dynamo effect thus reduces to a linear problem, which is entirely determined by the induction equation (Moffatt 1978)

$$\partial_t \boldsymbol{B} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \kappa \boldsymbol{\nabla}^2 \boldsymbol{B}, \tag{3.1}$$

supplemented by the solenoidality condition $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ and suitable boundary conditions for the magnetic field. In the above equation, κ represents the magnetic diffusivity, inversely proportional to the electrical conductivity of the fluid. The initial correlation-length of the magnetic field is supposed to lie within the inertial range of turbulence.

In the absence of diffusion ($\kappa = 0$), equation (3.1) can be solved by characteristics. The characteristics of the field $\boldsymbol{v}$ are the family of trajectories defined by the equation $\dot{\boldsymbol{a}} = \boldsymbol{v}(\boldsymbol{a}(s), s)$ and parameterized by their starting point $\boldsymbol{a}(0) = \boldsymbol{x}_0$. Along characteristic trajectories, the magnetic field evolves according to the same equation as for the vectors belonging to the tangent space (see equation (2.2)). To compute the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$, one has to select the trajectory that passes through $\boldsymbol{x}$ at time t ($\boldsymbol{a}(t) = \boldsymbol{x}$) and trace it back to its starting point $\boldsymbol{x}_0$ ($\boldsymbol{a}(0) = \boldsymbol{x}_0$). The magnetic field at the position $\boldsymbol{x}$ at time t is then recovered as the initial magnetic field at the position $\boldsymbol{x}_0$ evolved along the selected trajectory:

$$\boldsymbol{B}(\boldsymbol{x}, t) = W(t, \boldsymbol{x}; 0, \boldsymbol{x}_0)\boldsymbol{B}(\boldsymbol{x}_0, 0). \tag{3.2}$$

If the magnetic diffusivity is non-zero, the above construction of the solution is unchanged apart from the fact that the characteristics are stochastic trajectories, $\dot{\boldsymbol{a}} = \boldsymbol{v}(\boldsymbol{a}(s), s) + \sqrt{2\kappa}\boldsymbol{\beta}(s)$, and the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ is recovered as the average of the right-hand side of equation (3.2) over all stochastic trajectories passing through $\boldsymbol{x}$ at time t. The method of characteristics shows that the evolution of the field $\boldsymbol{B}$ has to be in strict connection with the Lagrangian dynamics in the tangent space. The main difference with §2 is that to compute the magnetic field one has to trace the field evolution backwards along stochastic trajectories, whereas §2 dealt with the forward evolution of tangent vectors.

The Lagrangian interpretation of the dynamo effect relies on the investigation of the second-order magnetic correlation $C^{ij}(\boldsymbol{r}, t) \equiv \langle B^i(\boldsymbol{x} + \boldsymbol{r}, t)B^j(\boldsymbol{x}, t)\rangle_v$. Thanks to the linearity of equation (3.1), $C^{ij}(\boldsymbol{r}, t)$ can be written in the form

$$C^{ij}(\boldsymbol{r}, t) = C^{k\ell}(0, \boldsymbol{r}) \int \bar{F}^{ij}_{k\ell}(0, \boldsymbol{\rho}|t, \boldsymbol{r})\mathrm{d}\boldsymbol{\rho}, \tag{3.3}$$

where

$$\bar{F}^{ij}_{k\ell}(0, \boldsymbol{\rho}|t, \boldsymbol{r}) = \langle [W_{(1)}(t)]^i_k [W_{(2)}(t)]^j_\ell \delta(\boldsymbol{\rho}(0) - \boldsymbol{\rho})\rangle_{\mathcal{L}}.$$

The average is now taken over stochastic trajectories that terminates at position $\boldsymbol{r}$ at time t, as emphasized by the conditioning in the form of the propagator. The operator $\bar{F}$ propagates backwards in time the configurations in physical and tangent spaces, and its evolution is, therefore, ruled by $\mathcal{M}^\dagger$ (equation (2.7)). Equation (3.3) establishes the formal relation between magnetic field evolution and tangent-space dynamics (see equation (2.3)).

When the advecting flow belongs to the Kraichnan–Kazantsev statistical ensemble, the magnetic correlation satisfies a closed equation (Lanotte & Mazzino 1999):

$$\partial_t C^{ij}(\boldsymbol{r}, t) = [\mathcal{M}(\boldsymbol{r})]^{ij}_{k\ell} C^{k\ell}(\boldsymbol{r}, t), \tag{3.4}$$

where $\mathcal{M}$ is exactly the operator defined in equation (2.5). The inertial-range dynamo for a δ-correlated Gaussian flow was firstly studied by Kazantsev (1967) and has been more recently investigated by Vergassola (1996), Vincenzi (2002), Boldyrev & Cattaneo (2004) and Horvai (in preparation). Equation (3.4) can be recast as a one-dimensional Schrödinger-like equation for the trace of the tensor C^{ij}. The form of the 'quantum' potential depends on the statistics of velocity increments, hence on the scaling exponent ξ. The value of ξ plays a fundamental role. For $\xi < 1$, the second-order correlation of the magnetic field decays in time; for $\xi > 1$ it grows exponentially (dynamo effect). In other words, the roughness of the flow inhibits the amplification of the magnetic field, $\xi = 1$ being the threshold for the dynamo effect (Kazantsev 1967).

In §2, we have shown that the time correlation between vectors belonging to tangent spaces coupled to different Lagrangian trajectories depends critically on ξ. The value $\xi=1$ marks the transition from the power-law decay to the exponential growth of Lagrangian correlations. Therefore, we can interpret the dynamo effect as follows. When there is not dynamo effect the correlations between tangent vectors decrease with time as power laws despite the fact that the absolute value of each tangent vector increases exponentially. This effect has to be ascribed to the fast decorrelation between angular degrees of freedom. On the contrary, in the dynamo regime the amplification of the absolute value prevails and the correlations between tangent vectors grow exponentially in time.

4. Zero modes and statistical conservation laws

The above results can be reformulated in terms of statistical conservation laws for the Lagrangian dynamics. The time behaviour of the correlations $\langle \boldsymbol{\tau}_{(1)}(t)\cdot\boldsymbol{\tau}_{(2)}(t)\rangle_{\mathcal{L}}$ is indeed strictly related to the existence of special functions of particle positions and tangent vectors that are statistically preserved by the Lagrangian dynamics.

We first consider the decay regime, $0\le\xi<1$. We denote by $\bar{Z}$ the zero mode of the adjoint operator $\mathcal{M}^{\dagger}$: $[\mathcal{M}^{\dagger}]^{ij}_{pq}\bar{Z}_{ij}=0$. From equation (2.4), it is easily seen that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \bar{Z}_{ij}(\boldsymbol{\rho})F^{ij}_{k\ell}(t,\boldsymbol{\rho}|0,\boldsymbol{r})\mathrm{d}\boldsymbol{\rho}=0.$$

Consequently, the Lagrangian dynamics admits the statistical integral of motion:

$$\langle \tau^{i}_{(1)}(t)\tau^{j}_{(2)}(t)\bar{Z}_{ij}(\boldsymbol{\rho}(t))\rangle_{\mathcal{L}}=\text{const.} \tag{4.1}$$

Under the assumption of statistical isotropy and parity invariance, the inertial-range expression of the zero mode is

$$\bar{Z}_{ij}(\boldsymbol{\rho})=\rho^{\bar{\zeta}}\left(K_1\delta_{ij}+K_2\frac{\rho_i\rho_j}{\rho^2}\right),\quad \eta\ll\rho\ll L,$$

where $K_1/K_2=-2(1+\xi)(1+\xi+\bar{\zeta})/(\xi^3+\xi^2+6+6\bar{\zeta})$ and $\bar{\zeta}$ has been defined in equation (2.9). Tangent vectors evolve on time-scales of the order of λ^{-1}, where λ is the maximum Lyapunov exponent of the flow. This is the shortest time-scale into play since it is shorter than diffusive time-scales, which are themselves shorter than inertial-range ones. In contrast, within the inertial range particle separations evolve on extremely longer time-scales, of the order of $\rho^{2-\xi}/D_1$. Because of this scale separation, statistical conservation law (4.1) implies

$$\langle \tau^{i}_{(1)}(t)\tau^{j}_{(2)}(t)\rangle_{\mathcal{L}}\langle \bar{Z}_{ij}(\boldsymbol{\rho}(t))\rangle_{\mathcal{L}}\sim\text{const.}$$

Since within the inertial range $\bar{Z}_{ij}(\boldsymbol{\rho})$ scales as $\rho^{\bar{\zeta}}$ and $\rho\sim t^{1/(2-\xi)}$, we finally obtain

$$\langle \tau^{i}_{(1)}(t)\tau^{j}_{(2)}(t)\rangle_{\mathcal{L}}\sim t^{-\bar{\zeta}/(2-\xi)},\quad 0\le\xi<1.$$

Therefore, the power-law decay of time correlations, equation (2.10), can be regarded as a result of statistical conservation law (4.1). The characteristic exponent $\bar{\zeta}$ is the scaling exponent of the zero mode of $\mathcal{M}^{\dagger}$. It is worth noticing that $\bar{\zeta}-2$ equals the scaling exponent for the zero mode of $\mathcal{M}$, i.e. the dominant contribution to the magnetic correlation at small-scales (Vergassola 1996).

In the range $1<\xi\leq 2$, the spectrum of $\mathcal{M}^{\dagger}$ is known to admit negative energies, which are on the contrary forbidden for $0\leq\xi<1$ (Kazantsev 1967; Vergassola 1996). This entails an exponential growth of magnetic field correlations, i.e. the dynamo effect. Let ψ^{0}_{ij} be the 'ground state' of $\mathcal{M}^{\dagger}$ and $E_0<0$ the corresponding eigenvalue:

$$[\mathcal{M}^{\dagger}]^{ij}_{pq}\psi^{0}_{ij} = -E_0\psi^{0}_{pq}.$$

Then, the following statistical conservation law holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int F^{ij}_{k\ell}(t,\boldsymbol{\rho}|0,\boldsymbol{r})\psi^{0}_{ij}(\boldsymbol{\rho})\mathrm{e}^{E_0 t}\,\mathrm{d}\boldsymbol{\rho} = 0,$$

or equivalently

$$\langle \tau^{i}_{(1)}(t)\tau^{j}_{(2)}(t)\psi^{0}_{ij}(\boldsymbol{\rho}(t))\mathrm{e}^{E_0 t}\rangle_{\mathcal{L}} = \mathrm{const.}$$

It should be noted that in each realization of the turbulent flow the statistical constant of motion depends explicitly on time through the decreasing exponential $\exp(E_0 t)$. It becomes constant only after the average over all realizations of the velocity field. On the basis of the aforementioned time-scale separation we expect that

$$\langle \tau^{i}_{(1)}(t)\tau^{j}_{(2)}(t)\rangle_{\mathcal{L}}\langle\psi^{0}_{ij}(\boldsymbol{\rho}(t))\rangle_{\mathcal{L}}\,\mathrm{e}^{E_0 t} \sim \mathrm{const.}$$

Since for large separations within the inertial range $\psi^{0}_{ij}(\boldsymbol{\rho})$ is likely to decay as a stretched exponential (Vincenzi 2002), we recover equation (2.11):

$$\langle \tau^{i}_{(1)}(t)\tau^{j}_{(2)}(t)\rangle_{\mathcal{L}} \sim \mathrm{e}^{|E_0|t}, \quad 1<\xi\leq 2.$$

Once more, the law ruling the time behaviour of correlations between tangent vectors can be interpreted as a by-product of a statistical conservation law.

5. Conclusions

We have formulated a Lagrangian interpretation of the turbulent dynamo effect. Our interpretation is based on a systematic analysis for the Kraichnan–Kazantsev model. The evolution of the magnetic field is connected to tangent-space dynamics along Lagrangian trajectories. The magnetic field results from the interplay between the exponential amplification of the norm of tangent vectors along single trajectories and the angular decorrelation between tangent vectors attached to different trajectories. The dynamo effect occurs when the former effect prevails and the correlations between tangent vectors grow exponentially in time. On the contrary, if the angular decorrelation prevails, the two-vector correlations decay in time as power laws and the magnetic energy decreases. The above behaviours depend on the existence of statistical integrals of motion for the Lagrangian dynamics. This supports the generality of the results proved for the Kraichnan–Kazantsev model. We indeed expect that the tangent-space dynamics along Lagrangian trajectories should not qualitatively change for different turbulent or chaotic flows (Childress & Gilbert 1995) or in the presence of a mean magnetic field (Biferale & Procaccia 2005). Therefore, we believe that our analysis of the kinematic dynamo in an isotropic short-correlated flow can be qualitatively applied to more realistic situations.

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