# Tensor and homotopy criteria for functional equations of $\ell$-adic and classical iterated integrals

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## 1 Introduction

The purpose of this paper is to show equivalence of two criteria for functional equations of (complex and $\ell$-adic) iterated integrals, one given by D. Zagier in the case of polylogarithms which we generalize to arbitrary iterated integrals and the other given by the second named author. We establish a device for computing a functional equation from a family of morphisms on fundamental groups of varieties, and present some examples showing how our device works commonly both in complex and $\ell$-adic cases. Some of our $\ell$-adic examples already supply non-trivial arithmetic relations between "$\ell$-adic polylogarithmic characters" – functions on the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ defined by Kummer properties along towers of certain arithmetic sequences – which were introduced in [NW] as generalization of the so-called Soulé characters studied by Ch. Soulé [S1], [S2].

Let $V := \mathbf{P}^1 - \{\text{several points}\}$ be a punctured projective line defined over a subfield $K$ of $\mathbb{C}$. In [W0,W2,W5], the second named author gave conditions to have functional equations of iterated integrals on $V$ in terms of induced morphisms on fundamental groups. In fact, in [W0], he formulated a complex iterated integral as the image of the (universal) unipotent period along a chain from $x$ to $z$ on $V(\mathbb{C})$ by a 1-form on the Lie algebra of the pro-unipotent fundamental group of $V$. Also in [W5], introduced is an $\ell$-adic iterated integral using the action of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ on the torsor of paths from $x$

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to $z$ (see Section 4 below). Then, the following result has been proved for their functional equations.

**Theorem 1.1** ([W0,W2,W5]) *Let $K$ be a subfield of $\mathbb{C}$, and let $\{a_1, \ldots, a_N\}$, $\{b_1, \ldots, b_M\}$ be respectively $N$- and $M$-point subsets of $K$. Consider $X := \mathbf{P}^1_K - \{a_1, \ldots, a_N, \infty\}$, $Y := \mathbf{P}^1_K - \{b_1, \ldots, b_M, \infty\}$, and pick any $K$-rational (possibly tangential) base point $v$ on $X$. Suppose we have algebraic morphisms $f_i\colon X \to Y$ $(i = 1, \ldots, m)$ together with homomorphisms $\psi_i\colon \mathrm{gr}^n_\Gamma \pi_1(Y(\mathbb{C}), f_i(v)) \to \mathbb{Z}$ and constants $c_1, \ldots, c_m \in \mathbb{Z}$ satisfying*

$$\sum_{i=1}^{m} c_i \, \psi_i \circ \mathrm{gr}^n_\Gamma(f_{i*}) = 0.$$

*Here, $f_{i*}\colon \pi_1(X(\mathbb{C}), v) \to \pi_1(Y(\mathbb{C}), f_i(v))$ denotes the induced homomorphism, and $\mathrm{gr}^n_\Gamma$ denotes the nth graded piece with respect to the lower central filtration. Then, we have a functional equation*

$$\sum_{i=1}^{m} c_i \, \mathcal{L}^{\psi}_Y(f_i(z), f_i(x)) \equiv 0$$

*modulo lower degree terms, where $\mathcal{L}^{\psi}_Y(f_i(z), f_i(x))$ denote, depending on the context, complex or $\ell$-adic iterated integrals on $Y$. (The lower degree terms will be specified later in this paper.)*

On the other hand, in [Z], D. Zagier gave conditions for the functional equations of classical polylogarithms in terms of (generalized) Bloch group [Bl1] that is a certain tensor of symmetric and wedge products of multiplicative groups of fields (cf. also [Ga] (1.10)). We generalize Zagier's conditions from [Z] to arbitrary iterated integrals in terms of tensor algebra of the abelianization of $\pi_1$.

The aim of this paper is, first to show that the condition on fundamental groups from [W0,W2,W5] and that on tensor algebras are essentially equivalent in a generalized setting of the above iterated integrals. Also we generalize results from [W5] to the case where $X$ is an arbitrary nonsingular variety (not necessarily a punctured projective line) in $\ell$-adic case. See Theorem 4.13 and Theorem 4.14 for our main statement of this paper.

Our main tool is a multi-linearized version of the classical Kummer pairing

$$\pi_1(X)^{\mathrm{ab}} \times \mathcal{O}_X^\times \longrightarrow \mathbb{Z}$$

for an algebraic variety $X$, where $\pi_1(X)^{\mathrm{ab}}$ is the abelianized fundamental group of $X$ and $\mathcal{O}_X^\times$ is the unit group of the ring of regular functions on $X$. The pairing in the complex case is given by

$$(\gamma, f(x)) \mapsto \frac{1}{2\pi i}\int_\gamma d\log f(x),$$

and in this paper, both components of $\gamma$ and $f(x)$ will be multi-linearized to study informations appearing in the higher graded quotients of the fundamental group. We make use of this tool to establish a device computing a functional equation from a family of morphisms satisfying the criteria of Theorem 4.13 and Theorem 4.14.

One new aspect of our device is that it enables us to compute "lower degree terms" of a functional equation of polylogarithms explicitly from given data in both complex and $\ell$-adic cases. A difference between complex and $\ell$-adic cases appears in that a complex iterated integral is canonically graded while an $\ell$-adic iterated integral is not. This causes us, in $\ell$-adic case, to need to introduce an extra notion of "($\ell$-adic) error term" whose computation involves a choice of splitting of the lower central filtration in an $\ell$-adic fundamental group. Applying our method developed in this paper, we shall deduce in Section 6 the following examples of typical functional equations.

| Complex case | $\ell$-adic case |
|---|---|
| $Li_2(z) + Li_2(1-z) + \log z \log(1-z) = \frac{\pi^2}{6}$ | $\tilde{\chi}_2^z + \tilde{\chi}_2^{1-z} + \rho_z\rho_{1-z} = \frac{1}{24}(\chi^2 - 1)$ |
| $Li_2(z) + Li_2(\frac{z}{z-1}) = -\frac{1}{2}\log^2(1-z)$ | $\tilde{\chi}_2^z + \tilde{\chi}_2^{\frac{z}{z-1}} = \frac{\rho_{1-z}}{2}\big(\chi - \rho_{1-z}\big)$ |
| $Li_m(z) + (-1)^m Li_m(\frac{1}{z})$ $= -\frac{(2\pi i)^m}{m!}B_m(\frac{\log z}{2\pi i})$ | $\tilde{\chi}_m^z + (-1)^m\tilde{\chi}_m^{1/z}$ $= -\frac{1}{m}\{B_m(-\rho_z) - B_m\chi^m\}$ |
| $Li_2(\frac{\xi\eta}{(1-\xi)(1-\eta)}) = Li_2(\frac{\xi}{1-\eta}) + Li_2(\frac{\eta}{1-\xi})$ $-Li_2(\xi) - Li_2(\eta) - \log(1-\xi)\log(1-\eta)$ | $\tilde{\chi}_2^{\frac{\xi\eta}{(1-\xi)(1-\eta)}} = \tilde{\chi}_2^{\frac{\xi}{1-\eta}} + \tilde{\chi}_2^{\frac{\eta}{1-\xi}}$ $-\tilde{\chi}_2^\xi - \tilde{\chi}_2^\eta - \rho_{1-\xi}\rho_{1-\eta}$ |

Here, $\tilde{\chi}_m^z\colon G_K \to \mathbb{Z}_\ell$ ($m \geq 1$) (resp. $\chi\colon G_K \to \mathbb{Z}_\ell^\times$; resp. $\rho_z\colon G_K \to \mathbb{Z}_\ell$) denotes the $\ell$-adic polylogarithmic character introduced in [NW] (resp. $\ell$-adic cyclotomic character; resp. Kummer 1-cocycle along $\ell$-power roots of $z$) for any field $K(\subset \mathbb{C})$ containing $z$ (cf. §5.2), and $B_m$ (resp. $B_m(X)$) is the $m$th Bernoulli number (resp. polynomial).

The contents of the present paper will be ordered as follows. In Section 2, we review and study basic properties of the multi-Kummer pairing, and in Section 3, we detect a tensor of functions as the multi-Kummer dual of a form on the Lie algebra of the fundamental group. In Section 4, we rephrase conditions of [W0,W2,W4] on a collection of homomorphisms of $\pi_1(X)$ in terms of conditions on tensor and wedge products of functions, and prove our main statements Theorem 4.13 and Theorem 4.14. As a special case, in Section 5, we closely consider the case of polylogarithms. We will present a more refined statement than Theorems 4.13–4.14 specialized to this case. Section 6 is devoted to present several typical examples of functional equations (listed in the above table) exhibiting computation using our device of this paper.

There is also an important family of functional equations of polylogarithms called the *distribution equations*:

$$Li_k(z^n) = n^{k-1}\left(\sum_{i=0}^{n-1} Li_k(\zeta_n^i z)\right) \qquad (\zeta_n = e^{2\pi i/n})$$

which, together with their $\ell$-adic analogs, will be treated from our point of view in the forthcoming subsequent paper [NW2].

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## 2 Multi-Kummer characters

Let $X$ be a (nonsingular, absolutely irreducible) algebraic variety defined over a number field $K$, a subfield of the field of complex numbers $\mathbb{C}$. We denote by $X^{\mathrm{an}} = X(\mathbb{C})$ the analytic manifold of the complex points of $X$, and by $X_{\overline{K}}$ the algebraic variety obtained as the fiber product $X \times_K \overline{K}$, where $\overline{K}$ is the algebraic closure of $K$ in $\mathbb{C}$. Fix a $K$-rational (tangential) base point $v$ on $X$. (For a definition of tangential base points, see, e.g., [N0].)

**2.1 Complex case** We shall write $\mathcal{O}(X^{\mathrm{an}})$ for the ring of holomorphic functions on $X^{\mathrm{an}}$. If a function $f \in \mathcal{O}(X^{\mathrm{an}})$ is everywhere non-vanishing on $X(\mathbb{C})$, i.e., $f$ belongs to the unit group $\mathcal{O}(X^{\mathrm{an}})^\times$, then it gives an analytic morphism of $X$ to the multiplicative group $f\colon X \to \mathbb{G}_m(\mathbb{C})$. Any topological loop $\gamma$ based at $v$ on $X(\mathbb{C})$ is mapped by $f$ to a loop $f(\gamma)$ on the punctured plane $\mathbb{G}_m(\mathbb{C}) = \mathbb{C} - \{0\}$ with the winding number

$$\kappa_f(\gamma) := \frac{1}{2\pi i}\int_{f(\gamma)} \frac{dw}{w} = \frac{1}{2\pi i}\int_\gamma d\log f(x)$$

that obviously belongs to $\mathbb{Z}$. From this arises the Kummer pairing (in the complex case)

$$\pi_1(X^{\mathrm{an}})^{\mathrm{ab}} \times \mathcal{O}(X^{\mathrm{an}})^\times \longrightarrow \mathbb{Z}.$$

If the function $f$ is of the form $\exp(g)$ for some $g \in \mathcal{O}(X^{\mathrm{an}})$, then $\kappa_f$ kills all loops $\gamma$ of $\pi_1(X(\mathbb{C}))$. Define

$$\mathcal{O}_h^\times(X^{\mathrm{an}}) := \mathcal{O}(X^{\mathrm{an}})^\times / \exp(\mathcal{O}(X^{\mathrm{an}})).$$

Then, for each positive integer $n$, we obtain a natural mapping

$$\kappa^{\otimes n} : \bigotimes_{i=1}^{n} \mathcal{O}_h^\times(X^{\mathrm{an}}) \longrightarrow \mathrm{Hom}(\bigotimes_{i=1}^{n} \pi_1(X(\mathbb{C}))^{\mathrm{ab}}, \mathbb{Z}),$$

which will play a central role below.

*Remark.* In the sequel, we shall use both notations $A^{\otimes n}$ and $\bigotimes_{i=1}^n A$ to denote the $n$-times tensor product of $A$.

**2.2 $\ell$-adic case** By abuse of notation, for an algebraic variety $X$, we understand $\mathcal{O}(X)$ to be the ring of regular (algebraic) functions on $X$. Recall from [Roq] (cf. also [La1] chapter II), the unit group $\mathcal{O}(X)^\times$ modulo $K^\times$ is finitely generated, torsion free abelian group.

Let $f \in \mathcal{O}(X_{\overline{K}})^\times$ and pick any $\gamma$ from the étale fundamental group $\pi_1(X_{\overline{K}}, v)$. Then, we have an algebraic winding number $\kappa_f(\gamma) \in \hat{\mathbb{Z}}$ as follows. Form the fibre product $X' = X \times_{\mathbb{G}_m} \mathbb{G}_m$ induced from the $n$-power map $\mathbb{G}_m \to \mathbb{G}_m$:

$$\begin{array}{ccc} X' & \xrightarrow{f_n} & \mathbb{G}_m \\ {\scriptstyle p_n(f)}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow{f} & \mathbb{G}_m, \end{array}$$

then $p_n = p_n(f) \colon X' \to X$ is a (not necessarily connected) finite étale cover of $X$ of degree $n$. The fundamental group $\pi_1(X, v)$ acts, by definition, on the $n$ point set $p_n^{-1}(v)$, and the action of each loop $\gamma \in \pi_1(X, v)$ is represented by a certain residue $\kappa_n \in (\mathbb{Z}/n\mathbb{Z})$ that "rotates" the upper $\mathbb{G}_m$ through the angle $2\pi\kappa_n/n$. Actually, $\kappa_n$ is determined up to the residue class of $f \in \mathcal{O}(X_{\overline{K}})^\times$ modulo $\mathcal{O}(X_{\overline{K}})^{\times n}$. Letting $n$ run over all positive integers multiplicatively, the sequence $\{\kappa_n\}$ defines a coherent element $\kappa_f(\gamma) \in \hat{\mathbb{Z}} = \varprojlim_n (\mathbb{Z}/n\mathbb{Z})$ according to any given class of $\widehat{\mathcal{O}^\times}(X_{\overline{K}}) := \varprojlim_n \mathcal{O}(X_{\overline{K}})^\times / \mathcal{O}(X_{\overline{K}})^{\times n}$. Thus, we obtain the Kummer pairing

$$\pi_1(X_{\overline{K}})^{\mathrm{ab}} \times \widehat{\mathcal{O}^\times}(X_{\overline{K}}) \longrightarrow \hat{\mathbb{Z}}.$$

If we fix a prime number $\ell$ and look only at the $\ell$-power maps of $\mathbb{G}_m$, then the above pairing terminates at $\mathbb{Z}_\ell$ and $\pi_1(X_{\overline{K}})$ and $\widehat{O}^\times(X_{\overline{K}})$ factor through

$$\pi_1^\ell(X_{\overline{K}}) := \text{the maximal pro-}\ell\text{ quotient of } \pi_1(X_{\overline{K}}),$$
$$\widehat{O}_\ell^\times(X_{\overline{K}}) := \varprojlim_n (O(X_{\overline{K}})^\times / O(X_{\overline{K}})^{\times \ell^n})$$

respectively. We also obtain the natural analog of $\kappa^{\otimes n}$ (written by the same symbol, for simplicity):

$$\kappa^{\otimes n} : \bigotimes_{i=1}^n \widehat{O}_\ell^\times(X_{\overline{K}}) \longrightarrow \mathrm{Hom}(\bigotimes_{i=1}^n \pi_1^\ell(X_{\overline{K}})^{\mathrm{ab}}, \mathbb{Z}_\ell),$$

where the $\otimes$ are understood to be taken over $\mathbb{Z}_\ell$.

**Lemma 2.1** *In both complex and ℓ-adic cases, $\kappa^{\otimes n}$ is injective.*

*Proof* The complex case: taking the cohomology associated to the exact sequence of sheaves $0 \to \mathbb{Z} \to O \overset{\exp}{\to} O^\times \to 0$ on $X^{\mathrm{an}}$ (and the universal coefficient theorem for cohomology and the Hurewicz theorem), we have an injection

$$O_h^\times(X^{\mathrm{an}}) \hookrightarrow H^1(X^{\mathrm{an}}, \mathbb{Z}) \tilde{\to} \mathrm{Hom}(\pi_1(X^{\mathrm{an}})^{\mathrm{ab}}, \mathbb{Z}).$$

This settles the case $n = 1$. For the case $n > 1$, noting that both $O_h^\times(X^{\mathrm{an}})$ and $\mathrm{Hom}(\pi_1(X^{\mathrm{an}})^{\mathrm{ab}}, \mathbb{Z})$ are torsion free, i.e. flat $\mathbb{Z}$-modules, we obtain the injection

$$\bigotimes_{i=1}^n O_h^\times(X^{\mathrm{an}}) \longrightarrow \bigotimes_{i=1}^n \mathrm{Hom}(\pi_1(X^{\mathrm{an}})^{\mathrm{ab}}, \mathbb{Z}) = \mathrm{Hom}(\bigotimes_{i=1}^n \pi_1(X^{\mathrm{an}})^{\mathrm{ab}}, \mathbb{Z})$$

(cf. [B-1] chapter 2 §4 (23) for the latter equality). The ℓ-adic case: since $O(X_{\overline{K}})^{\times \ell^n}$ contains $\overline{K}^\times$, $\widehat{O}_\ell^\times(X_{\overline{K}})$ is a torsion-free $\mathbb{Z}_\ell$-module. This is injectively mapped into $\mathrm{Hom}(\pi_1^\ell(X_{\overline{K}})^{\mathrm{ab}}, \mathbb{Z}_\ell)$, as $\ell^n$th roots of each non-constant function give non-trivial extensions of the function field of $X$ when $n \to \infty$. Thus, the case of $n = 1$ follows. For the case $n > 1$, noting again that these are both flat $\mathbb{Z}_\ell$-modules, we complete the proof in the same way. □

**Remark 2.2** It is known that $\pi_1(X(\mathbb{C}))$ is a finitely generated group (see [Ra]). Therefore, the domain and the target modules of $\kappa^{\otimes n}$ are finitely generated torsion free $\mathbb{Z}$- or $\mathbb{Z}_\ell$-modules. Moreover, in the complex case, the cohomology sequence in the above proof implies that the cokernel of $O_h^\times(X^{\mathrm{an}}) \hookrightarrow H^1(X^{\mathrm{an}}, \mathbb{Z})$ is injectively mapped into the complex vector space $H^1(X^{\mathrm{an}}, O)$, hence is a (finitely generated) torsion-free $\mathbb{Z}$-module (in particular, it is a "pure" submodule in the sense of [B-2] chapter 1 §2 ex. 24). Meanwhile, in the ℓ-adic case, torsion possibility in the cokernel of $O_\ell^\times(X_{\overline{K}}) \hookrightarrow \mathrm{Hom}(\pi_1^\ell(X_{\overline{K}})^{\mathrm{ab}}, \mathbb{Z}_\ell)$ is a more subtle question.

**Remark 2.3** In our argument below, use of $O_h^\times(X^{\mathrm{an}})$ may be replaced by $O^\times(X_{\mathbb{C}}^{\mathrm{alg}})/\mathbb{C}^\times$, the multiplicative group of the algebraic unit functions modulo constants, as this group is also injectively mapped in $O_h^\times(X^{\mathrm{an}})$. This injectivity follows easily from the fact $\exp(O(X^{\mathrm{an}})) \cap O^\times(X_{\mathbb{C}}^{\mathrm{alg}}) = \mathbb{C}^\times$. In fact, if an analytic function $f$ on $X^{\mathrm{an}}$ has $\exp(f)$ being an algebraic regular function on $X$, then, all $\exp(f/n)$ $(n \geq 1)$ must be univalent algebraic functions on $X$, while $O^\times(X_{\mathbb{C}}^{\mathrm{alg}})/\mathbb{C}^\times$ has no nontrivial divisible elements by the above mentioned result by Roquette [Roq]. Alternatively, one can use the Kummer sequence in Galois cohomology to show the injectivity of $O^\times(X_{\mathbb{C}}^{\mathrm{alg}})/O^\times(X_{\mathbb{C}}^{\mathrm{alg}})^{\ell^n}$ into $\mathrm{Hom}(\pi_1^\ell(X_{\overline{K}})^{\mathrm{ab}}, \mathbb{Z}/\ell^n\mathbb{Z})$, and then take the projective limit $n \to \infty$.

## 3 Multi-Kummer duals

We recall that the lower central series of a group $\pi$ is defined inductively by setting

$$\Gamma^1\pi := \pi, \quad \Gamma^n\pi := (\pi, \Gamma^{n-1}\pi) \quad n > 1.$$

The commutator bracket $(x, y) = xyx^{-1}y^{-1}$ then induces the Lie algebra structure on the graded sum

$$\mathrm{Gr\,Lie}\,\pi = \bigoplus_{n=1}^{\infty} \mathrm{gr}^n\pi = \bigoplus_{n=1}^{\infty} (\Gamma^n\pi/\Gamma^{n+1}\pi)$$

in such a way that, for $\alpha \in \mathrm{gr}^n\pi, \beta \in \mathrm{gr}^m\pi$, the Lie bracket is given by

$$[\alpha,\beta] := (\alpha,\beta) \bmod \Gamma^{n+m+1}\pi.$$

We denote by $\bar{x}$ the image of $x \in \pi$ in the abelianization $\pi^{\mathrm{ab}} = \mathrm{gr}^1\pi$.

**Definition 3.1** Define a natural map

$$\boldsymbol{a}_n(\pi)\colon \bigotimes_{i=1}^{n} \pi^{\mathrm{ab}} \longrightarrow \mathrm{gr}^n\pi$$

by induction on $n$ by setting $\boldsymbol{a}_1(\pi) :=$ identity on $\pi^{\mathrm{ab}}$ and

$$\boldsymbol{a}_n(\pi)(\alpha_1 \otimes \cdots \otimes \alpha_n) := [\alpha_1, \boldsymbol{a}_{n-1}(\pi)(\alpha_2 \otimes \cdots \otimes \alpha_n)]$$

for $n > 1$. In the case when $\pi$ is a pro-$\ell$ group, we define $\boldsymbol{a}_n(\pi)$ in exactly the same way after taking the lower central series and the tensor products respectively as topological pro-$\ell$ groups and as $\mathbb{Z}_\ell$-modules. It is also important to consider the case where $\pi$ is replaced by its quotient by the (closure of, in the pro-$\ell$ case) double commutator subgroup $\pi''$ of $\pi$. It is obvious that the maps $\boldsymbol{a}_n(\pi), \boldsymbol{a}_n(\pi/\pi'')$ are *surjective*, therefore they induce injective homomorphisms

$$\mathrm{Hom}(\mathrm{gr}^n_\Gamma(\pi/\pi''), \mathbb{Z}) \hookrightarrow \mathrm{Hom}(\mathrm{gr}^n_\Gamma\pi, \mathbb{Z}) \hookrightarrow \mathrm{Hom}(\bigotimes_{i=1}^{n} \pi^{\mathrm{ab}}, \mathbb{Z}).$$

We denote by $\boldsymbol{a}^*_n(\pi)$ (resp. $\boldsymbol{a}^*_n(\pi/\pi'')$) the injection of $\mathrm{Hom}(\mathrm{gr}^n_\Gamma\pi, \mathbb{Z})$ (resp. $\mathrm{Hom}(\mathrm{gr}^n_\Gamma(\pi/\pi''), \mathbb{Z})$) into $\mathrm{Hom}(\bigotimes_{i=1}^n \pi^{\mathrm{ab}}, \mathbb{Z})$ induced by $\boldsymbol{a}_n(\pi)$ (resp. $\boldsymbol{a}_n(\pi/\pi'')$).

Now, let us consider permutations of components of $\bigotimes_{i=1}^n \pi^{\mathrm{ab}}$. We introduce special permutation actions $\sigma$, $\tau$ by

$$\begin{cases} \sigma(\eta_1 \otimes \cdots \otimes \eta_{n-3} \otimes a \otimes b \otimes c) = \eta_1 \otimes \cdots \otimes \eta_{n-3} \otimes b \otimes c \otimes a, \\ \tau(\eta_1 \otimes \cdots \otimes \eta_{n-2} \otimes a \otimes b) = \eta_1 \otimes \cdots \otimes \eta_{n-2} \otimes b \otimes a, \end{cases}$$

and let any permutation $\rho \in S_{n-2}$ act by

$$\rho(\eta_1 \otimes \cdots \otimes \eta_{n-2} \otimes a \otimes b) = \eta_{\rho(1)} \otimes \cdots \otimes \eta_{\rho(n-2)} \otimes a \otimes b.$$

**Proposition 3.2** *If a homomorphism* $\varphi\colon \bigotimes_{i=1}^n \pi^{\mathrm{ab}} \to \mathbb{Z}$ *belongs to the image of* $\boldsymbol{a}^*_n(\pi)$*, then, for* $\boldsymbol{\eta} \in \bigotimes_{i=1}^n \pi^{\mathrm{ab}}$,

(i) $$\varphi(\boldsymbol{\eta}) + \varphi(\tau(\boldsymbol{\eta})) = 0,$$

(ii) $$\varphi(\boldsymbol{\eta}) + \varphi(\sigma(\boldsymbol{\eta})) + \varphi(\sigma^2(\boldsymbol{\eta})) = 0.$$

*If* $\varphi : \bigotimes_{i=1}^n \pi^{\mathrm{ab}} \to \mathbb{Z}$ *belongs moreover to the image of* $\boldsymbol{a}^*_n(\pi/\pi'')$*, then*

(iii) $$\varphi(\boldsymbol{\eta}) = \varphi(\rho(\boldsymbol{\eta})) \text{ for all } \rho \in S_{n-2}.$$

*Proof* Equations (i), (ii) follow immediately from the Lie identities $[A, B] + [B, A] = 0$, $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ respectively. Equation (iii) follows by observing that

$$[A, [B, [C, D]]] = [[A, B], [C, D]] + [B, [A, [C, D]]$$

and the fact that $[[A, B], [C, D]] = 0$ in $\mathrm{Gr}\,\mathrm{Lie}(\pi/\pi'')$. □

**Remark 3.3** When $\pi$ is a free group with free generators $y_1, \ldots, y_N$, the Lie algebra $\mathrm{Gr}(\pi) := \oplus_n \mathrm{gr}^n_\Gamma\pi$ can be regarded as the Lie part of the graded free associative algebra $A(\pi) := \oplus_n (\pi^{\mathrm{ab}})^{\otimes n}$. Therefore, there is a natural embedding $\iota_n\colon \mathrm{gr}^n_\Gamma\pi \hookrightarrow (\pi^{\mathrm{ab}})^{\otimes n}$. It is known that $\iota_n$ gives a "$\frac{1}{n}$-splitting" of the above surjection $\boldsymbol{a}_n\colon (\pi^{\mathrm{ab}})^{\otimes n} \to \mathrm{gr}^n_\Gamma\pi$ (cf. [MKS] Theorem 5.17). For a general group $\pi$, we do not have an analog of the mapping $\iota_n$, so we choose to argue in the principal use of $\boldsymbol{a}_n$.

**Notation 3.4** For an abelian group $A$, let the symmetric group $S_n$ of degree $n$ act on $A^{\otimes n}$ by the component permutations. We write $\mathfrak{Sym}^n A$ (resp. $\mathfrak{Alt}^n A$) the usual symmetric (resp. alternate) tensor product defined as the maximal quotient of $A^{\otimes n}$ on which $S_n$ acts trivially (resp. by sign-alterations). On the other hand, we regard the wedge tensor product $\bigwedge^n A$ (resp. the symmetric tensor product $\mathrm{Sym}^n A$) as the submodule of $\bigotimes_{i=1}^n A$ consisting of those elements which are sign-alternated (resp. invariant) under the component permutations by $S_n$. We use the following notations for elements in the latter type of symmetric and alternative tensor products of an abelian group $A$:

$$a \wedge b := a \otimes b - b \otimes a \quad \in A \wedge A \ \subset \bigotimes_{i=1}^{2} A,$$
$$a^{\odot n} := \underbrace{a \otimes \cdots \otimes a}_{n} \quad \in \mathrm{Sym}^n A \ \subset \bigotimes_{i=1}^{n} A.$$

(The last notation is borrowed from e.g. [Bl2], [Ga], and will be used only later in Sections 5 and 6. In fact, since our $\mathrm{Sym}^n A$ is now given as the invariant subspace of $A^{\otimes n}$ (unlike in the quotient symmetric tensor space $\mathfrak{Sym}^n A$ of $A^{\otimes n}$), how to define a "canonical $a_1 \odot \cdots \odot a_n$" in $\mathrm{Sym}^n A$ becomes a nontrivial question, even if it may be restricted to our necessary case $A \cong \mathbb{Z}^r$. We leave the question as a problem for future study when necessity arises.) If $A$ is a $\mathbb{Z}_\ell$-module, we shall understand the tensor $\otimes$ as taken in the category of $\mathbb{Z}_\ell$-modules.

Now, suppose that the above $\pi$ is given as the fundamental group $\pi_1(X(\mathbb{C}))$ of an algebraic variety $X$ over a field $K \subset \mathbb{C}$ as in Section 2. The above properties (i), (iii) of Proposition 3.2 would lead us to consider the following type of commutative diagram:

$$\begin{array}{ccccc}
\bigotimes_{i=1}^n \mathcal{O}_h^\times & \xrightarrow[inj.]{\kappa^{\otimes n}} & \mathrm{Hom}((\pi_1^{\mathrm{ab}})^{\otimes n}, \mathbb{Z}) & & \\
\uparrow{\scriptstyle incl.} & & \uparrow{\scriptstyle incl.} & & \\
(\mathcal{O}_h^\times)^{\otimes n-2} \otimes \wedge^2 \mathcal{O}_h^\times & \xrightarrow[inj.]{} & \mathrm{Hom}((\pi_1^{\mathrm{ab}})^{\otimes n-2} \otimes \mathfrak{Alt}^2 \pi_1^{\mathrm{ab}}, \mathbb{Z}) & \xleftarrow{incl.} & \mathrm{Hom}(\mathrm{gr}_\Gamma^n \pi_1, \mathbb{Z}) \\
\uparrow{\scriptstyle incl.} & & \uparrow{\scriptstyle incl.} & & \uparrow{\scriptstyle incl.} \\
\mathrm{Sym}^{n-2} \mathcal{O}_h^\times \otimes \wedge^2 \mathcal{O}_h^\times & \xrightarrow[inj.]{} & \mathrm{Hom}(\mathfrak{Sym}^{n-2} \pi_1^{\mathrm{ab}} \otimes \mathfrak{Alt}^2 \pi_1^{\mathrm{ab}}, \mathbb{Z}) & \xleftarrow{incl.} & \mathrm{Hom}(\mathrm{gr}^n(\pi_1/\pi_1''), \mathbb{Z})
\end{array}$$

Here, we should like to relate the homomorphisms in $\mathrm{Hom}(\mathrm{gr}_\Gamma^n \pi_1, \mathbb{Z})$ and $\mathrm{Hom}(\mathrm{gr}^n(\pi_1/\pi_1''), \mathbb{Z})$ with tensors of the unit functions. But since $\kappa^{\otimes n}$ is not necessarily surjective in general, we shall first restrict ourselves to the case where surjectivity of $\kappa^{\otimes n}$ is available: to the special case where $\pi$ is given as the fundamental group of the complex points of the projective $t$-line $Y =$

$\mathbf{P}^1_t - \{b_1, \ldots, b_N, \infty\}$ ($N \geq 2$) with $y_i$ a standard loop running once around the puncture $b_i$ based at a fixed (tangential) base point $v$ on $Y$. Then, $\pi$ is a free group, freely generated by the $y_1, \ldots, y_N$. Set $[1, N] = \{1, ..., N\}$, and, for any $n$-tuple $\boldsymbol{k} = (k_1, \ldots, k_n) \in [1, N]^n$, write $\bar{y}_{\boldsymbol{k}} = \bar{y}_{k_1} \otimes \cdots \otimes \bar{y}_{k_n} \in \bigotimes_{i=1}^n \pi^{\mathrm{ab}}$. Observe that the group of units on $Y$ is given by

$$(\mathcal{O}(Y^{an})^\times \supset)\, \mathcal{O}(Y_{\overline{K}})^\times = \overline{K}\Bigg[t,\ \prod_{i=1}^N (t - b_i)^{-1}\Bigg]^\times = \overline{K}^\times \prod_{i=1}^N (t - b_i)^{\mathbb{Z}}.$$

**Proposition 3.5** *Given the above notation for* $Y = \mathbf{P}^1_t - \{b_1, \ldots, b_N,\ \infty\}$ $(N \geq 2)$*, for any* $\varphi\colon \bigotimes_{i=1}^n \pi_1(Y(\mathbb{C}), v)^{\mathrm{ab}} \to \mathbb{Z}$*, we have*

$$\varphi = \kappa^{\otimes n}\Bigg(\sum_{\boldsymbol{k} \in [1,N]^n} \varphi(\bar{y}_{\boldsymbol{k}})(t - b_{k_1}) \otimes \cdots \otimes (t - b_{k_n})\Bigg).$$

*Also, its obvious ℓ-adic analog holds, where* $\pi_1(Y(\mathbb{C}), v)$*,* $\mathbb{Z}$ *are replaced by* $\pi_1^\ell(Y_{\overline{K}}, v)$*,* $\mathbb{Z}_\ell$ *respectively. In particular,* $\kappa^{\otimes n}$ *gives an isomorphism:*

$$\mathcal{O}_h^\times(Y^{\mathrm{an}})^{\otimes n} \xrightarrow{\sim} \mathrm{Hom}(\pi_1^{\mathrm{ab}}(Y(\mathbb{C}))^{\otimes n}, \mathbb{Z})$$
$$\Big(\textit{resp. } \mathcal{O}_\ell^\times(Y_{\overline{K}})^{\otimes n} \xrightarrow{\sim} \mathrm{Hom}(\pi_1^{\mathrm{ab}}(Y_{\overline{K}})^{\otimes n}, \mathbb{Z}_\ell)\Big).$$

*Proof* As the right-hand side equals

$$\sum_{\boldsymbol{k} \in [1,N]^n} \varphi(\bar{y}_{\boldsymbol{k}})\ \kappa_{t-b_{k_1}} \otimes \cdots \otimes \kappa_{t-b_{k_n}},$$

the multi-linearity of $\varphi$ reduces the proof to showing

$$\kappa_{t-b_i}(\bar{y}_j) = \delta_{ij} \quad (1 \leq i, j \leq N;\ \delta = \text{Kronecker's delta}).$$

But this is immediate from the definition of the loops $y_j$. The ℓ-adic case follows in the same way as above. □

For $Y = \mathbf{P}^1_t - \{b_1, \ldots, b_N, \infty\}$ ($N \geq 2$), Proposition 3.2 may be rephrased in terms of the coefficients of the above proposition as follows. We let the permutations $\sigma, \tau \in S_n$ and $\rho \in S_{n-2}$ act on $\boldsymbol{k} = (k_1, \ldots, k_n) \in [1, N]^n$ by $\tau(\boldsymbol{k}) = (k_1, \ldots, k_{n-2}, k_n, k_{n-1})$, $\sigma(\boldsymbol{k}) = (k_1, \ldots, k_{n-3}, k_{n-1}, k_n, k_{n-2})$ and $\rho(\boldsymbol{k}) = (k_{\rho(1)}, \ldots, k_{\rho(n-2)}, k_{n-1}, k_n)$.

**Lemma 3.6** *If a homomorphism* $\varphi\colon \bigotimes_{i=1}^n \pi^{\mathrm{ab}} \to \mathbb{Z}$ *belongs to the image of* $\boldsymbol{a}_n^*(\pi)$*, then,*

(i) $$\varphi(\bar{y}_{\boldsymbol{k}}) + \varphi(\bar{y}_{\tau(\boldsymbol{k})}) = 0,$$

(ii) $$\varphi(\bar{y}_{\boldsymbol{k}}) + \varphi(\bar{y}_{\sigma(\boldsymbol{k})}) + \varphi(\bar{y}_{\sigma^2(\boldsymbol{k})}) = 0.$$

*If* $\varphi\colon \bigotimes_{i=1}^{n}\pi^{\mathrm{ab}} \to \mathbb{Z}$ *belongs moreover to the image of* $\boldsymbol{a}_n^*(\pi/\pi'')$*, then,*

(iii) $$\varphi(\bar{y}_{\boldsymbol{k}}) = \varphi(\bar{y}_{\rho(\boldsymbol{k})}) \quad \textit{for all} \quad \rho \in S_{n-2}.$$

**Corollary 3.7** *Using the same notation as in Proposition 3.5, let* $\pi$ *denote* $\pi_1(Y(\mathbb{C}), v)$ *in the complex case, and suppose that* $\varphi$ *lies in the image of* $\boldsymbol{a}_n^*(\pi)$*, i.e.,* $\varphi \in \mathrm{Hom}(\mathrm{gr}_\Gamma^n\pi, \mathbb{Z})$*. Then,*

$$\varphi = \kappa^{\otimes n}\Bigg(\sum_{\boldsymbol{k}} \sum_{\substack{l_1<l_2\\ l_1,l_2\in[1,N]}} \varphi(\bar{y}_{(\boldsymbol{k},l_1,l_2)})(t-b_{k_1})\otimes\cdots\otimes(t-b_{k_{n-2}})\otimes\big((t-b_{l_1})\wedge(t-b_{l_2})\big)\Bigg),$$

*where* $\boldsymbol{k} = (k_1,\ldots,k_{n-1})$ *runs over all tuples belonging to* $[1,N]^{n-2}$*. If moreover* $\varphi$ *lies in the image of* $\boldsymbol{a}_n^*(\pi/\pi'')$*, i.e.,* $\varphi \in \mathrm{Hom}(\mathrm{gr}_\Gamma^n(\pi/\pi''), \mathbb{Z})$*, then the coefficients* $\varphi(\bar{y}_{(\boldsymbol{k},l_1,l_2)})$ *are invariant under the action of* $S_{n-2}$ *on the first* $n-2$ *tensor components, so that the above inside of* $\kappa^{\otimes n}$ *lies in* $\mathrm{Sym}^{n-2}\mathcal{O}_h^\times(Y^{\mathrm{an}}) \otimes \wedge^2\mathcal{O}_h^\times(Y^{\mathrm{an}})$*. Exactly parallel statements also hold in the* $\ell$*-adic case after substituting* $\pi_1^\ell(Y_{\overline{K}}, v)$*,* $\mathbb{Z}_\ell$ *and* $\mathcal{O}_\ell^\times(Y_{\overline{K}})$ *in the obvious way.*

*Proof* This follows as a simple combination of Proposition 3.2(i) and Proposition 3.5. □

**Definition 3.8** We shall call the inside of the right-hand side of the above corollary the *Kummer dual* of $\varphi$ and write $\widehat{\kappa_{\otimes n}}(\varphi)(t)$. Regarded as an element of $(\bigotimes_{i=1}^{n}\mathcal{O}_h^\times(Y^{\mathrm{an}}))\otimes\bigwedge^2\mathcal{O}_h^\times(Y^{\mathrm{an}})$ in the complex case, and of $(\bigotimes_{i=1}^{n}\widehat{\mathcal{O}}_\ell^\times(Y_{\overline{K}}))\otimes\bigwedge^2\widehat{\mathcal{O}}_\ell^\times(Y_{\overline{K}})$ in the $\ell$-adic case, the Kummer dual $\widehat{\kappa_{\otimes n}}(\varphi)(t)$ of $\varphi$ is uniquely determined by the equality

$$\varphi = \kappa^{\otimes n}(\widehat{\kappa_{\otimes n}}(\varphi))$$

by virtue of the injectivity of $\kappa^{\otimes n}$ shown in Lemma 2.1. Moreover, given a $k$-morphism of an algebraic variety $X$ with base point $v$ to the punctured projective $t$-line $Y$:

$$f\colon X \longrightarrow Y = \mathbf{P}_t^1 - \{b_1,\ldots,b_M,\infty\}$$

and $\varphi \in \mathrm{Hom}(\mathrm{gr}_\Gamma^n(\pi_1(Y(\mathbb{C}), f(v))), \mathbb{Z})$ (or $\in \mathrm{Hom}(\mathrm{gr}_\Gamma^n(\pi_1(Y_{\overline{K}}, f(v))), \mathbb{Z}_\ell)$ in the $\ell$-adic case), we shall denote by $\widehat{\kappa_{\otimes n}}(\varphi)(f)$ the pulled-back image of the Kummer dual $\widehat{\kappa_{\otimes n}}(\varphi)(t)$ induced by the mapping $g \mapsto g\circ f$ of the unit functions on $Y$ to those on $X$.

## 4 Iterated integrals and their functional equations

We first review the definition of complex and $\ell$-adic iterated integrals. For the reader's convenience, we make the paper as complete as possible even if this means that we repeat some arguments from earlier papers of the second named author. Let $X$ be an algebraic variety defined over $K \subset \mathbb{C}$. We assume, for simplicity, that $X$ is nonsingular and absolutely irreducible over $K$.

**4.1 Complex iterated integrals** In this subsection, we set $K = \mathbb{C}$ and pick complex points $v, z \in X(\mathbb{C})$ and a path $p$ from $v$ to $z$ on $X(\mathbb{C})$. Given a collection of holomorphic 1-forms $w_1, \ldots, w_n$ on the smooth analytic manifold $X(\mathbb{C})$, we can form an iterated integral $\int_p w_1 \cdots w_n$.

Take a smooth compactification $X^*$ of $X$ with $D := X^* - X$ a normal crossing divisor, and let $\Omega^i (= \Omega^i_{\log}(X)) := \Omega^i(X \log D)(X^*)$ be the space of (global sections of) meromorphic $i$-forms on $X^*$, holomorphic on $X$, with logarithmic singularities along $D$. It is known that the spaces $\Omega^i_{\log}(X)$ are determined independently of the choice of the compactification $X^*$ of $X$ as above (cf. [Ii] chapter 11).

Let $V_i$ be the dual space of $\Omega^i$ and $K^\perp \subset V_1 \wedge V_1$ be the orthogonal space to the kernel of the cup product $\Omega^1 \wedge \Omega^1 \to \Omega^2$.

**Definition 4.1** Let $\mathrm{Lie}(V_1) = \bigoplus_{n=1}^\infty \mathrm{Lie}(V_1)_n$ be the free Lie algebra generated by $V_1$ equipped with natural gradation by $\mathrm{Lie}(V_1)_n$ – the part of homogenous Lie polynomials of degree $n$. Let $L(V_1, K^\perp)$ be the quotient of $\mathrm{Lie}(V_1)$ modulo the ideal generated by $K^\perp \subset \mathrm{Lie}(V_1)_2 = V_1 \wedge V_1$. We denote by $L_X$ the completion of $L(V_1, K^\perp)$ with respect to the lower central series, and by $U_X$ (resp. $\pi(X)$) the complete Hopf algebra given as the universal enveloping algebra of $L_X$ (resp. the group of the group-like elements of $U_X$).

Note that $L(V_1, K^\perp)$ has a natural gradation $L(V_1, K^\perp) = \bigoplus_{n=1}^{\infty} L(V_1, K^\perp)_n$ inherited from that of $\mathrm{Lie}(V_1)$. There is a natural bijection between $L_X$ and $\pi(X)$ given by exp and log that also preserves the lower central filtrations of $L_X$ and of $\pi(X)$ mutually. Thus, we have a *canonical* identification (cf. also [De2] §12):

$$\pi(X)/\Gamma^{N+1}\pi(X) \underset{\exp}{\overset{\log}{\rightleftarrows}} L_X/\Gamma^{N+1}L_X \cong \bigoplus_{n=1}^{N} L(V_1, K^\perp)_n. \tag{4.1}$$

It is known that the 1-form $\omega_X \in V_1 \otimes \Omega^1 = \mathrm{Hom}(\Omega^1, \Omega^1)$ corresponding to the identity element gives an integrable connection on the trivial principal bundle $X \times U_X \to X$, i.e., it satisfies $d\omega_X = \omega_X \wedge \omega_X = 0$ (cf. [H] §4, [W2] §1).

Given a path $\gamma$ from $v$ to $z$, the associated horizontal section starting from $(v, 1) \in X \times U_X$ over $\gamma$ terminates at a point $(z, \Lambda_\gamma(z, v))$ that is uniquely determined as long as $\gamma\colon v \rightsquigarrow z$ changes in the same homotopy class. From this, one can define the parallel transport mapping

$$\theta_{v,z,X}\colon \pi_1(X, v, z) \longrightarrow \pi(X) \quad (\gamma \mapsto \Lambda_\gamma(z, v)). \tag{4.2}$$

This construction is compatible with composition of paths, i.e., for paths $\alpha\colon v \rightsquigarrow y, \beta\colon y \rightsquigarrow z$, we have

$$\Lambda_\alpha(y,v)\Lambda_\beta(z,y) = \Lambda_{\alpha\beta}(z,v). \tag{4.3}$$

It is shown by K.-T. Chen that $\Lambda_\gamma(z,v)$ can be expressed in terms of iterated integrals as

$$\Lambda_\gamma(z,v) = 1 + \int_\gamma \omega_X + \int_\gamma \omega_X\omega_X + \int_\gamma \omega_X\omega_X\omega_X + \cdots . \tag{4.4}$$

The above constructions of $L_X$, $\pi(X)$, $U_X$ together with $\Lambda_\gamma(z,v)$ are functorial with respect to morphisms, i.e., for any morphism $f\colon X \to Y$, the naturally induced homomorphism $f_*\colon L_X \to L_Y$ (that can be extended to the homomorphism $f_*\colon U_X \to U_Y$ which also gives $f_*\colon \pi(X) \to \pi(Y)$) maps each component of the gradation of $L_X$ into the corresponding component of $L_Y$, and keeps

$$f_*(\Lambda_\gamma(z,v)) = \Lambda_{f(\gamma)}(f(z), f(v)) \tag{4.5}$$

(see e.g. corollary 1.7 of [W3]). In fact, the pull-back of differentials by a morphism $f\colon X \to Y$ keeps the differentials with log poles, i.e., it holds that $f^*\Omega^1_{\log}(Y) \subset \Omega^1_{\log}(X)$ (cf. [Ii] §11.1c). This, together with the fact $f_*\omega_X = f^*\omega_Y$ (where, $f_*$, $f^*$ act on $V_1 \otimes \Omega^1$ respectively by $f_* \otimes \mathrm{id}$, $\mathrm{id} \otimes f^*$) implies (4.5).

When $v = z$, the monodromy map extends to a surjective homomorphism of complete Hopf algebras

$$\theta_{v,X}\colon \mathbb{C}\pi_1(X,v)^\wedge \longrightarrow U_X. \tag{4.6}$$

Notably, R. Hain [H] determined the kernel of $\theta_{v,X}$ to be the ideal $I := F^0 \cap J + F^{-1} \cap J^2 + F^{-2} \cap J^3 + \cdots$, where $J$ is the augmentation ideal and $F^i$ are Hodge filtrations on the Malcev fundamental group algebra $\mathbb{C}\pi_1(X,v)^\wedge$.

**Remark 4.2** Note that $\Omega^1_{\log}(X) \subset H^1_{DR}(X/\mathbb{C})$ ([De1]). In the above construction, we need not assume the equality of this inclusion. In particular, we allow the case $H^1(X^*, \mathcal{O}_{X^*}) \neq \{0\}$.

**Example 4.3** Let $X$ be an elliptic curve $E := \{y^2 = x^3 + ax + b\} \cup \{\infty\}$ minus a set of several points $D = \{p_1, \dots, p_N\} \cup \{\infty\}$. By Abel's theorem (cf. [Rob] p. 134), every holomorphic 1-form $\omega$ with poles included in $D$ is uniquely written in the form

$$\omega = \underbrace{\left(df + \alpha\frac{xdx}{y}\right)}_{\omega_2} + \underbrace{\left(\beta\frac{dx}{y}\right)}_{\omega_1} + \underbrace{\left(\sum_{i=1}^N \gamma_i \frac{y + y(p_i)}{x - x(p_i)}\frac{dx}{y}\right)}_{\omega_3},$$

with $f \in \mathcal{O}(E \setminus D)^\times$, $\alpha, \beta, \gamma_i \in \mathbb{C}$. If we write $\omega_2$, $\omega_1$, $\omega_3$ respectively for the above three terms (. . . .), then $\omega_1$ is a differential of the 1st kind, $\omega_2+\omega_1$ is of the 2nd kind and $\omega_1+\omega_3$ is of the 3rd kind. (Residues of the differential $\frac{y+y(p_i)}{x-x(p_i)}\frac{dx}{y}$ are $\pm 2$.) The space $\Omega^1_{\log}(E\setminus D)$ consists of the differential forms $\omega$ with no $\omega_2$, i.e., only of the 1st and 3rd differentials, that is a finite dimensional vector space. (If we allow the part $\omega_2$ to be alive, then we would need to deal with an infinite dimensional space of forms.) Meanwhile the map $\Omega^1(X\log D)(X^*)\to H^1_{DR}(X/\mathbb{C})$ is injective, but generally has a non-trivial cokernel (corresponding to the class of anti-holomorphic differentials). For any given morphism $f\colon E\setminus D\to \mathbf{P}^1_t\setminus\{a_1,\dots,a_M,\infty\}$, the pullback by $f$ sends each $\frac{dt}{t-a_i}$ into $\Omega^1_{\log}(E\setminus D)$. This observation exhibits a point that enables us to carry out the above construction in a functorial way with respect to morphisms, and leads us to generalize our previous result (cf. Theorem 1.1) in the setting of an arbitrary variety $X$ accompanied with morphisms to a punctured line.

Now, suppose $X=\mathbf{P}^1-\{a_1,\dots,a_N,\infty\}$ $(N\geq 2)$ and that we are given a form $\psi\colon \mathrm{gr}^n_\Gamma\pi_1(X,v)\to\mathbb{Z}$. In this special case, Hain's kernel is trivial and the monodromy representation $\theta_{v,X}$ gives a canonical isomorphism between $\mathbb{C}\pi_1(X,v)^\wedge$ and $U_X$. Moreover, since $K^\perp=0$, one can identify

$$\mathrm{gr}^n\theta_{v,X}\colon\ \mathrm{gr}^n_\Gamma\pi_1(X,v)\otimes\mathbb{C}\xrightarrow{\sim}\mathrm{Lie}(V_1)_n. \tag{4.7}$$

Then, the linear form $\psi_{\mathbb{C}}=\psi\otimes\mathbb{C}$ on $\mathrm{gr}^n_\Gamma\pi_1(X,v)\otimes\mathbb{C}$ composed with (the splitting of $L_X/\Gamma^{N+1}L_X$ and) log of (4.1) can be regarded as a polynomial function on $\pi(X)/\Gamma^{n+1}\pi(X)$. Note also that one can follow the same story even when $v$ is a tangential base point on $X$ (See [W3] §3).

**Definition 4.4** Assume $X=\mathbf{P}^1-\{a_1,\dots,a_N,\infty\}$ $(N\geq 2)$. We may think of $\psi_{\mathbb{C}}\circ\log$ as a polynomial function on $\pi(X)/\Gamma^{n+1}\pi(X)$ as above. We shall call

$$\mathcal{L}^\psi_{\mathbb{C}}(z,v;\gamma):=\psi_{\mathbb{C}}\left(\log\Lambda_\gamma(z,v)^{-1}\bmod\Gamma^{n+1}\right)$$

the *complex iterated integral* associated to the form $\psi\in\mathrm{Hom}(\mathrm{gr}^n_\Gamma\pi_1(X,v),\mathbb{Z})$ and the path $\gamma\colon v\rightsquigarrow z$ on $X$.

**4.2 ℓ-adic iterated integrals ([W4])** Let $v$, $z$ be $K$-rational (possibly tangential) points on $X$, and $p$ be an étale path from $v$ to $z$.

There is an isomorphism of pro-$\ell$ spaces between the pro-$\ell$ fundamental group $\pi:=\pi_1^\ell(X_{\overline{K}},v)$ and the pro-$\ell$ torsor of paths $\pi_1^\ell(X_{\overline{K}},v,z)$ from $v$ to $z$ given by

$$\pi=\pi_1^\ell(X_{\overline{K}},v)\xrightarrow{\sim}\pi_1^\ell(X_{\overline{K}},v,z)\quad(\gamma\mapsto p\gamma^{-1}). \tag{4.8}$$

Through this identification, the natural Galois action on $\pi_1^\ell(X_{\overline{K}},v,z)$ induces a $\mathbb{Q}_\ell$-linear Galois representation $G_K\to\mathrm{GL}(\hat{U}(\pi))$ $(\sigma\mapsto\sigma_p)$ in the universal enveloping algebra $\hat{U}(\pi)$ of the (complete) $\ell$-adic Lie algebra $L(\pi)$ of $\pi$. This new action generally depends on the choice of path $p\colon v \rightsquigarrow y$, and is determined by the formula

$$\sigma_p(S) := \mathfrak{f}_\sigma^p \cdot \sigma(S) \qquad (S \in \pi_1^\ell(X_{\overline{K}}, v),\ \sigma \in G_K), \tag{4.9}$$

where $\mathfrak{f}_\sigma^p := p \cdot \sigma(p)^{-1}$. To be distinguished from the above action $\sigma_p$, we shall also write $\sigma_v$ to designate the standard Galois action at $v\colon G_K \to \mathrm{Aut}(\hat{U}(\pi))$ $(\sigma \mapsto \sigma_v)$.

**Definition 4.5** Let $\{\Gamma_i\}_{i=1,2,\dots}$ denote the lower central filtrations of $L(\pi)$, and set

$$G_i(v) := \{\sigma \in G_K \mid \sigma_v : \text{trivial on } L(\pi)/\Gamma^{i+1}L(\pi)\};$$
$$H_i(z, v) := \{\sigma \in G_i(v) \mid \sigma_p : \text{trivial on } L(\pi)/\Gamma^{i}L(\pi)\}.$$

It follows from [W4], lemma 1.0.5, that $H_i(z, v)$ does not depend on the choice of $p$. Note that, for $i = 1$, $G_1(v) = H_1(z, v)$.

Now, recall that we have a canonical group homomorphism of $\pi$ into $\pi(\mathbb{Q}_\ell)$ – the ($\mathbb{Q}_\ell$-valued points) of the pro-algebraic group formed by the group-like elements of $U(\pi)$. There is also a bijective correspondence between $\pi(\mathbb{Q}_\ell)$ and $L(\pi)$ given by log and exp. In this paper, for any element $\mathfrak{f} \in \pi$, we shall write simply $\log(\mathfrak{f}) \in L(\pi)$ to denote the log of the image of $\mathfrak{f}$ in $\pi(\mathbb{Q}_\ell)$. On the other hand, the universal enveloping algebra $U(\pi)$ has the augmentation ideal $I$ and we have $I/I^2 \xrightarrow{\sim} \mathrm{gr}^1_\Gamma(\pi) \otimes \mathbb{Q}_\ell$. A $\mathbb{Q}_\ell$-linear automorphism $\varepsilon$ of $\hat{U}(\pi)$ is called unipotent if it acts trivially on $I/I^2$. We write $\mathrm{Log}(\varepsilon)$ for the $\mathbb{Q}_\ell$-linear endomorphism obtained as the logarithm of a $\mathbb{Q}_\ell$-linear unipotent automorphism $\varepsilon$ of $\hat{U}(\pi)$. The following basic facts were proved in [W4].

**Proposition 4.6** *Suppose $\sigma \in G_1(v)$. Then:*

*(i) The actions of $\sigma_v$ and $\sigma_p$ on $\hat{U}(\pi)$ are unipotent.*
*(ii)* $(\mathrm{Log}\sigma_p)(1) \in \hat{U}(\pi)$ *is a Lie element, i.e., belongs to the Lie part $L(\pi) \subset \hat{U}(\pi)$.*
*(iii)* $\mathrm{Log}(\sigma_p) = L_{(\mathrm{Log}\sigma_p)(1)} + \mathrm{Log}(\sigma_v)$*, where $L_\lambda$ means the left multiplication by $\lambda$.*
*(iv) If $\sigma \in H_i(z, v)$, then* $(\mathrm{Log}\sigma_p)(1)$ *belongs to* $\Gamma^i L(\pi)$.

**Definition 4.7** For each $\mathbb{Q}_\ell$-valued form $\psi$ on $L(\pi)$, we define the *naive $\ell$-adic iterated integral* $\mathcal{L}^\psi_{\mathrm{nv}}(z, v; p)$ to be the function on $G_K$ given by

$$\mathcal{L}^\psi_{\mathrm{nv}}(z, v; p)(\sigma) := \psi(\log(\mathfrak{f}_\sigma^p)^{-1}) \quad (\sigma \in G_K),$$

where $\mathfrak{f}_\sigma^p = p \cdot \sigma(p)^{-1}$. The *big ℓ-adic iterated integral* $\mathcal{L}^\psi(z, v; p)$ is defined to be the function on $G_1(v)$ by

$$\mathcal{L}^\psi(z, v; p)(\sigma) := \psi((\mathrm{Log}\sigma_p)(1)) \quad (\sigma \in G_1(v)).$$

If $\psi$ is a(n induced) form on $\mathrm{gr}^i_\Gamma \pi \otimes \mathbb{Q}_\ell \cong \Gamma^i L(\pi)/\Gamma^{i+1} L(\pi)$, then we define the *ℓ-adic iterated integral* $\mathcal{L}^\psi(z, v)$ to be the restriction of $\mathcal{L}^\psi(z, v; p)$ on $H_i(z, v)$, that is,

$$\mathcal{L}^\psi(z, v)(\sigma) := \psi((\mathrm{Log}\sigma_p)(1)) \quad (\sigma \in H_i(z, v)).$$

**Proposition 4.8** *$\mathcal{L}^\psi(z, v)$ does not depend on the choice of the path p.* □

In the special case of $X = \mathbf{P}^1_K - \{a_1, \dots, a_N, \infty\}$ $(N \geq 2)$, we may proceed to some more construction. Fix a generator system $\vec{x} = (x_1, \dots, x_N)$ of the topological fundamental group $\pi_1(X(\mathbb{C}), v)$ so that $x_i$ $(1 \leq i \leq N)$ is a loop based at $v$ running around the puncture $a_i$ once in the anti-clockwise way. Since $\pi = \pi_1^\ell(X_{\overline{K}}, v)$ is canonically isomorphic to the pro-ℓ completion of $\pi_1(X(\mathbb{C}), v)$, we may regard $\vec{x}$ as the topological generator system of $\pi$. Then, $X_i = \log(x_i)$ freely generate the complete ℓ-adic Lie algebra $L(\pi)$, i.e., every element of $L(\pi)$ has an expansion as a formal Lie series in $X_1, \dots, X_N$. In particular, we may define the homogeneous degree $n$ part $L(\pi)_{n,\vec{x}} \subset L(\pi)$ and the decomposition

$$L(\pi) = \prod_{n=1}^{\infty} L(\pi)_{n,\vec{x}} \tag{4.10}$$

depending on the choice of $\vec{x}$. Given a $\mathbb{Z}$-valued form $\psi\colon \mathrm{gr}^n \pi_1(X(\mathbb{C}), v) \to \mathbb{Z}$ and a generator system $\vec{x} = (x_1, \dots, x_N)$ of $\pi_1(X(\mathbb{C}), v)$, we obtain a $\mathbb{Q}_\ell$-valued form $\psi_{\vec{x}}$ (or just written $\psi$) on $L(\pi)$ as the composition of

$$\psi_{\vec{x}} : L(\pi) \xrightarrow{\text{proj.}} L(\pi)_{n,\vec{x}} \xrightarrow{\sim} \mathrm{gr}^n \pi_1(X, v) \otimes \mathbb{Q}_\ell \xrightarrow{\psi \otimes \mathrm{id}} \mathbb{Q}_\ell, \tag{4.11}$$

where the middle isomorphism is the one induced by mapping $X_i \mapsto \bar{x}_i \in \mathrm{gr}^1_\Gamma(\pi_1(X, v))$. For this isomorphism $\psi_{\vec{x}}$, we may apply Definition 4.7 above to define the ℓ-adic naive iterated integral $\mathcal{L}^{\psi_{\vec{x}}}_{\mathrm{nv}}(z, v; p)$, the ℓ-adic big iterated integral $\mathcal{L}^{\psi_{\vec{x}}}(z, v; p)$ and ℓ-adic iterated integral $\mathcal{L}^{\psi_{\vec{x}}}(z, v)$.

The graded quotients $G_n(v)/G_{n+1}(v)$ and $H_n(z, v)/H_{n+1}(z, v)$ have natural $G_K$-module structures by conjugation. This action is shown to be factored through $G_K/G_1(v)$. Sometimes useful is, denoting by $\mathcal{T}$ a finite collection of pairs $(z_j, v_j)$ of $K$-rational (tangential) points on $X$, to consider the intersection

$$H_n(\mathcal{T}) := \bigcap_{(z_j, v_j) \in \mathcal{T}} H_n(z_j, v_j). \tag{4.12}$$

In the case of $X = \mathbf{P}^1_K - \{a_1, \dots, a_N, \infty\}$, we see that $G_1(v) = G_{K(\mu_{\ell^\infty})}$ and that

$G_K/G_1(v) = \mathrm{Gal}(K(\mu_{\ell^\infty})/K)$ acts on each graded quotient $H_n(\mathcal{T})/H_{n+1}(\mathcal{T})$ via multiplication by the $n$th power of $\chi\colon G_K \to \mathbb{Z}_\ell^\times$, the cyclotomic character, i.e. makes it isomorphic to a finite sum of the Tate twist $\mathbb{Z}_\ell(n)$. The standard weight argument assures

**Proposition 4.9** *Notations being as above for $X = \mathbf{P}^1_K - \{a_1, \ldots, a_N, \infty\}$ $(N \geq 2)$, the natural homomorphisms*

$$H_n(\mathcal{T})/H_{n+1}(\mathcal{T}) \to H_n(\mathcal{S})/H_{n+1}(\mathcal{S}) \to G_n(v)/G_{n+1}(v)$$

$(n = 1, 2, ...)$ *are almost surjective (i.e., have finite cokernels) for any subset $\mathcal{S} \subset \mathcal{T}$.* □

Let us now take $\mathcal{T}$ to be a collection of $(z, v)$ and $(\vec{a_i}, v)$ with $\vec{a_i}$ being any $K$-rational tangential base point at $a_i$ for $1 \leq i \leq N$.

**Proposition 4.10** *Notations being as above, the $\ell$-adic iterated integral $\mathcal{L}^{\psi_{\vec{x}}}(z, v)$ on $H_n(\mathcal{T})$ is independent of the choice of the generator system $\vec{x}$ (i.e., depending only on the order of missing points on $\mathbf{P}^1_K$).*

According to this proposition, we may without ambiguity write $\mathcal{L}^{\psi}(z, v)$, abbreviating the reference to $\vec{x}$ for the $\ell$-adic iterated integral on $H_n(\mathcal{T})$. For details of the above propositions, see [W4].

Before closing this subsection, we note the following lemma that will be applied to control a behavior of naive $\ell$-adic iterated integrals under changes of the choice of paths.

**Lemma 4.11** *Let $X/K$ be any algebraic variety with $K$-rational base point $v$, and let $\sigma_v$ denote the action of $\sigma \in G_K$ on $L(\pi)$ at $v$. Then, for every $\sigma \in G_K$ and $s \in \Gamma^n \pi_1^\ell(X_{\overline{K}}, v)$, we have*

$$\log \mathfrak{f}_\sigma^{sp} \equiv \log \mathfrak{f}_\sigma^{p} + (1 - \sigma_v) \cdot (\log s) \ mod\ \Gamma^{n+1} L(\pi).$$

*In particular,* $\log \mathfrak{f}_\sigma^{p}$ *is invariant modulo $\Gamma^n L(\pi)$ under the change of $p \mapsto sp$ $(s \in \Gamma^n \pi_1^\ell(X_{\overline{K}}, v))$. In the case $X = \mathbf{P}^1_K - \{a_1, \ldots, a_N, \infty\}$ $(N \geq 2)$, it holds that*

$$\log \mathfrak{f}_\sigma^{sp} \equiv \log \mathfrak{f}_\sigma^{p} + (1 - \chi(\sigma)^n)(\log s) \ mod\ \Gamma^{n+1} L(\pi),$$

*where $\chi\colon G_K \to \mathbb{Z}_\ell^\times$ is the cyclotomic character.*

*Proof* We have $\mathfrak{f}_\sigma^{sp} = sp \cdot \sigma(sp)^{-1} = s\mathfrak{f}_\sigma^{p} s^{-1} \cdot s\sigma(s)^{-1}$. Taking $\log\colon \pi \to L(\pi)$, we obtain

$$\log \mathfrak{f}_\sigma^{sp} = s(\log \mathfrak{f}_\sigma^{p})s^{-1} \boxplus \log(s \cdot \sigma(s)^{-1}),$$

where $\boxplus$ denotes the Baker–Campbell–Hausdorff sum: $S \boxplus T = \log(e^S e^T)$. Since $s$ lies in the center of $\pi/\Gamma^{n+1}\pi$, after taking modulo $\Gamma^{n+1} L(\pi)$, we see the above right-hand side is congruent to $\log \mathfrak{f}_\sigma^p + \log(s) - \log(\sigma(s))$. From this follows the first formula. In the special case of the punctured projective line, $\sigma$ acts on $\mathrm{gr}_\Gamma^n \pi$ by $\chi(\sigma)^n$-multiplication. Thus, the proof of the lemma is completed. □

Applying $\psi_{\vec{x}}\colon L(\pi) \to \mathbb{Q}_\ell$ to the above lemma gives a generalization of [DW2] Lemma 2.1 to the naive $\ell$-adic iterated integrals.

**4.3 Functional equations for iterated integrals** Suppose now that we are given a $K$-morphism to a punctured projective $t$-line:

$$f\colon X \longrightarrow Y = \mathbf{P}_t^1 - \{b_1, \ldots, b_M, \infty\}.$$

Then, induced are the homomorphisms

$$\mathrm{gr}^n(f_*)\colon \mathrm{gr}_\Gamma^n(\pi_1(X, v)) \to \mathrm{gr}_\Gamma^n(\pi_1(Y, f(v)))$$

$(n = 1, 2, ...)$, where $\pi_1$ denotes the topological (resp. pro-$\ell$) fundamental group of the complex points of $X, Y$ (resp. of $X_{\overline{K}}$, $Y_{\overline{K}}$) in the complex (resp. $\ell$-adic) case. Also, the pull-back of functions $g \mapsto g \circ f$ gives rise to the mappings $O_h^\times(Y^{\mathrm{an}}) \to O_h^\times(X^{\mathrm{an}})$ and $\widehat{O}_\ell^\times(Y_{\overline{K}}) \to \widehat{O}_\ell^\times(X_{\overline{K}})$ in respective cases.

**Lemma 4.12** *We have*

$$\kappa^{\otimes n}(g_1 \otimes \cdots \otimes g_n)(f_*(\alpha_1) \otimes \cdots \otimes f_*(\alpha_n)) = \kappa^{\otimes n}(g_1 \circ f \otimes \cdots \otimes g_n \circ f)(\alpha_1 \otimes \cdots \otimes \alpha_n)$$

*for any $g_1, \ldots, g_n \in O_h^\times(Y^{\mathrm{an}})$ (resp. $\in \hat{O}_\ell^\times(Y_{\overline{K}})$) and for any $\alpha_1, \ldots, \alpha_n \in \pi_1(X(\mathbb{C}), v)^{\mathrm{ab}}$ (resp. $\in \pi_1^\ell(X_{\overline{K}}, v)^{\mathrm{ab}}$) in the complex (resp. $\ell$-adic) case.*

*Proof* This follows immediately from the definition of Kummer pairing given in §2. □

Now we shall state our main theorem. Recall from Definition 3.8 that for any homomorphism $\varphi\colon \mathrm{gr}_\Gamma^n(\pi_1(Y, f(v))) \to \mathbb{Z}$ or $\mathbb{Z}_\ell$, the symbol $\widehat{\kappa_{\otimes n}}(\varphi)(f)$ denotes the image of the Kummer dual $\widehat{\kappa_{\otimes n}}(\varphi)(t)$ by the pull-back mapping $g \mapsto g \circ f$.

**Theorem 4.13** *Let $X$ be an arbitrary algebraic variety over a subfield $K$ of $\mathbb{C}$, and let $Y := \mathbf{P}_K^1 - \{b_1, \ldots, b_M, \infty\}$, where $\{b_1, \ldots, b_M\}$ is a subset of $K$ with cardinality $M$. Fix any $K$-rational (possibly tangential) base point $v$ on $X$. Then, for a collection of algebraic $K$-morphisms $f_i\colon X \to Y$ ($i = 1, \ldots, m$), homomorphisms $\psi_i\colon \mathrm{gr}_\Gamma^n \pi_1(Y(\mathbb{C}), f_i(v)) \to \mathbb{Z}$ and integers $c_1, \ldots, c_m \in \mathbb{Z}$, the* *following conditions* (i)$_{\mathbb{C}}$, (ii)$_{\mathbb{C}}$ and (iii)$_{\mathbb{C}}$ *are equivalent:*

(i)$_{\mathbb{C}}$ $$\sum_{i=1}^{m} c_i\, \psi_i \circ \mathrm{gr}^n_\Gamma(f_{i*}) = 0 \quad \textit{in } \mathrm{Hom}\,(\mathrm{gr}^n_\Gamma \pi_1(X(\mathbb{C}), v), \mathbb{Z}),$$

(ii)$_{\mathbb{C}}$ $$\sum_{i=1}^{m} c_i\, \widehat{\kappa_{\otimes n}}(\psi_i)(f_i) = 0$$

$$\textit{in } \Big(\bigotimes_{i=1}^{n-2} \mathcal{O}_h^\times(X^{\mathrm{an}})\Big) \otimes (\mathcal{O}_h^\times(X^{\mathrm{an}}) \wedge \mathcal{O}_h^\times(X^{\mathrm{an}})),$$

(iii)$_{\mathbb{C}}$ $$\sum_{i=1}^{m} c_i\, \mathcal{L}_{\mathbb{C}}^{\psi_i}(f_i(z), f_i(v); f_i(\gamma)) = 0$$

*for each path* $\gamma\colon v \rightsquigarrow z$ *on* $X(\mathbb{C})$.

The proof of the above theorem will be given soon after stating the next theorem. In the following $\ell$-adic analogs (i)$_\ell$, (ii)$_\ell$ and (iii)$_\ell$ of the above conditions, we have, in general, the equivalence (i)$_\ell$ $\Leftrightarrow$ (ii)$_\ell$ and the implication (i)$_\ell$ (ii)$_\ell$ $\Rightarrow$ (iii)$_\ell$, (iii)$_\ell^{\mathrm{nv}}$. If moreover $\mu_{\ell^\infty} \not\subset K$, then we also have (i)$_\ell$ $\Leftrightarrow$ (ii)$_\ell$ $\Leftrightarrow$ (iii)$_\ell^{\mathrm{nv}}$:

In order to state the condition (iii)$_\ell$, (iii)$_\ell^{\mathrm{nv}}$, we need to introduce a precise notion of "error term" which complement (part of) "lower degree terms" of $\ell$-adic iterated integrals.

**Definition of $\ell$-adic error terms** Let $X$, $v$, $z$ be as in §4.2. We shall call a $\mathbb{Q}_\ell$-valued function $E(\sigma, p)$ on $(\sigma, p) \in G_K \times \pi_1^\ell(X_{\overline{K}}, v, z)$ an *error term of degree n*, if it satisfies

(E1) $E(\sigma, p) = 0$ for $\sigma \in H_n(v, z)$,

(E2) $E(\sigma, p) = E(\sigma, sp)$ for $s \in \Gamma^n \pi_1^\ell(X_{\overline{K}}, v)$.

We also introduce some more notations for the above system of a collection of algebraic morphisms $f_i\colon X \to Y$ $(i = 1, \ldots, m)$ with $Y := \mathbf{P}^1_K - \{b_1, \ldots, b_M, \infty\}$. Fix a $K$-rational (*tangential*) base point $w$ on $Y$, a generator system $\vec{y} = (y_1, \ldots, y_M)$ of $\pi_1(Y(\mathbb{C}), w)$ such that $y_j$ turns only around the puncture $b_j$ once, and paths $\gamma_i\colon w \rightsquigarrow f_i(v)$ for $1 \le i \le m$. Let $\vec{y}_i := (\gamma_i^{-1} y_j \gamma_i)_{j=1}^M$ be the generator system of $\pi_1(Y(\mathbb{C}), f_i(v))$ induced from $\gamma_i$ and $\vec{y}$. We then apply the process of (4.10–4.11) to obtain a $\mathbb{Q}_\ell$-valued form $\psi_{i,\vec{y}_i}$ on the $\ell$-adic Lie algebra $L(\pi_{Y, f_i(v)})$ of $\pi_1^\ell(Y_{\overline{K}}, f_i(v))$ for each $1 \le i \le m$.

The following theorem refines [W5], theorem 10.0.7, concerning the functional equations of $\ell$-adic iterated integrals. One of our new ingredients of this paper is to formulate those functional equations for arbitrary algebraic varieties $X$, generalizing results of [W4] and [W5], which were concerned with projective lines minus finite number of points. If we restrict ourselves only to $\sigma \in H_n(v,z) \cap \bigcap_{i=1}^m H_n(f_i(v), f_i(z))$, then we can deduce those functional equations from $(\mathrm{i})_\ell$, $(\mathrm{ii})_\ell$ by simply applying the given conditions to definitions of $\ell$-adic iterated integrals as in loc.cit. But to treat more general $\sigma$, we need to employ some elaborate arguments as presented in the next subsection.

**Theorem 4.14** *Let $X$ be an arbitrary algebraic variety over a subfield $K$ of $\mathbb{C}$, and let $Y := \mathbf{P}^1_K - \{b_1, \dots, b_M, \infty\}$, where $\{b_1, \dots, b_M\}$ is a subset of $K$ with cardinality $M$. Fix any $K$-rational (possibly tangential) base point $v$ on $X$. Then, for a collection of algebraic $K$-morphisms $f_i\colon X \to Y$ $(i = 1, \dots, m)$, homomorphisms $\psi_i\colon \mathrm{gr}^n_\Gamma \pi_1(Y(\mathbb{C}), f_i(v)) \to \mathbb{Z}$ and integers $c_1, \dots, c_m \in \mathbb{Z}$, the following conditions* $(\mathrm{i})_\ell$ and $(\mathrm{ii})_\ell$ *are equivalent:*

$$(\mathrm{i})_\ell \qquad \sum_{i=1}^m c_i\, \psi_i \circ \mathrm{gr}^n_\Gamma(f_{i*}) = 0 \quad \textit{in } \mathrm{Hom}\,(\mathrm{gr}^n_\Gamma \pi^\ell_1(X_{\overline{K}}, v), \mathbb{Z}_\ell),$$

$$(\mathrm{ii})_\ell \qquad \sum_{i=1}^m c_i\, \widehat{\kappa_{\otimes n}}(\psi_i)(f_i) = 0 \quad \textit{in } \Big(\bigotimes_{i=1}^{n-2} \widehat{\mathcal{O}}^\times_\ell(X_{\overline{K}})\Big) \otimes \Big(\widehat{\mathcal{O}}^\times_\ell(X_{\overline{K}}) \wedge \widehat{\mathcal{O}}^\times_\ell(X_{\overline{K}})\Big).$$

*Moreover, these conditions imply the following* $(\mathrm{iii})_\ell$ and $(\mathrm{iii})^{\mathrm{nv}}_\ell$.

$(\mathrm{iii})_\ell$ *There exists an error term $E\colon G_{K(\mu_{\ell^\infty})} \times \pi^\ell_1(X_{\overline{K}}, v, z) \to \mathbb{Q}_\ell$ of degree $n$ such that $\sum_{i=1}^m c_i\, \mathcal{L}^{\psi_i, \vec{y}_i}(f_i(z), f_i(v); f_i(p))(\sigma) = E(\sigma, p)$ $(\sigma \in G_{K(\mu_{\ell^\infty})})$ for each étale path $p\colon v \rightsquigarrow z \in X(K)$.*

$(\mathrm{iii})^{\mathrm{nv}}_\ell$ *There exists an error term $E_{\mathrm{nv}}\colon G_K \times \pi^\ell_1(X_{\overline{K}}, v, z) \to \mathbb{Q}_\ell$ of degree $n$ such that $\sum_{i=1}^m c_i\, \mathcal{L}^{\psi_i, \vec{y}_i}_{\mathrm{nv}}(f_i(z), f_i(v); f_i(p))(\sigma) = E_{\mathrm{nv}}(\sigma, p)$ $(\sigma \in G_K)$ for each étale path $p\colon v \rightsquigarrow z \in X(K)$.*

*Finally, if $\mu_{\ell^\infty} \not\subset K$, then* $(\mathrm{iii})^{\mathrm{nv}}_\ell$ *implies* $(\mathrm{i})_\ell$, $(\mathrm{ii})_\ell$.

**4.4 Proof of Theorems 4.13 and 4.14** The equivalence of (i) and (ii) follows from Lemma 2.1, Lemma 4.12 and Corollary 3.7 for both complex and $\ell$-adic case. We prove $(\mathrm{i})_\mathbb{C}$ implies the functional equation $(\mathrm{iii})_\mathbb{C}$ by essentially tracing lines of [W0], theorem E, and [W5], theorem 11.2.1. Suppose $(\mathrm{i})_\mathbb{C}$ holds. By (4.5), we have $\Lambda_{f_i(\gamma)}(f(z), f(v)) = f_{i*}(\Lambda_\gamma(z, v))$ for each $i = 1, ..., m$. Then,

$$\begin{aligned}\sum_{i=1}^m c_i\, \mathcal{L}^{\psi_i}_\mathbb{C}(f_i(z), f_i(v); f_i(\gamma)) &= \sum_{i=1}^m c_i\, \psi_{i\mathbb{C}}\big(f_{i*}(\log \Lambda_\gamma(z,v)^{-1} \bmod \Gamma^{n+1})\big) \\ &= \sum_{i=1}^m c_i\, \psi_{i\mathbb{C}} \circ \mathrm{gr}^n_\Gamma(f_{i*})([\log \Lambda_\gamma(z,v)^{-1}]_n) = 0.\end{aligned}$$

Here $[\log \Lambda_\gamma(z,v)^{-1}]_n$ denotes the $n$th homogeneous component of $\log \Lambda_\gamma(z,v)^{-1} \bmod \Gamma^{n+1}$. Note that we used here the property that $f_{i*}$ preserves the gradation in the complex case, as remarked just before (4.5). Thus the functional equation $(\mathrm{iii})_{\mathbb{C}}$ follows.

Conversely, if $(\mathrm{iii})_{\mathbb{C}}$ holds, then we may argue as in the proof of [W0], theorem 10.5.1. The composition law (4.3) and the surjectivity of $\theta_{v,X}$ (4.6) insure the surjectivity of

$$\theta_{v,z,X}\colon \pi_1(X,v,z) \to \pi(X) \quad (\gamma \mapsto \Lambda_\gamma(z,v)),$$

hence the images of paths $\gamma\colon v \rightsquigarrow z$ in $\pi(X)$ modulo the lower central subgroups form Zariski dense subsets. Noticing the remark before Definition 4.4, we see that $\psi_{i\mathbb{C}} \circ f_{i*} \circ \log$ give polynomial functions on $\pi(X)/\Gamma^{n+1}\pi(X)$. From this follows that the functional equations for all $\gamma$ implies the equation $(\mathrm{i})_{\mathbb{C}}$.

Next, suppose $(\mathrm{i})_\ell$ holds. Then, since $f_i$ are defined over $K$, we have

$$\log \mathfrak{f}_\sigma^{f_i(p)} = f_i(\log \mathfrak{f}_\sigma^p) \quad (\sigma \in G_K). \tag{4.13}$$

Let $L(\pi_{X,v})$, $L(\pi_{Y,f_j(v)})$ denote the $\ell$-adic (complete) Lie algebras of $\pi_1^\ell(X_{\overline{K}}, v)$, $\pi_1^\ell(Y_{\overline{K}}, f_i(v))$ respectively, and denote the composition of $\psi_{i,\vec{y}_i}$ with $f_{i*}\colon L(\pi_{X,v}) \to L(\pi_{Y,f_i(v)})$ by

$$\psi_{i,\vec{y}_i} \circ f_i\colon L(\pi_{X,v}) \to L(\pi_{Y,f_i(v)}) \to \mathbb{Q}_\ell. \tag{4.14}$$

Then, from the above (4.13) we see

$$\mathcal{L}_{\mathrm{nv}}^{\psi_{i,\vec{y}_i}}(f_i(z), f_i(v); f_i(p)) = \mathcal{L}_{\mathrm{nv}}^{\psi_{i,\vec{y}_i}\circ f_i}(z, v; p). \tag{4.15}$$

However, the necessity of error terms occurs from the fact that, in the $\ell$-adic case, we do not have a canonical gradation in $L(\pi_{X,v})$, although we have chosen a collection of splittings of $L(\pi_{Y,f_i(v)})$ (via the path system $\{\gamma_i, y_j\}$ as above) which are compatible with each other for $i = 1, \ldots, m$ but not with $L(\pi_{X,v})$. We anyway take and fix one of the (vector space) complements $L_{<n}$ to $\Gamma^n L(\pi_{X,v}) \subset L(\pi_{X,v})$:

$$L(\pi_{X,v}) = L_{<n} \oplus \Gamma^n L(\pi_{X,v}), \tag{4.16}$$

and write

$$\log(\mathfrak{f}_\sigma^p)^{-1} = [\log(\mathfrak{f}_\sigma^p)^{-1}]_{<n} + [\log(\mathfrak{f}_\sigma^p)^{-1}]_{\geq n} \quad (\sigma \in G_K). \tag{4.17}$$

Putting this decomposition of $\log \mathfrak{f}_\sigma^p$ into the map $\sum_i c_i \psi_{i,\vec{y}_i} \circ f_i$, and applying the condition $(\mathrm{i})_\ell$, we obtain

$$(4.18)\qquad \begin{aligned}&\sum_i c_i \mathcal{L}_{\mathrm{nv}}^{\psi_{i,\vec{y}_i}\circ f_i}(z,v;p)\\ &= \sum_i c_i\,\psi_{i,\vec{y}_i}\circ f_i([\log(\mathfrak{f}_\sigma^p)^{-1}]_{<n}) + \sum_i c_i\,\psi_{i,\vec{y}_i}\circ f_i([\log(\mathfrak{f}_\sigma^p)^{-1}]_{\geq n})\\ &= \sum_i c_i\,\psi_{i,\vec{y}_i}\circ f_i([\log(\mathfrak{f}_\sigma^p)^{-1}]_{<n}) + 0.\end{aligned}$$

Then, noticing that $[\log(\mathfrak{f}_\sigma^p)^{-1}]_{<n}$ vanishes for $\sigma \in H_n(v,z)$ and is invariant under the change $p \mapsto sp$ for $s \in \Gamma^n\pi_1^\ell(X_{\overline{K}}, v)$ (Lemma 4.11), we find the right-hand side above satisfies the conditions (E1) and (E2) of an error term of degree $n$. This proves $(\mathrm{i})_\ell \Rightarrow (\mathrm{iii})_\ell^{\mathrm{nv}}$. To deduce $(\mathrm{iii})_\ell$, we wish to consider the logarithm of $\sigma_p$ for $\sigma \in G_{K(\mu_{\ell^\infty})}$, but $\sigma_p$ may generally not be a unipotent operator on the total $\hat{U}\pi_1^\ell(X_{\overline{K}}, v)$. Consider now the diagonal homomorphism into the direct product:

$$(4.19)\quad \Delta\left(:= \Pi_{i=1}^m f_i\right) : \pi_1^\ell(X_{\overline{K}}, v) \longrightarrow \prod_{i=1}^m \pi_1^\ell(Y_{\overline{K}}, f_i(v))$$

$$(S \mapsto (f_1(S), \ldots, f_m(S))),$$

which naturally induces homomorphisms of the ℓ-adic Lie algebra and of the universal enveloping algebra into the corresponding products (denoted also Δ):

$$(4.20)\quad L(\pi_{X,v}) \longrightarrow \bigoplus_{i=1}^m L(\pi_{Y,f_i(v)}), \qquad \hat{U}\pi_1^\ell(X_{\overline{K}}, v) \longrightarrow \prod_{i=1}^m \hat{U}\pi_1^\ell(Y_{\overline{K}}, f_i(v)).$$

Let $\overline{L}_X$, $\overline{U}_X$ denote the images of the last two homomorphisms, and denote by $\bar{f}_{i*}$, for simplicity, the both maps $\overline{L}_X \to L(\pi_{Y,f_i(v)})$ and $\overline{U}_X \to \hat{U}\pi_1^\ell(Y_{\overline{K}}, f_i(v))$ factoring $f_i$ for respective $i = 1, \ldots, m$. Introduce the filtration $\{\Gamma^k\overline{L}_X\}_{k=1}^\infty$ as the pull-back of the lower central filtration on $\bigoplus_i L(\pi_{Y,f_i(v)})$. Then, for $\psi_i$ extended to the unique $\mathbb{Q}_\ell$-linear form on $\mathrm{gr}_\Gamma^n L(\pi_{Y,f_i(v)})$ for each $1 \leq i \leq m$, the condition $(\mathrm{i})_\ell$ insures

$$(4.21)\qquad \sum_{i=1}^m c_i\,\psi_i \circ \mathrm{gr}_\Gamma^n(\bar{f}_{i*}) = 0 \text{ in } \mathrm{Hom}(\overline{L}_X, \mathbb{Q}_\ell),$$

whereas the induced operations $\bar{\sigma}_v$ and $\bar{\sigma}_p$ on $\overline{U}_X$ are unipotent for $\sigma \in G_{K(\mu_{\ell^\infty})}$ (as their operations on the $i$th component are unipotent for every $i$, cf. Proposition 4.6). Now, since $f_i$ are defined over $K$, we have

$$(4.22)\qquad \sigma_{f_i(p)} \circ f_i = \bar{f}_i \circ \bar{\sigma}_p \circ \Delta = f_i \circ \sigma_p \quad (\sigma \in G_K,\ i = 1, \ldots, m),$$

hence it holds on $\overline{U}_X$ that

$$(\mathrm{Log}\sigma_{f_i(p)})(1) = \bar{f}_i(\mathrm{Log}\bar{\sigma}_p(1)) \quad (\sigma \in G_{K(\mu_{\ell^\infty})},\ i = 1, \ldots, m).$$

**Lemma 4.15** $(\mathrm{Log}\bar{\sigma}_p)(1)$ *belongs to* $\overline{L}_X$.

*Proof* In fact, this follows in exactly the same way as the proof of Proposition 4.6(ii) by taking the logarithm of both sides of $\bar{\sigma}_p = L_{\bar{\mathfrak{f}}} \circ \bar{\sigma}_v$, where $\bar{\mathfrak{f}}$ denotes the image of $\mathfrak{f}_\sigma^p$ in $\bar{U}_X$. □

Again, although we have a collection of compatible splittings of $L(\pi_{Y,f_i(v)})$ (via the path system $\{\gamma_i, y_j\}$ as above), it generally does not induce a compatible splitting on the subspace $\overline{L}_X \subset \bigoplus_i L(\pi_{Y,f_i(v)})$. So we take and fix one (vector space) splitting

$$\overline{L}_X = \overline{L}_{<n} \oplus \Gamma^n \overline{L}_X, \tag{4.23}$$

and write

$$(\mathrm{Log}\bar{\sigma}_p)(1) = [\mathrm{Log}\bar{\sigma}_p(1)]_{<n} + [\mathrm{Log}\bar{\sigma}_p(1)]_{\geq n} \quad (\sigma \in G_{K(\mu_{\ell^\infty})}). \tag{4.24}$$

To deduce $(\mathrm{iii})_\ell$ from $(\mathrm{i})_\ell$, it then only remains to repeat the same argument as above with applying the role of $\log \mathfrak{f}_\sigma^p$ to $\mathrm{Log}\bar{\sigma}_p(1)$. We may leave the rest to the reader.

Finally, assume $\mu_{\ell^\infty} \not\subset K$ and $(\mathrm{iii})_\ell^{\mathrm{nv}}$. Let us fix a path $p\colon v \rightsquigarrow z$ on $X_{\overline{K}}$. For each $s \in \Gamma^n \pi_1^\ell(X_{\overline{K}}, v)$ and $\sigma \in G_K$, we have from Lemma 4.11:

$$\begin{aligned}\mathcal{L}_{\mathrm{nv}}^{\psi_{i,\vec{y}_i}}&(f_i(z), f_i(v); f_i(sp))(\sigma)\\ &= \mathcal{L}_{\mathrm{nv}}^{\psi_{i,\vec{y}_i}}(f_i(z), f_i(v); f_i(p))(\sigma) + (\chi(\sigma)^n - 1)\psi_{i,\vec{y}_i}(f_i(\log s))\end{aligned}$$

for $i = 1, \ldots, m$. Since the error term $E_{\mathrm{nv}}(\sigma, p)$ of $(\mathrm{iii})_\ell^{\mathrm{nv}}$ is invariant under the change $p \mapsto sp$, putting the above equation into the functional equation yields

$$(\chi(\sigma)^n - 1)\left(\sum_{i=1}^m c_i\, \psi_{i,\vec{y}_i} \circ f_i\right)(\log(s)) = 0.$$

By assumption of $\mu_{\ell^\infty} \not\subset K$, the factor $\chi(\sigma)^n - 1$ ($\sigma \in G_K$) runs over a non-trivial subset of $\mathbb{Z}_\ell$. From this together with the observation that the images of $\log(s)$ ($s \in \Gamma^n \pi_1^\ell(X_{\overline{K}}, v)$) generate the vector space $\mathrm{gr}_\Gamma^n \pi_1^\ell(X_{\overline{K}}, v) \otimes \mathbb{Q}_\ell$, we conclude the equation $(\mathrm{i})_\ell$.

The proof of Theorems 4.13 and 4.14 is thus completed. The above proof of Theorem 4.14 also implies the following

**Corollary 4.16** *Notations being as in Theorem 4.14 and its proof, suppose condition* $(\mathit{iii})_\ell^{\mathrm{nv}}$ *holds. Pick a vector space splitting* $L(\pi_{X,v}) = L_{<n} \oplus \Gamma^n L(\pi_{X,v})$

*and decompose* $\log(\mathfrak{f}_\sigma^p)^{-1} = [\log(\mathfrak{f}_\sigma^p)^{-1}]_{<n} + [\log(\mathfrak{f}_\sigma^p)^{-1}]_{\geq n}$ *according to it. Then the error term of the functional equation is given by*

$$E_{\mathrm{nv}}(\sigma, p) = \sum_i c_i \, \psi_{i,\vec{y}_i} \circ f_i([\log(\mathfrak{f}_\sigma^p)^{-1}]_{<n}) \quad (\sigma \in G_K,\ p\colon v \rightsquigarrow z).$$

*A similar formula also holds for condition (iii)$_\ell$ with simple replacements of* $L(\pi_{X,v})$*,* $\log(\mathfrak{f}_\sigma^p)^{-1}$*,* $G_K$ *by* $\overline{L}_X$*,* $(\mathrm{Log}\bar{\sigma}_p)(1)$*,* $G_{K(\mu_{\ell^\infty})}$ *respectively.*

The formula above gives us a way to calculate the error term $E(\sigma, p)$ in a concrete way after picking a splitting of $L(\pi)$. Note that the error term itself is independent of the choice of splitting $L(\pi_{X,v}) = L_{<n} \oplus \Gamma^n L(\pi_{X,v})$, since the left-hand sides of the functional equations (iii)$_\ell$ and (iii)$_{\mathrm{nv}}$ are independent of this choice.

In Theorems 4.13 and 4.14, we restricted the coefficients $c_1, \ldots, c_m$ only to integers. One may ask if a functional equation with more general coefficients exists among complex or $\ell$-adic iterated integrals. In fact, it turns out that there are essentially no new such functional equations as shown next.

**Definition 4.17** Taking notation as in Theorems 4.13 and 4.14, we say that a family of complex iterated integrals $\mathcal{L}_{\mathbb{C}}^{\psi_i}(f_i(z), f_i(v); f_i(\gamma))$ (resp. $\ell$-adic iterated integrals $\mathcal{L}^{\psi_{i,\vec{y}_i}}(f_i(z), f_i(v); f_i(p))$, resp. $\mathcal{L}_{\mathrm{nv}}^{\psi_{i,\vec{y}_i}}(f_i(z), f_i(v); f_i(p))$ ) $(i = 1, \ldots, m)$ *has a linear functional equation*, if a non-trivial linear combination of them with coefficients in some field $F$ of characteristic 0 becomes zero (resp. becomes an error term) for all paths $\gamma\colon v \rightsquigarrow z$ (resp. for all étale paths $p\colon v \rightsquigarrow z$) on $X$ with $v$ fixed, $z$ and $\gamma$ (resp. $p$) vary.

Let us here consider the following four conditions:

(a) The family $\{\mathcal{L}_{\mathbb{C}}^{\psi_i}(f_i(z), f_i(v); f_i(\gamma))\}_i$ has a linear functional equation.
(b) The family $\{\mathcal{L}^{\psi_{i,\vec{y}_i}}(f_i(z), f_i(v); f_i(p))\}_i$ has a linear functional equation for all $\ell$.
(c) The family $\{\mathcal{L}_{\mathrm{nv}}^{\psi_{i,\vec{y}_i}}(f_i(z), f_i(v); f_i(p))\}_i$ has a linear functional equation for all $\ell$.
(d) The family $\{\mathcal{L}_{\mathrm{nv}}^{\psi_{i,\vec{y}_i}}(f_i(z), f_i(v); f_i(p))\}_i$ has a linear functional equation for one $\ell$.

**Proposition 4.18** *(i) Condition (a) implies all the other conditions (b), (c), (d) and, in each case, the linear combination can be replaced by a (nontrivial) combination with coefficients in $\mathbb{Z}$.*

*(ii) If $\mu_{\ell^\infty} \not\subset K$, then conditions (a), (c), (d) are equivalent and, in each case, the linear combination can be replaced by a (nontrivial) combination with coefficients in $\mathbb{Z}$.*

*Proof* (i) Suppose condition (a) holds, i.e., that $(\text{iii})_{\mathbb{C}}$ holds with coefficients $c_1, \ldots, c_m \in F$. Then tracing the same argument as in the proof of Theorem 4.13, one can easily see that

$$\sum_{i=1}^{m} c_i \psi_i \circ \mathrm{gr}_\Gamma^n(f_{i*}) = 0$$

in $\mathrm{Hom}(\mathrm{gr}_\Gamma^n \pi_1(X(\mathbb{C}), v), F)$. But since $\psi_i \circ \mathrm{gr}_\Gamma^n(f_{i*})$ are defined over $\mathbb{Z}$, it follows that the coefficients of this linear equation can be replaced by rational numbers, hence by rational integers. This remark has already been pointed out in [W0], corollary 10.6.7. Then we obtain $(\text{i})_\ell$ for these integer coefficients, which implies (b), (c), (d) with the same coefficients.

(ii) Suppose that condition (d) holds, i.e., that $(\text{iii})_\ell^{\text{nv}}$ holds with coefficients $c_1, \ldots, c_m \in F$. Then, again, tracing the argument in the proof of Theorem 4.14, one sees

$$\sum_{i=1}^{m} c_i \psi_i \circ \mathrm{gr}_\Gamma^n(f_{i*}) = 0$$

in $\mathrm{Hom}(\mathrm{gr}_\Gamma^n \pi_1^\ell(X_{\overline{K}}, v), F\mathbb{Q}_\ell)$. But recalling that $\mathrm{gr}_\Gamma^n \pi_1^\ell(X_{\overline{K}}, v)$ is isomorphic to $\mathrm{gr}_\Gamma^n \pi_1(X(\mathbb{C}), v) \otimes \mathbb{Z}_\ell$, we may regard $\psi_i \circ \mathrm{gr}_\Gamma^n(f_{i*})$ as objects coming from $\mathrm{Hom}(\mathrm{gr}_\Gamma^n \pi_1(X(\mathbb{C}), v), \mathbb{Z})$. Therefore, by the same reasoning as in (i), we obtain a linear equation $(\text{i})_{\mathbb{C}}$ with rational integers. This and part (i) conclude the proof. □

**Remark 4.19** The condition "(iii) $\sum_{i=1}^{N} n_i\, b_Y(e_i)([f_i]) = 0$" in [W2], theorem 5.1, should be replaced by the condition:

$$\text{“(iii)} \quad \sum_{i=1}^{N} n_i\, \widehat{\kappa_{\otimes n}}(e_i^*)(f_i) = 0 \text{ in } \left( \bigotimes_{i=1}^{n-2} \mathcal{O}_h^\times(X^{\text{an}}) \right) \otimes \left( \mathcal{O}_h^\times(X^{\text{an}}) \wedge \mathcal{O}_h^\times(X^{\text{an}}) \right)\text{”.}$$

In the proof of [W2], theorem 5.1, we state that, passing with $\mathrm{Lie}(H(X))$ – free Lie algebra on a vector space $H(X)$ – to a dual object, we get $\mathrm{Lie}(H(X)^*) = \mathrm{Lie}(A^1(X))$ – free Lie algebra on $H(X)^* = A^1(X)$. This is obviously not correct, as a dual of a free Lie algebra on $H(X)$ is not naturally isomorphic to a free Lie algebra on a dual $H(X)^*$. The same remark concerns theorem 10.8.2 in [W0]; condition (i) should be replaced by Zagier's condition.

## 5 Case of polylogarithms

Now we shall apply results of the previous section in the polylogarithmic case.

**5.1 Review of classical polylogarithms** Let us set

$$P_0 := \mathbf{P}^1_K - \{0, 1, \infty\}$$

defined over a field $K \subset \mathbb{C}$. With notation as in 4.1, the space $\Omega^1 = \Omega^1_{\log}(X)$, for $X = P_0(\mathbb{C})$, $D = \{0, 1, \infty\}$ and $X^* = \mathbf{P}^1$, is a two-dimensional vector space generated by the differential form $\omega_0 = \frac{dz}{z}$ and $\omega_1 = \frac{dz}{z-1}$. We take a basis $(X, Y)$ of $V_1 = (\Omega^1)^*$ dual to the basis $(\omega_0, \omega_1)$ of $\Omega^1$. In this case, $K^\perp = 0$ and $L(V_1, K^\perp)$ is just a free Lie algebra generated by the $X$ and $Y$.

On the other hand, the topological fundamental group $\pi_1(P_0(\mathbb{C}), \overrightarrow{01})$ is a free group, freely generated by the standard loops $x, y$ running around the punctures $0, 1$ once anticlockwise respectively, so that the Lie algebra $\mathrm{Gr\,Lie}\,\pi_1(P_0(\mathbb{C}), \overrightarrow{01})$ is freely generated by the images $\bar{x}$ and $\bar{y}$. The natural isomorphism $\mathrm{gr}^1_\Gamma \pi_1(P_0, \overrightarrow{01}) \otimes \mathbb{C} \xrightarrow{\sim} L(V_1, K^\perp)_1 = V_1$ of (4.7) gives then the identification:

$$\frac{\bar{x}}{2\pi i} = X, \qquad \frac{\bar{y}}{2\pi i} = Y.$$

Let us fix a Hall basis of the free Lie algebra $\mathrm{Gr\,Lie}\,\pi_1(P_0(\mathbb{C}), \overrightarrow{01})$ corresponding to the ordering $\bar{x}, \bar{y}$ of generators. Then, the special elements $e_1, e_2, ...$ defined by

$$e_1 := \bar{y}, \quad e_n := [\bar{x}, e_{n-1}] = (\mathrm{ad}\,\bar{x})^{n-1}(\bar{y}) \quad \text{for } n > 1$$

belong to the Hall basis. Write

$$\varphi_n\colon \mathrm{gr}^n_\Gamma \pi_1(P_0(\mathbb{C}), \overrightarrow{01}) \longrightarrow \mathbb{Z}$$

for the linear form dual to $e_n$ with respect the above Hall basis. Since $\varphi_n$ kills the basis elements other than $e_n$, especially those of double commutator type, it belongs to $\mathrm{Hom}(\mathrm{gr}_\Gamma(\pi_1/\pi_1''), \mathbb{Z})$. We find its Kummer dual is then given by

$$\widehat{\kappa_{\otimes n}}(\varphi_n) = z^{\odot n-2} \otimes (z \wedge (z-1)) \in (\mathrm{Sym}^{n-2}\mathcal{O}_h^\times) \otimes (\wedge^2 \mathcal{O}_h^\times).$$

**Definition 5.1** We define the *complex polylogarithm function* $\mathrm{li}_n(z, \gamma)$ as the complex iterated integral associated to the above $\varphi_n$ and $\gamma\colon \overrightarrow{01} \rightsquigarrow z$ (cf. Definition 4.4):

$$\mathrm{li}_n(z, \gamma) := \mathcal{L}^{\varphi_n}_{\mathbb{C}}(z, \overrightarrow{01}; \gamma) \; (= \varphi_{n,\mathbb{C}}(\log \Lambda_\gamma(z, \overrightarrow{01})^{-1} \bmod \Gamma^{n+1})).$$

We also recall that *classical polylogarithm functions* $Li_n(z, \gamma)$ $(n = 1, 2, ...)$ for a path $\gamma$ from $\overrightarrow{01}$ to $z$ on $P_0(\mathbb{C})$ are defined as the iterated integrals

$$Li_n(z, \gamma) := \int_\gamma (-w_1) \cdot \underbrace{w_0 \cdots w_0}_{n-1}.$$

Note that $Li_1(z, \gamma) = -\log(1 - z)$.

The following proposition complements lemma 10.6.5 of [W0].

**Proposition 5.2** *With notation as above, the complex polylogarithm function is given in terms of the classical polylogarithm functions by the formula:*

$$\mathrm{li}_n(z,\gamma) = \frac{(-1)^{n+1}}{(2\pi i)^n}\sum_{k=0}^{n-1}\frac{B_k}{k!}(\log z)^k Li_{n-k}(z,\gamma) \qquad (n \geq 1).$$

*Here,* $\log z$ *takes the principal branch along* $\gamma$*, and* $\{B_n\}_{n=0}^{\infty}$ *is the sequence of Bernoulli numbers defined by* $\sum_{n=0}^{\infty}\frac{B_n}{n!}T^n = \frac{T}{e^T-1}$.

*Proof* (cf. [W1], lemma 3.4). We calculate $\Lambda(z) := \Lambda_\gamma(z,\overrightarrow{01})$ for the 1-form $\omega_{P_0} := \frac{dz}{z}X + \frac{dz}{z-1}Y$. This is a group-like element in the non-commutative power series ring $\mathbb{C}\langle\langle X,Y\rangle\rangle$ and its coefficients of the terms $X^i, YX^i$ $(i = 0,1,2,...)$ look like

$$\Lambda(z) = 1 + \sum_{i=1}^{\infty}\frac{(\log z)^i}{i!}X^i - \sum_{i=0}^{\infty}Li_{i+1}(z)YX^i + \cdots \text{ other terms.}$$

(See [F], §3.1, for shapes of coefficients of other terms such as $XY$, $Y^2$, $XYX$.) Noting that the above "$X^i$-part" can be written as $e^{(\log z)X} - 1$ and taking the logarithm $\log\Lambda(z) = (\Lambda(z)-1) - \frac{1}{2}(\Lambda(z)-1)^2 + \frac{1}{3}(\Lambda(z)-1)^3 - + \cdots$, we find that the coefficient of $YX^{n-1}$ of $\log\Lambda(z)$ comes from the expansion of the product of the series

$$-Li_1(z)Y - Li_2(z)YX - Li_3(z)YX^2 - \cdots$$

with

$$1 - \frac{1}{2}(e^{(\log z)X}-1)^2 + \frac{1}{3}(e^{(\log z)X}-1)^3 - + \cdots = \frac{(\log z)X}{e^{(\log z)X}-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}(\log z)^n X^n.$$

On the other hand, if we express the Lie element $\log\Lambda(z)$ as the Lie series with respect to the above fixed Hall basis, the term $YX^{n-1}$ arises only as $(-1)^{n-1}$-multiple of the term appearing in the expansion of $e_n = \mathrm{ad}(\bar{x})^{n-1}(\bar{y}) = (2\pi i)^n\,\mathrm{ad}(X)^{n-1}(Y)$. The formula follows from this observation and our definition $\mathrm{li}_n(z,\gamma) = \varphi_{n,\mathbb{C}}(\log\Lambda(z)^{-1})$. □

**Remark 5.3** As seen from the above proof, we have the equation

$$\log\Lambda_\gamma(z,\overrightarrow{01})^{-1} = -\frac{\log z}{2\pi i}\bar{x} - \frac{\log(1-z)}{2\pi i}\bar{y} + \sum_{m\geq 2}\mathrm{li}_m(z,\gamma)\,e_m + \text{other terms.}$$

With regard to this equation, we shall write

$$\mathrm{li}_0(z,\gamma) = -\frac{\log z}{2\pi i}, \quad \mathrm{li}_1(z,\gamma) = -\frac{\log(1-z)}{2\pi i}.$$

(The latter also follows from Definition 5.1 and Proposition 5.2.)

**5.2 ℓ-adic polylogarithms ([W5])** Notations being as in the previous subsection, let $K$ be a subfield of $\mathbb{C}$, $z \in P_0(K)$ and pick an étale path $p\colon \overrightarrow{01} \rightsquigarrow z$. Regard the pro-ℓ fundamental group $\pi_1^\ell(P_0/\bar{K}, \overrightarrow{01})$ as the pro-ℓ completion of $\pi_1(P_0(\mathbb{C}), \overrightarrow{01})$. Let $L(\pi_{\overrightarrow{01}})$ denote the associated complete ℓ-adic Lie algebra consisting of all the formal Lie series in $X := \log(x)$ and $Y := \log(y)$; the fixed generator system $\vec{x} = (x, y)$ of $\pi_1(P_0(\mathbb{C}), \overrightarrow{01})$ as in §5.1 defines the natural extension $\varphi_{n,\vec{x}}\colon L(\pi_{\overrightarrow{01}}) \to \mathbb{Q}_\ell$ of $\varphi_n$ as in §4 (4.10–4.11).

**Definition 5.4** ([W5] §11) Write $I_Y$ for the ideal of $L(\pi_{\overrightarrow{01}})$ generated by the Lie monomials involving $Y$ twice or more. The ℓ-adic polylogarithm function $\ell i_n(z, p, \vec{x})\colon G_K \to \mathbb{Q}_\ell$ $(n = 1, 2, ...)$ for a path $p$ from $\overrightarrow{01}$ to $z$ on $P_0(K)$ is defined as the naive ℓ-adic iterated integral associated to $\varphi_{n,\vec{x}}$:

$$\ell i_n(z, p, \vec{x})(\sigma) := \mathcal{L}_{\mathrm{nv}}^{\varphi_n,\vec{x}}(z, \overrightarrow{01}; p) \ (= \varphi_{n,\vec{x}}(\log(\mathfrak{f}_\sigma^p)^{-1})) \qquad (\sigma \in G_K).$$

More directly, the ℓ-adic polylogarithms can be defined as coefficients of the Lie formal expansion of $\log(\mathfrak{f}_\sigma^p)^{-1}$ in $X = \log x$, $Y = \log y$, i.e.,

$$\text{(5.1)} \quad \log \mathfrak{f}_\sigma^p(x, y)^{-1} \equiv \rho_z(\sigma)X + \rho_{1-z}(\sigma)Y + \sum_{m\geq 1} \ell i_{m+1}(z, p, \vec{x})(\sigma)\, \mathrm{ad}(X)^m(Y) \bmod I_Y$$

for $\sigma \in G_K$, where $I_Y$ denotes the ideal generated by the terms with two or more $Y$, and $\rho_z$ (resp. $\rho_{1-z}$) is the Kummer cocycle along a carefully chosen system of power roots of $z$ (resp. $1 - z$). With regard to this formula, we shall also define

$$\ell i_0(z, p, \vec{x})(\sigma) = \rho_z(\sigma), \quad \ell i_1(z, p, \vec{x})(\sigma) = \rho_{1-z}(\sigma).$$

In [NW], we related the above $\ell i_n(z, p, \vec{x})$ with the *ℓ-adic polylogarithmic character* $\tilde{\chi}_m^z\colon G_K \to \mathbb{Z}_l$ $(m \geq 1)$ defined by the Kummer properties for $n \geq 1$:

$$\text{(5.2)} \qquad \zeta_{\ell^n}^{\tilde{\chi}_m^z(\sigma)} = \sigma\left(\prod_{a=0}^{\ell^n-1}(1 - \zeta_{\ell^n}^{\chi(\sigma)^{-1}a} z^{1/\ell^n})^{\frac{a^{m-1}}{\ell^n}}\right) \Big/ \prod_{a=0}^{\ell^n-1}(1 - \zeta_{\ell^n}^{a+\rho_z(\sigma)} z^{1/\ell^n})^{\frac{a^{m-1}}{\ell^n}},$$

where $(1 - \zeta_{\ell^n}^\alpha z^{1/\ell^n})^{\frac{\beta}{\ell^n}}$ means the $\beta$th power of a carefully chosen $\ell^n$th root of $(1 - \zeta_{\ell^n}^\alpha z^{1/\ell^n})$ as in loc. cit. depending on $p\colon \overrightarrow{01} \rightsquigarrow z$. Note, in particular, that $\tilde{\chi}_1^z = \rho_{1-z}$. The formula of [NW], p. 293 corollary, gives an expression for $\ell i_m(z, p, \vec{x})$ in terms of $\tilde{\chi}_m^z$ exactly as in a similar way to Proposition 5.2 of the complex case:

$$\text{(5.3)} \quad \ell i_m(z, p, \vec{x})(\sigma) = (-1)^{m+1} \sum_{k=0}^{m-1} \frac{B_k}{k!}(-\rho_z(\sigma))^k \frac{\tilde{\chi}_{m-k}^z(\sigma)}{(m-k-1)!} \qquad (m \geq 1).$$

(Here we point out a misprint $\sum_{k=0}^{m}$ in the formula of loc. cit. that should have read $\sum_{k=0}^{m-1}$ as above.) In particular, for $\sigma \in G_{K(z^{1/\ell^\infty})}$, we have

$$\ell i_m(z, p, \vec{x})(\sigma) = (-1)^{m-1} \frac{\tilde{\chi}_m^z(\sigma)}{(m-1)!} \qquad (m \geq 1). \tag{5.4}$$

**Remark 5.5** The $\ell$-adic polylogarithms $\ell i_m(z, p, \vec{x})$ introduced in this paper are same as $\ell_m^z(\sigma)$ defined in [NW]. The functions $\ell i_n(z, p, \vec{x})$ give homomorphisms on $G_{K(\mu_{\ell^\infty}, z^{1/\ell^\infty})}$, while, by [NW], prop.1, $H_2(\overrightarrow{01}, z) = G_{K(\mu_{\ell^\infty}, z^{1/\ell^\infty}, (1-z)^{1/\ell^\infty})}$. On $G_1(\overrightarrow{01}) = G_{K(\mu_{\ell^\infty})}$, one can easily see their relation to the big $\ell$-adic iterated integral:

$$-\ell i_n(z, p, \vec{x})(\sigma) = \mathcal{L}^{\varphi_n, \vec{x}}(z, \overrightarrow{01}; p)(\sigma) \qquad (\sigma \in G_1(\overrightarrow{01}) = G_{K(\mu_{\ell^\infty})}).$$

(Cf. [W5] cor. 11.0.16, or proof of [NW], lemma 2.)

**Remark 5.6** The Galois representation of the pro-$\ell$ fundamental group of $\mathbf{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ has been studied intensively by Y. Ihara, P. Deligne and other authors (see, e.g., [Ih1], [Ih2], [De2], [HM], [MS]). In particular, the filtration $\{G_n(\overrightarrow{01})\}_n$ of $G_{\mathbb{Q}}$ is deeply relevant to the arithmetic of (higher) cyclotomic fields. This would suggest that what is needed is a more consistent study of "$z$-version" filtrations of $G_{\mathbb{Q}}$ also depending on various quotients of the $\pi_1$-torsor. One such important aspect is an $\ell$-adic analog of Zagier's conjecture ([W6], [DW2]). Motivic interpretation of this conjecture by Beilinson–Deligne [BD] has greatly inspired creative ideas in our work. The case $\mathbf{P}^1 - \{0, \mu_n, \infty\}$ has already been studied in depth by A. B. Goncharov [Gon] and the second named author ([DW1],[DW2], [W6],[W7],[W8]).

**5.3 Functional equations for polylogarithms** Given a $K$-morphism $f\colon X \to P_0$ of an arbitrary algebraic variety $X$ to $P_0$ and a $K$-rational (tangential) base point $v$ on $X$, choose a path $\delta$ from $\overrightarrow{01}$ to $f(v)$ so that we obtain an isomorphism

$$\iota_\delta\colon \pi_1(P_0, f(v)) \xrightarrow{\sim} \pi_1(P_0, \overrightarrow{01})$$

by $x \mapsto \delta x \delta^{-1}$. The induced map $\mathrm{gr}^n_\Gamma(\iota_\delta)$ of $\mathrm{gr}^n_\Gamma(\pi_1(P_0, f(v))) \xrightarrow{\sim} \mathrm{gr}^n_\Gamma(\pi_1(P_0, \overrightarrow{01}))$ is *independent of the choice of* $\delta$, as inner automorphisms act trivially on the graded quotients via the (lower) central filtration. From Corollary 3.7 and the above definition of $\varphi_n$ it follows that the composition $\varphi_n(f) := \varphi_n \circ \mathrm{gr}^n_\Gamma(\iota_\delta)$ yields the identity

$$\varphi_n(f) \circ \mathrm{gr}^n_\Gamma(f_*) = \kappa^{\otimes n}\left(f^{\odot n-2} \otimes (f \wedge (f-1))\right), \tag{5.5}$$
$$\text{i.e.,} \qquad \widehat{\kappa_{\otimes n}}(\varphi_n(f))(f) = f^{\odot n-2} \otimes (f \wedge (f-1)),$$

where $\widehat{\kappa_{\otimes n}}(\varphi_n(f))(f)$ designates the pulled-back Kummer dual in the sense of Definition 3.8.

In the $\ell$-adic case, we use the natural extensions $\varphi_{n,\vec{x}}\colon L(\pi_{\overrightarrow{01}}) \to \mathbb{Q}_\ell$ and $\varphi_n(f)_{\vec{x}}\colon L(\pi_{f(v)}) \to \mathbb{Q}_\ell$ with respect to the fixed generator system $\vec{x} = (x, y)$ (cf. equation (4.11)). They are related by the (naturally induced) isomorphism $\iota_\delta\colon \hat{U}(\pi_{f(v)}) \xrightarrow{\sim} \hat{U}(\pi_{\overrightarrow{01}})$ by

$$\varphi_n(f)_{\vec{x}}(\lambda) = \varphi_{n,\vec{x}}(\delta\lambda\delta^{-1}) \quad (\lambda \in L(\pi_{f(v)})). \tag{5.6}$$

Now, given a topological or an étale path $q$ from $v$ to $z$ on $X$, we have classical and $\ell$-adic polylogarithms

$$\mathrm{li}_n(f(z), \delta f(q)),\ \mathrm{li}_n(f(v), \delta); \quad \ell i_n(f(z), \delta f(q), \vec{x}),\ \ell i_n(f(v), \delta, \vec{x}),$$

where $\delta f(q)$ denotes the composition of paths $\delta\colon \overrightarrow{01} \rightsquigarrow f(v)$ and $f(q)\colon f(v) \rightsquigarrow f(z)$. Later in Proposition 5.10, these will be related to the iterated integrals

$$\mathcal{L}_{\mathbb{C}}^{\varphi_n}(f(z), f(v); f(q)), \quad \mathcal{L}_{\mathrm{nv}}^{\varphi_n(f)_{\vec{x}}}(f(z), f(v); f(q)).$$

that appear in the direct application of the functional equations of Theorems 4.13 and 4.14 to this special case of polylogarithms, that is:

**Theorem 5.7** *Let $V$ be an arbitrary algebraic variety over a subfield $K$ of $\mathbb{C}$ with a $K$-rational (possibly tangential) base point $v$ on $V$, and let $P_0 = \mathbf{P}^1_K - \{0, 1, \infty\}$. Then, for algebraic morphisms $f_i\colon V \to P_0$ $(i = 1, \ldots, m)$ and integers $c_1, \ldots, c_m \in \mathbb{Z}$ together with paths $\delta_i\colon \overrightarrow{01} \rightsquigarrow f_i(v)$, the following conditions* (i)$_\mathbb{C}$, (ii)$_\mathbb{C}$ and (iii)$_\mathbb{C}$ *are equivalent:*

(i)$_\mathbb{C}$ $$\sum_{i=1}^m c_i\, \varphi_n(f_i) \circ \mathrm{gr}^n_\Gamma(f_{i*}) = 0 \quad \textit{in } \mathrm{Hom}\,(\mathrm{gr}^n_\Gamma \pi_1(V(\mathbb{C}), v), \mathbb{Z}),$$

(ii)$_\mathbb{C}$ $$\sum_{i=1}^m c_i\, f_i^{\odot n-2} \otimes (f_i \wedge (f_i - 1)) = 0$$
*in* $\mathrm{Sym}^{n-2}\mathcal{O}_h^\times(V^{an}) \otimes (\mathcal{O}_h^\times(V^{an}) \wedge \mathcal{O}_h^\times(V^{an}))$,

(iii)$_\mathbb{C}$ $$\sum_{i=1}^m c_i\, \mathcal{L}_{\mathbb{C}}^{\varphi_n}(f_i(z), f_i(x); f_i(\gamma)) = 0$$
*for each path $\gamma : x \rightsquigarrow z$ on $V(\mathbb{C})$.*

*For the following $\ell$-adic analogs* (i)$_\ell$, (ii)$_\ell$ and (iii)$_\ell$ *of the conditions above, we have, in general, the equivalence* (i)$_\ell$ ⇔ (ii)$_\ell$ *and the implication* (i)$_\ell$ (ii)$_\ell$ ⇒

(iii)$_\ell$. *If* $\mu_{\ell^\infty} \not\subset K$, *then we have also* (i)$_\ell$ (ii)$_\ell$ $\Leftrightarrow$ (iii)$_\ell$:

(i)$_\ell$ $$\sum_{i=1}^{m} c_i\, \varphi_n(f_i) \circ \mathrm{gr}^n_\Gamma(f_{i*}) = 0 \quad \text{in } \mathrm{Hom}\,(\mathrm{gr}^n_\Gamma \pi_1^\ell(V_{\overline{K}}, v), \mathbb{Z}_\ell),$$

(ii)$_\ell$ $$\sum_{i=1}^{m} c_i\, f_i^{\odot n-2} \otimes (f_i \wedge (f_i - 1)) = 0$$

*in* $\mathrm{Sym}^{n-2}\widehat{\mathcal{O}}^\times_\ell(V_{\overline{K}}) \otimes (\widehat{\mathcal{O}}^\times_\ell(V_{\overline{K}}) \wedge \widehat{\mathcal{O}}^\times_\ell(V_{\overline{K}}))$,

(iii)$_\ell$ *There exists an error term* $E\colon G_K \times \pi_1^\ell(V_{\overline{K}}, v, z) \to \mathbb{Q}_\ell$ *of degree n such that*

$$\sum_{i=1}^{m} c_i\, \mathcal{L}_{\mathrm{nv}}^{\varphi_n(f_i)_{\vec{x}}}(f_i(z), f_i(v); f_i(p))(\sigma) = E(\sigma, p) \quad (\sigma \in G_K)$$

*for each étale path* $p\colon x \rightsquigarrow z$.

*Proof* This is only a special case of Theorem 4.13 and Theorem 4.14. □

We remark that, in conditions (i)$_\mathbb{C}$ and (i)$_\ell$, $\mathrm{Hom}(\mathrm{gr}^n_\Gamma \pi_1, -)$ may be replaced by $\mathrm{Hom}(\mathrm{gr}^n_\Gamma(\pi_1/\pi_1''), -)$ according to the last half statement of Corollary 3.7. For later convenience when computing the error term $E(\sigma, p)$ in $\ell$-adic cases, we shall rephrase Corollary 4.16 in this special case using (5.6). Let $\pi_{V,v}$ be the pro-unipotent completion of the pro-$\ell$ fundamental group $\pi_1^\ell(V_{\overline{K}}, v)$ over $\mathbb{Q}_\ell$, and let $L(\pi_{V,v})$ denote its (complete) Lie algebra equipped with the lower central filtration $L(\pi_{V,v}) = \Gamma^1 L(\pi_{V,v}) \supset \Gamma^2 L(\pi_{V,v}) \supset \cdots$.

**Corollary 5.8** *Using notation as in Theorem 5.7, suppose condition (iii)*$_\ell$ *holds. Pick a vector space splitting* $L(\pi_{V,v}) = L_{<n} \oplus \Gamma^n L(\pi_{V,v})$ *and decompose* $\log(\mathfrak{f}^p_\sigma)^{-1} = [\log(\mathfrak{f}^p_\sigma)^{-1}]_{<n} + [\log(\mathfrak{f}^p_\sigma)^{-1}]_{\geq n}$ *according to it. Then the error term of the functional equation is given by*

$$E(\sigma, p) = \sum_i c_i\, \varphi_{n,\vec{x}}(\delta_i \cdot f_i([\log(\mathfrak{f}^p_\sigma)^{-1}]_{<n}) \cdot \delta_i^{-1}) \quad (\sigma \in G_K,\ p\colon v \rightsquigarrow z).$$

Next, we shall consider the problem of expressing the iterated integrals in the functional equations (iii)$_\mathbb{C}$, (iii)$_\ell$ by polylogarithms explicitly. For this purpose, we shall introduce a useful series of polynomials in several variables as follows.

**Proposition 5.9** (Polylog-BCH formula) *Let* $\{a_i\}_{i=0}^\infty$, $\{b_i\}_{i=0}^\infty$ *be countably many mutually commuting variables, and let* $X, Y$ *be non-commutative variables. Consider the Lie algebra consisting of the Lie formal series in X and Y with coefficients in the polynomial ring* $\mathbb{Q}[a_i, b_i]_{i=0}^\infty$. *Then, there exists a unique se-*

*quence of polynomials*

$$\mathsf{P}_n = \mathsf{P}_n(\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n) \qquad (n = 0, 1, 2, \dots)$$

*characterized by*

$$\Big(a_0X + a_1Y + \sum_{i=1}^{\infty} a_{i+1}\,\mathrm{ad}(X)^i(Y)\Big) \boxplus \Big(b_0X + b_1Y + \sum_{i=1}^{\infty} b_{i+1}\,\mathrm{ad}(X)^i(Y)\Big)$$
$$\equiv \mathsf{P}_0X + \mathsf{P}_1Y + \sum_{i=1}^{\infty} \mathsf{P}_i\,\mathrm{ad}(X)^i(Y) \quad \mathit{mod}\ I_Y,$$

*where* $\boxplus$ *denotes the Baker–Campbell–Hausdorff sum:* $S \boxplus T = \log(e^S e^T)$*, and* $I_Y$ *is the ideal generated by the Lie monomials having* $Y$ *twice or more.*

*Proof* We put $S = a_0X + \sum_{i\geq 0} a_i\,\mathrm{ad}(X)^i(Y)$ and $T = b_0X + \sum_{i\geq 0} b_i\,\mathrm{ad}(X)^i(Y)$ into the Baker–Campbell–Hausdorff formula:

$$S \boxplus T = S + T + \frac{1}{2}[S,T] + \frac{1}{12}[S[S,T]] - \frac{1}{12}[T[S,T]] - \frac{1}{24}[S[T[S,T]]] + \cdots .$$

Observing $[S,T] \equiv (a_0b_1 - b_0a_1)[X,Y] + a_0\sum_{i\geq 2} b_i\,\mathrm{ad}(X)^i(Y)$ (mod $I_Y$), we have only to treat the terms of the form $[U_1[U_2[\cdots[U_n,[S,T]]\cdots]]]$ where $U_i = S$ or $T$, because the other type of terms vanish modulo $I_Y$. Further calculations show that each of these terms with $k+1$-times of $S$ and $l+1$-times of $T$ contributes $b_0^l\sum_{j=0}^{\infty}(a_0b_{j+1} - b_0a_{j+1})\,\mathrm{ad}(X)^{k+l+j}(Y)$ regardless of the order of $S$ and $T$. So these terms having the same numbers of $S$ and $T$ can be counted together; their appearances are summed up as the coefficient of $S^{k+1}T^{l+1}$ of $S \boxplus T$ (in the non-commutative power series ring in $S$ and $T$) multiplied by $(-1)^l$. The coefficients of these terms are explicitly calculated by K. Goldberg [Gol, th. 3 and (10)] (see also [K]) showing that those coefficients $c(s,t)$ of $S^sT^t$ in $S \boxplus T$ and their generating functions are given by

(G1) $$c(s,t) = \frac{(-1)^s}{s!\,t!}\sum_{i=1}^{t}\binom{t}{i}B_{s+t-i} \quad (s,t \geq 1),$$

(G2) $$uv\sum_{k,l=0}^{\infty} c(k+1,l+1)u^kv^l = \frac{ue^u(e^v-1) - ve^v(e^u-1)}{e^u - e^v} \quad (=: uv\,G(u,v)).$$

Putting these together, if we write $G(a_0t, -b_0t) = \sum_{i=0}^{\infty} C_i(a_0,b_0)t^i$, then we obtain

$$\mathsf{P}_n = a_n + b_n + (a_0b_1 - b_0a_1)C_{n-2}(a_0,b_0) + \cdots + (a_0b_{n-1} - b_0a_{n-1})C_0(a_0,b_0)$$

for $n \geq 2$. This completes the proof of our proposition. □

**Example 5.10** Taking notation as in the proof above, one can write

$$G(a_0t,-b_0t)=\sum_{i=0}^{\infty}C_i(a_0,b_0)t^i=\frac{1}{b_0t}\left(1-\frac{e^{a_0t}-1}{a_0}\cdot\frac{a_0+b_0}{e^{(a_0+b_0)t}-1}\right), \tag{5.7}$$

which is more useful in computer calculations (The form of the right-hand side given above is derived in [K].) The last equation in the proof above yields a formula for the generating function of the polynomials $\{\mathsf{P}_n\}$ of the above proposition as follows:

$$\sum_{n=0}^{\infty}\mathsf{P}_nt^n=\sum_{n=0}^{\infty}(a_n+b_n)t^n+t\Big(a_0\sum_{n=1}^{\infty}b_nt^n-b_0\sum_{n=1}^{\infty}a_nt^n\Big)\cdot G(a_0t,-b_0t). \tag{5.8}$$

The first several polynomials $\mathsf{P}_n$ read

$$\begin{aligned}
\mathsf{P}_0&=a_0+b_0,\\
\mathsf{P}_1&=a_1+b_1,\\
\mathsf{P}_2&=a_2+b_2+\frac{1}{2}(a_0b_1-b_0a_1),\\
\mathsf{P}_3&=a_3+b_3+\frac{1}{2}(a_0b_2-b_0a_2)+\frac{1}{12}(a_0^2b_1-a_0a_1b_0-a_0b_0b_1+a_1b_0^2),\\
\mathsf{P}_4&=a_4+b_4+\frac{1}{2}(a_0b_3-b_0a_3)+\frac{1}{12}(a_0^2b_2-a_0a_2b_0-a_0b_0b_2+a_2b_0^2)\\
&\quad-\frac{1}{24}(a_0^2b_0b_1-a_0a_1b_0^2),\\
\mathsf{P}_5&=a_5+b_5+\frac{1}{2}(a_0b_4-b_0a_4)+\frac{1}{12}(a_0^2b_3-a_0a_3b_0-a_0b_0b_3+a_3b_0^2)\\
&\quad-\frac{1}{24}(a_0^2b_0b_2-a_0a_2b_0^2)-\frac{1}{180}(a_0a_1b_0^3-a_0^2a_1b_0^2-a_0^2b_0^2b_1+a_0^3b_0b_1)\\
&\quad-\frac{1}{720}(a_1b_0^4-a_0b_0^3b_1-a_0^3a_1b_0+a_0^4b_1).
\end{aligned}$$

Concerning symmetric properties of our polynomials $\mathsf{P}_n$, it firstly follows from $(e^Se^T)^{-1}=e^{-T}e^{-S}$ that

$$\mathsf{P}_n(\{a_i\}_{i=0}^n,\{b_i\}_{i=0}^n)=-\mathsf{P}_n(\{-b_i\}_{i=0}^n,\{-a_i\}_{i=0}^n). \tag{5.9}$$

For our later calculations, useful are also the formulas of $G(a_0t,b_0t)$ in special cases of $a_0$ or $b_0=0$. These are easily obtained from de l'Hospital's formula as follows:

$$G(0,-b_0t)=-\sum_{n=1}^{\infty}\frac{B_n}{n!}(b_0t)^{n-1},\quad G(a_0t,0)=1+\sum_{n=1}^{\infty}\frac{B_n}{n!}(a_0t)^{n-1}. \tag{5.10}$$

Note $B_1=-\frac{1}{2}$ so that the above series both start from the constant term $\frac{1}{2}$.

Now we arrive at the stage where translations of the functional equations $(\mathrm{iii})_{\mathbb{C}}$, $(\mathrm{iii})_\ell$ of Theorem 5.6 are available in terms of polylogarithms.

**Proposition 5.11** *Given a $K$-morphism $f\colon V \to P_0$ as in the beginning of this subsection, we can express the complex and $\ell$-adic iterated integrals associated to $\varphi_n$ and the image of a path $q\colon v \rightsquigarrow z$ on $V$ in $P_0$ in terms of polylogarithms along the paths $\delta\colon \overrightarrow{01} \rightsquigarrow f(v)$ and $\delta f(q)\colon \overrightarrow{01} \rightsquigarrow f(z)$ as follows:*

(i) $\mathcal{L}_{\mathbb{C}}^{\varphi_n}(f(z), f(v); f(q)) = \mathsf{P}_n(\{\mathrm{li}_i(f(z), \delta f(q))\}_{i=0}^n, \{-\mathrm{li}_i(f(v), \delta)\}_{i=0}^n)$,

(ii) $\mathcal{L}_{\mathrm{nv}}^{\varphi_n(f)_{\vec{x}}}(f(z), f(v); f(q)) = \mathsf{P}_n(\{-\ell i_i(f(v), \delta, \vec{x})\}_{i=0}^n, \{\ell i_i(f(z), \delta f(q), \vec{x})\}_{i=0}^n)$.

*For the definitions of* $\mathrm{li}_0$, $\mathrm{li}_1$, *$\ell i_0$ and $\ell i_1$, see Remark 5.3 and Definition 5.4.*

*Proof* By the chain rule (4.3), we have

$$\Lambda_{f(q)}(f(z), f(q))^{-1} = \Lambda_{\delta f(q)}(f(z), \overrightarrow{01})^{-1} \cdot \left(\Lambda_\delta(f(v), \overrightarrow{01})^{-1}\right)^{-1}.$$

Hence

$$\begin{aligned}&\log(\Lambda_{f(q)}(f(z), f(q))^{-1})\\ &\qquad = \left(\log(\Lambda_{\delta f(q)}(f(z), \overrightarrow{01})^{-1})\right) \boxplus \left(-\log(\Lambda_\delta(f(v), \overrightarrow{01})^{-1})\right).\end{aligned}$$

Expanding both sides above as Lie formal series in $\bar{x}$ and $\bar{y}$ modulo $I_Y$, we obtain (i). For the $\ell$-adic case, since $\mathfrak{f}_\sigma^{\delta f(q)} = \delta \mathfrak{f}_\sigma^{f(q)} \delta^{-1} \mathfrak{f}_\sigma^{\delta}$, we have

$$\log(\delta(\mathfrak{f}_\sigma^{f(q)})^{-1}\delta^{-1}) = \left(-\log(\mathfrak{f}_\sigma^{\delta})^{-1}\right) \boxplus \left(\log(\mathfrak{f}_\sigma^{\delta f(q)})^{-1}\right).$$

By Definition 4.7 and (5.6), the left-hand side of (ii) may be written as

$$\begin{aligned}\mathcal{L}_{\mathrm{nv}}^{\varphi_n(f)_{\vec{x}}}(f(z), f(v); f(q))(\sigma) &= \varphi_n(f)_{\vec{x}}(f(\log(\mathfrak{f}_\sigma^{q})^{-1}))\\ &= \varphi_{n,\vec{x}}(\delta \log(\mathfrak{f}_\sigma^{f(q)})^{-1}\delta^{-1}).\end{aligned}$$

These equations complete the proof of (ii). □

The following proposition computes the "BCH-conjugation" of polylog Lie series.

**Proposition 5.12** *Suppose that $A$, $B$ are Lie formal power series in $X, Y$ in the form modulo $I_Y$:*

$$\begin{cases} A \equiv a_0 X + a_1 Y + \sum_{i=1}^\infty a_{i+1}\, \mathrm{ad}(X)^i(Y),\\ B \equiv b_0 X + b_1 Y + \sum_{i=1}^\infty b_{i+1}\, \mathrm{ad}(X)^i(Y).\end{cases}$$

*Then, we have*

$$\log(e^A e^B e^{-A}) \equiv B + \sum_{n=1}^\infty \sum_{k=1}^\infty (a_0 b_n - b_0 a_n) \frac{a_0{}^{k-1}}{k!} (\mathrm{ad}\, X)^{n+k-1}(Y) \quad \textit{mod } I_Y.$$

*In particular, if* $a_0 = 0$*, then,* $\log(e^A e^B e^{-A}) \equiv B - b_0[X, A]$ *mod* $I_Y$.

*Proof* This follows from the well-known formula $e^A e^B e^{-A} = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}\, A)^n(B)}{n!}$ by simple computation. □

**5.4 Drinfeld associators under $S_3$** We may consider $\Lambda_\gamma(w, v)$ for any path $\gamma\colon v \rightsquigarrow w$ between tangential basepoints

$$v, w \in \mathfrak{B} := \{\overrightarrow{01}, \overrightarrow{0\infty}, \overrightarrow{\infty 1}, \overrightarrow{10}, \overrightarrow{1\infty}, \overrightarrow{\infty 0}\}.$$

See [De2], [W3] for precise definitions. We shall call $\Lambda_\gamma := \Lambda_\gamma(w, v)$ the Drinfeld associator for $\gamma$. The fundamental groupoid $\pi_1(P_0(\mathbb{C}), \mathfrak{B})$ is generated by the standard paths $\langle a, b\rangle$, $[a_b^c]$ ($\{a, b, c\} = \{0, 1, \infty\}$), where $\langle a, b\rangle$ denotes the path from $\overrightarrow{ab}$ to $\overrightarrow{ba}$ along $\mathbf{P}^1(\mathbb{R})$, and $[a_b^c]$ denotes the half anticlockwise rotation from $\overrightarrow{ab}$ to $\overrightarrow{ac}$ (cf. [N2-I]). By the chain rule (4.3) and the pushforward property (4.5) (extended to those $\Lambda_\gamma$ between tangential base points), we may compute any Drinfeld associator by compositions of the $S_3$-transforms of $\Lambda_{\langle 0,1\rangle}$, $\Lambda_{[0_1^\infty]} = e^{\pi i X}$. Usually, the non-commutative power series $\Lambda(X, Y) := \Lambda_{\langle 0,1\rangle}$ is called 'the' Drinfeld associator whose coefficients are given by multiple zeta values ([Dr]; see, e.g., [F]). For the sequel of this article, we only recall its polylogarithmic part:

$$\log(\Lambda_{\langle 0,1\rangle})^{-1} \equiv \sum_{m=2}^{\infty} (-1)^{m+1} \zeta(m) (\mathrm{ad}\, X)^{m-1}(Y) \qquad \mathrm{mod}\ I_Y.$$

It is useful to recall the following action of $S_3$-automorphisms of $P_0$ on $X, Y$ to compute the other $\Lambda_\gamma$.

| $f(z)$ | $z$ | $\frac{z}{z-1}$ | $\frac{1}{z}$ | $1-z$ | $\frac{1}{1-z}$ | $\frac{z-1}{z}$ |
|---|---|---|---|---|---|---|
| $f_*(X)$ | $X$ | $X$ | $-X-Y$ | $Y$ | $Y$ | $-X-Y$ |
| $f_*(Y)$ | $Y$ | $-X-Y$ | $Y$ | $X$ | $-X-Y$ | $X$ |

For our later application, let us illustrate the computation of the polylog part of $\log(\Lambda_\gamma)^{-1}$ for $\delta = \langle 0, 1\rangle[1_0^\infty]\langle 1, \infty\rangle$. By the chain rule and the pushforward property, it follows that

$$\Lambda_\delta = \Lambda_{\langle 0,1\rangle} e^{\pi i Y} f_*(\Lambda_{\langle 0,1\rangle})^{-1} = \Lambda(X, Y) e^{\pi i Y} \Lambda(-X - Y, Y)^{-1},$$

where $f(z) = \frac{1}{z}$. Evaluating $\log(\Lambda_\delta)^{-1}$ after the polylog BCH formula (Proposition 5.9), we find from $\zeta(2n) = (2\pi i)^{2n}\frac{-B_{2n}}{2\cdot(2n)!}$:

$$\text{(5.11)} \qquad \log(\Lambda_{\langle 0,1\rangle[1_0^\infty]\langle 1,\infty\rangle})^{-1} \equiv \sum_{n=1}^{\infty}(2\pi i)^n \frac{B_n}{n!}(\mathrm{ad}\, X)^{n-1}(Y)$$
$$\equiv \sum_{n=1}^{\infty}\frac{B_n}{n!}(\mathrm{ad}\,\bar{x})^{n-1}(\bar{y}) \quad \bmod I_Y.$$

As the $\ell$-adic correspondent of Drinfeld associators, we may consider the functions $\mathfrak{f}_\sigma^\gamma$ ($\sigma \in G_{\mathbb{Q}}$) for $\gamma \in \pi_1(P_0/\overline{\mathbb{Q}}, \mathfrak{B})$. The basic associator is $\mathfrak{f}_\sigma := \mathfrak{f}_\sigma^{\langle 0,1\rangle}$ whose polylog part is known essentially by Ihara's work (cf. [NW]) as

$$\text{(5.12)} \qquad \log(\mathfrak{f}_\sigma)^{-1} \equiv \sum_{m\geq 1} \ell i_{m+1}^{\overrightarrow{10}}(\sigma)(ad\, X)^m(Y)$$
$$\equiv \sum_{m\geq 1}(-1)^m \frac{\tilde{\chi}_{m+1}^{\overrightarrow{10}}(\sigma)}{m!}(ad\, X)^m(Y) \bmod I_Y$$

for $\sigma \in G_{\mathbb{Q}}$. The coefficient character $\tilde{\chi}_m^{\overrightarrow{10}}$ is the $(1-\ell^{m-1})^{-1}$-multiple of the so-called Soulé character $\chi_m$ which, over $G_{\mathbb{Q}(\zeta_{\ell^\infty})}$, vanishes for even $m \geq 2$, and is non-trivial for odd $m \geq 3$. Precise formulas for the Soulé characters of even degrees $m = 2k$ ($k = 1, 2, \dots$) over $G_{\mathbb{Q}}$ have also been calculated by several authors. For $m = 2$, [LS], pp. 582–583, expressed $\tilde{\chi}_2^{\overrightarrow{10}}(\sigma)$ as $\frac{1}{24}(\chi(\sigma)^2 - 1)$ by using the $\ell$-adic cyclotomic character $\chi : G_{\mathbb{Q}} \to \mathbb{Z}_\ell^\times$, and [Ih1] theorem (p. 115) remarked a complete formula for all $m$ (without proof). In [W10], the second named author gave a proof for the case of even degrees $m \geq 2$ which has essentially the same nature as the first one below. Note that these formulae should be regarded as the $\ell$-adic analogs of the classical formulae $\zeta(2k) = -(2\pi i)^{2k}\frac{B_{2k}}{2\cdot(2k)!}$ for which a "geometric proof" via the use of associators was presented in [De2], §18.17.

**Proposition 5.13** *For $k = 1, 2, \dots$, we have*

$$\tilde{\chi}_{2k}^{\overrightarrow{10}}(\sigma) = \frac{B_{2k}}{2(2k)}(\chi(\sigma)^{2k} - 1) \qquad (\sigma \in G_{\mathbb{Q}}).$$

*First proof* Fix $\sigma \in G_{\mathbb{Q}}$ and consider the power-conjugate form of the $\sigma$-action on the standard loop $z := (xy)^{-1}$ around the puncture $\infty$ in the pro-$\ell$ fundamental group $\pi_1^\ell(P_0/\overline{\mathbb{Q}}, \overrightarrow{01})$. This is known, for example, from [Ih1] p. 106, [N1] (A10), [N2-I] prop. 2.11, and the $\sigma$-actions on both sides of $z = y^{-1}x^{-1}$ yield the equation

$$G_\sigma\, z^{\chi(\sigma)}\, G_\sigma^{-1} = \mathfrak{f}_\sigma(y, x)\, y^{-\chi(\sigma)}\, \mathfrak{f}_\sigma(y, x)^{-1} \cdot x^{-\chi(\sigma)},$$

where $G_\sigma := \mathfrak{f}_\sigma(y,x)y^{\frac{1-\chi(\sigma)}{2}}\mathfrak{f}_\sigma(z,y)$. We shall evaluate the log of both sides of the above equation modulo $I_Y$. First, by a similar computation to (5.10), it follows that

$$\log G_\sigma \equiv \frac{1-\chi(\sigma)}{2}Y + \sum_{n=1}^{\infty} 2\ell i_{2n}^{\overrightarrow{10}}(\mathrm{ad}\,X)^{2n-1}(Y) \bmod I_Y.$$

Applying this and the well-known formula $\log(e^\alpha e^\beta) \equiv \beta + \sum_{n=0}^{\infty}\frac{B_n}{n!}(\mathrm{ad}\beta)^n(\alpha)$ mod $\deg(\alpha) \geq 2$ to Proposition 5.12, we obtain the following congruence modulo $I_Y$:

$$\begin{aligned}
&\log(G_\sigma \cdot (xy)^{-\chi(\sigma)} \cdot G_\sigma^{-1}) \\
&\quad \equiv \chi(\sigma)\log(y^{-1}x^{-1}) + \chi(\sigma)(\mathrm{ad}\,X)\log G_\sigma \\
&\quad \equiv -\chi(\sigma)X + \sum_{n=0}^{\infty}(-1)^{n+1}\chi(\sigma)\frac{B_n}{n!}(\mathrm{ad}\,X)^n(Y) \\
&\qquad + \chi(\sigma)(\mathrm{ad}\,X)\left\{\frac{1-\chi(\sigma)}{2}Y + 2\sum_{n=1}^{\infty}\ell i_{2n}^{\overrightarrow{10}}(\sigma)(\mathrm{ad}\,X)^{2n-1}(Y)\right\}.
\end{aligned}$$

On the other hand, since $\log \mathfrak{f}_\sigma$ and $\log(y^{\chi(\sigma)})$ have no terms of $X$, by Proposition 5.12, it follows that $\log(\mathfrak{f}_\sigma(y,x)\,y^{-\chi(\sigma)}\,\mathfrak{f}_\sigma(y,x)^{-1}) \equiv -\chi(\sigma)Y$, hence the log of LHS is congruent to

$$\begin{aligned}
&\log(y^{-\chi(\sigma)}x^{-\chi(\sigma)}) \\
&\quad \equiv -\chi(\sigma)X + \sum_{n=0}^{\infty}(-\chi(\sigma))^{n+1}\frac{B_n}{n!}(ad\,X)^n(Y) \\
&\quad \equiv -\chi(\sigma)X - \chi(\sigma)Y - \frac{\chi(\sigma)^2}{2}[X,Y] + \sum_{n=1}^{\infty}(-\chi(\sigma)^{2n}\frac{B_{2n}}{(2n)!}(\mathrm{ad}\,X)^{2n}(Y)
\end{aligned}$$

modulo $I_Y$. Comparing the coefficients of $(\mathrm{ad}\,X)^{2n}(Y)$ of the RHS's settles the desired formula. □

*Second proof* We shall make use of the explicit formula (5.2) for $\tilde{\chi}_{2k} := \tilde{\chi}_{2k}^{\overrightarrow{10}}$:

$$\zeta_{\ell^n}^{\tilde{\chi}_{2k}(\sigma)} = \frac{\sigma\left(\prod_{a=0}^{\ell^n-1}(1-\zeta_{\ell^n}^{\chi(\sigma)^{-1}a}z^{1/\ell^n})^{\frac{a^{2k-1}}{\ell^n}}\right)}{\prod_{a=0}^{\ell^n-1}(1-\zeta_{\ell^n}^{a+\rho_z(\sigma)}z^{1/\ell^n})^{\frac{a^{2k-1}}{\ell^n}}}. \tag{5.13}$$

Fix any $\sigma \in G_{\mathbb{Q}}$ and set $c := \chi(\sigma) \in \mathbb{Z}_\ell^\times$. Pick $\bar{c} \in \mathbb{Z}_{>0}$ such that $c\bar{c} \equiv 1$ mod $\ell^{2n}$. Choose any decomposition of the index set

$$S_+ \sqcup S_- \sqcup S_0 = \{1,\ldots,\ell^n-1\}, \quad S_0 := \begin{cases}\emptyset & (\ell \neq 2), \\ \{\frac{1}{2}\ell^n\} & (\ell = 2),\end{cases}$$

so that $a \in S_+$ if and only if $\ell^n - a \in S_-$. We are going to rewrite both of the numerator and the denominator of RHS of (5.13) by using

$$(1 - \zeta_{\ell^n}^{-a})^{\frac{1}{\ell^n}} = (1 - \zeta_{\ell^n}^{a})^{\frac{1}{\ell^n}} \cdot \zeta_{2\ell^{2n}}^{\ell^n - 2a}. \tag{5.14}$$

We remark that, for any $a \in S_+$, the quotient of factors corresponding to $\ell^n - a \in S_-$ in the numerator and denominator of (5.13) may be replaced by that of factors corresponding to $-a$. Therefore, we may and do regard the set $S_-$ to be $\{-a \mid a \in S_+\}$. First we shall consider the case $\ell \neq 2$. It is easy to see that the denominator of (5.13) amounts to the product

$$\prod_{a \in S_+} \zeta_{2\ell^{2n}}^{2a^{2k} - a^{2k-1}\ell^n}$$

To apply (5.14) for the numerator of (5.13), we first need to replace the exponent $c^{-1}a$ of $\zeta_{\ell^n}$ by the least residue modulo $\ell^n$, i.e., by $\bar{c}a - [\frac{\bar{c}a}{\ell^n}]\ell^n$ (we denote by $[*]$ the largest integer $\leq *$). From this remark, it easily amounts to

$$\prod_{a \in S_+} \zeta_{2\ell^{2n}}^{2a^{2k} - ca^{2k-1}\ell^n - 2c\ell^n a^{2k-1}[\frac{\bar{c}a}{\ell^n}]}.$$

Thus, writing the fractional part as $\{*\} := * - [*]$, we obtain the congruence modulo $\ell^n$

$$\begin{aligned} \tilde{\chi}_{2k}(\sigma) &\equiv \sum_{a \in S+} a^{2k-1} \left( -c \left[ \frac{\bar{c}a}{\ell^n} \right] - \frac{c}{2} + \frac{1}{2} \right) \\ &\equiv \sum_{a \in S+} a^{2k-1} \left( c \left\{ \frac{\bar{c}a}{\ell^n} \right\} - \left\{ \frac{a}{\ell^n} \right\} + \frac{1-c}{2} \right). \end{aligned}$$

Observe that in the above sum $S_+$ may be replaced by $S_-$ (i.e., giving the same sum). So we may take $\frac{1}{2}\sum_{a \in S}$ instead of $\sum_{a \in S_+}$. Then, applying [La2] p. 39 and then p. 36, we find

$$\begin{aligned} \tilde{\chi}_{2k}(\sigma) &\equiv \frac{1}{2} \sum_{a \in S} \frac{\ell^{n(2k-1)}}{2k} \left[ c^{2k} B_{2k}\left( \left\{ \frac{\bar{c}a}{\ell^n} \right\} \right) - B_{2k}\left( \left\{ \frac{a}{\ell^n} \right\} \right) \right] \\ &\equiv \frac{1}{2} \frac{c^{2k} - 1}{2k} B_{2k}(0) \end{aligned}$$

modulo $\frac{\ell^n}{2(2k)D_k}\mathbb{Z}$ (where $D_k$ is the least common multiple of the denominators of coefficients of the polynomial $B_{2k}(X)$). Letting then the projective limit $n \to \infty$, we obtain the desired formula. In the case of $\ell = 2$, we have to take care of the factor coming from the index set $S_0$. But it turns out to form only a bounded value on $S_0$ converging to "measure zero", having no essential effect on the final conclusion of the above argument. □

**Remark 5.14** Let $\delta$ be a path from $\overrightarrow{01}$ to $z=-1$ on $\mathbf{P}^1(\mathbb{C})-\{0,1,\infty\}$ defined as the composition of the anticlockwise half-turn around $z=0$ and the simple move to $z=-1$ along the reals. By a method similar to that in the second proof of Proposition 5.13, one can show

$$\tilde{\chi}_2^{z=-1}(\sigma)=-\frac{1}{48}(\chi(\sigma)^2-1)-\frac{\chi(\sigma)}{2}\rho_2(\sigma) \qquad (\sigma\in G_{\mathbb{Q}}),$$

where $\tilde{\chi}_2^{z=-1}$ is taken along the above $\delta$. (Changing $\delta$ to its complex conjugate $\bar{\delta}$ above changes the sign of $\frac{\chi(\sigma)}{2}\rho_2(\sigma)$ on the right-hand side.) We point out that $\frac{1}{48}(\chi(\sigma)^2-1)$ is generally not in $\mathbb{Z}_2$, while $\frac{1}{48}(\chi(\sigma)^2-1)\pm\frac{\chi(\sigma)}{2}\rho_2(\sigma)$ does always belong to $\in\mathbb{Z}_2$ . This is the $\ell$-adic analog of the well-known formula "$Li_2(-1)=-\frac{\pi^2}{12}$" in the complex case (cf. [Le]).

## 6 Examples

**6.1 $\mathbf{Li_2(z)+Li_2(1-z)}$** Let $V=P_0=\mathbf{P}^1_z-\{0,1,\infty\}$ defined over a subfield $K\subset\mathbb{C}$, and consider two morphisms $f_1,f_2\colon V\to P_0$ defined by $f_1(z)=z$, $f_2(z)=1-z$. Then, the images of $v=\overrightarrow{01}$ on $V$ by these morphisms are given by $f_1(v)=\overrightarrow{01}$ and $f_2(v)=\overrightarrow{10}$. Let $x,y$ be the standard loops based at $\overrightarrow{01}$ on $V=P_0$ taken as in §5.1, and $\bar{x},\bar{y}$ be their images in $\mathrm{gr}^1\pi_1(P_0(\mathbb{C}),\overrightarrow{01})$. The space $\mathrm{gr}^2\pi_1(P_0(\mathbb{C}),\overrightarrow{01})$ is a free $\mathbb{Z}$-module generated by $[\bar{x},\bar{y}]$, and $\varphi_2\colon\mathrm{gr}^2\pi_1(P_0(\mathbb{C}),\overrightarrow{01})\to\mathbb{Z}$ is just taking the coefficient of $[\bar{x},\bar{y}]$. To connect $\overrightarrow{01}$ to $f_i(v)$ $(i=1,2)$, we take the path $\delta_1\colon\overrightarrow{01}\rightsquigarrow f_1(v)$ to be trivial and the path $\delta_2\colon\overrightarrow{01}\rightsquigarrow f_2(v)$ to be the real segment $[0,1]$ (i.e., $\delta_2=\langle 0,1\rangle$ in the notation of §5.4). In the sequel, we shall write $\delta:=\delta_2$. It is easy to see that $\varphi_2(f_1)\circ\mathrm{gr}^2_\Gamma(f_{1*})([\bar{x},\bar{y}])=1$ and $\varphi_2(f_2)\circ\mathrm{gr}^2_\Gamma(f_{2*})([\bar{x},\bar{y}])=-1$, hence that the condition $(\mathrm{i})_{\mathbb{C}}$ of Theorem 5.7 holds in the above setting, i.e.,

$$\varphi_2(f_1)\circ\mathrm{gr}^2_\Gamma(f_{1*})+\varphi_2(f_2)\circ\mathrm{gr}^2_\Gamma(f_{2*})=0 \tag{6.1}$$

in $\mathrm{Hom}(\mathrm{gr}^2_\Gamma\pi_1(P_0(\mathbb{C}),\overrightarrow{01}),\mathbb{Z})$. This just reflects the simple equation $(\mathrm{ii})_{\mathbb{C}}$:

$$z\wedge(z-1)+(1-z)\wedge(-z)=0 \tag{6.2}$$

in $\wedge^2\mathcal{O}_h^\times(V^{an})$. The $\ell$-adic analogs $(\mathrm{i})_\ell$, $(\mathrm{ii})_\ell$ also hold in the obvious way.

Now, we shall consider the functional equation $(\mathrm{iii})_{\mathbb{C}}$ in Theorem 5.7. For any path $\gamma\colon v\rightsquigarrow z$, it reads

$$\mathcal{L}^{\varphi_2}_{\mathbb{C}}(z,\overrightarrow{01};\gamma)+\mathcal{L}^{\varphi_2}_{\mathbb{C}}(1-z,\overrightarrow{10};f_2(\gamma))=0. \tag{6.3}$$

Let us apply Proposition 5.11(i) to each term of the above. Since $f_1(v)=$

$\overrightarrow{01}$ and $\delta_1$ is trivial, for the first term, the sequence $\{b_i\} = \{0\}$. This implies $\mathcal{L}_{\mathbb{C}}^{\varphi_2}(z, \overrightarrow{01}; \gamma) = \mathrm{li}_2(z, \gamma)$. For the second term, to apply Proposition 5.11, we must calculate $\mathsf{P}_2(\{a_i\}_{i=0}^2, \{b_i\}_{i=0}^2)$ for

$$\{a_i\}_{i=0}^2 = \{\mathrm{li}_0(1-z, \delta f_2(\gamma)), \mathrm{li}_1(1-z, \delta f_2(\gamma)), \mathrm{li}_2(1-z, \delta f_2(\gamma))\}, \tag{6.4}$$

$$\{b_i\}_{i=0}^2 = \{0, 0, -\mathrm{li}_2(\overrightarrow{10}, \delta)\} \tag{6.5}$$

to get $\mathcal{L}_{\mathbb{C}}^{\varphi_2}(1-z, \overrightarrow{10}; f_2(\gamma)) = \mathrm{li}_2(1-z, \delta f_2(\gamma)) - \mathrm{li}_2(1, \delta)$. Thus we obtain a functional equation of complex dilogarithms:

$$\mathrm{li}_2(z, \gamma) + \Big(\mathrm{li}_2(1-z, \delta f_2(\gamma)) - \mathrm{li}_2(1, \delta)\Big) = 0. \tag{6.6}$$

We may further rewrite it in terms of classical dilogarithms using Proposition 5.2. Noting the Bernoulli numbers $B_0 = 1$, $B_1 = -1/2$, we find that

$$\mathrm{li}_2(z, \gamma) = \frac{1}{4\pi^2}\left(Li_2(z) + \frac{1}{2}\log(1-z)\log z\right), \tag{6.7}$$

$$\mathrm{li}_2(1-z, \delta f_2(\gamma)) = \frac{1}{4\pi^2}\left(Li_2(1-z) + \frac{1}{2}\log(1-z)\log z\right), \tag{6.8}$$

$$\mathrm{li}_2(\overrightarrow{10}, \delta) = \frac{1}{4\pi^2} Li_2(1). \tag{6.9}$$

Summing up, we obtain the well-known equation (cf. [Le]):

$$Li_2(z) + Li_2(1-z) + \log z \log(1-z) = Li_2(1). \tag{6.10}$$

Note that $Li_2(1) = \zeta(2) = \frac{\pi^2}{6}$.

Next, we shall consider the $\ell$-adic analog in this case. Theorem 5.7 (iii)$_\ell$ reads:

$$\mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_1)_{\vec{x}}}(z, \overrightarrow{01}; \gamma)(\sigma) + \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_2)_{\vec{x}}}(1-z, \overrightarrow{10}; f_2(\gamma))(\sigma) = E(\sigma, \gamma) \tag{6.11}$$

for $\sigma \in G_K$. Let us first examine the left-hand side above. From (5.3) it follows immediately that the first term is equal to

$$\ell i_2(z, p, \vec{x}) = -\left\{\tilde{\chi}_2^z(\sigma) + \frac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma)\right\}. \tag{6.12}$$

The second term can be calculated after Proposition 5.11 as $\mathsf{P}_2(\{a_i\}_{i=0}^2, \{b_i\}_{i=0}^2)$ with

$$a_i = -\ell i_i(\overrightarrow{10}, \delta, \vec{x}), \quad b_i = \ell i_i(1-z, \delta f_2(\gamma), \vec{x}) \qquad (i = 0, 1, 2).$$

Writing $X = \log x$, $Y = \log y$, we know (cf. (5.1), (5.4))

$$\log(\mathfrak{f}_\sigma^\delta)^{-1} \equiv \sum_{m\geq 1} \ell i_{m+1}^{\overrightarrow{10}}(\sigma)(adX)^m(Y)$$

$$\equiv \sum_{m\geq 1}(-1)^m \frac{\tilde{\chi}_{m+1}^{\overrightarrow{10}}(\sigma)}{m!}(adX)^m(Y) \quad \mod I_Y.$$

It follows then from Proposition 5.13 that $(a_0, a_1, a_2) = (0, 0, \frac{1}{24}(\chi(\sigma)^2 - 1))$ and hence, the second term on the left-hand side of (6.11) turns out to be

$$\ell i_2(1-z, \delta f_2(\gamma), \vec{x})(\sigma) - \ell i_2(\overrightarrow{10}, \delta, \vec{x})(\sigma)$$
$$= -\left(\tilde{\chi}_2^{1-z}(\sigma) + \frac{1}{2}\rho_z(\sigma)\rho_{1-z}(\sigma)\right) + \frac{1}{24}(\chi(\sigma)^2 - 1).$$

To estimate $E(\sigma, \gamma)$, the right-hand side of (6.11), we shall make use of Corollary 5.8. Let us take a decomposition of the $\ell$-adic Lie algebra

$$(6.13) \qquad L(\pi_{V,v}) = L_{<2} \oplus \Gamma^2 L(\pi_{V,v}), \quad L_{<2} := \mathbb{Q}_\ell \log x + \mathbb{Q}_\ell \log y$$

according to the Lie series expansion with respect to $(\log(x), \log(y))$ in the sense of (4.10). Now, for the generator system $\vec{x} = \vec{y}_1 = (x, y)$, it is easy to see $\varphi_{2,\vec{x}}([\log(\mathfrak{f}_\sigma^\gamma)^{-1}]_{<2}) = 0$, as by definition $\varphi_2(f_1)_{\vec{y}_1}$ quarries out the degree 2 part. For $\vec{y}_2 = \delta^{-1}(x, y)\delta = (f_2(y), f_2(x))$, since $f_2(\log \mathfrak{f}_\sigma^\gamma)$ is just obtained from $\log \mathfrak{f}_\sigma^\gamma$ after replacing $\log(x), \log(y)$ by $\log(f_2(x)), \log(f_2(y))$ respectively, we see also $\varphi_{2,\vec{x}}(\delta[\log(\mathfrak{f}_\sigma^\gamma)^{-1}]_{<2}\delta^{-1}) = 0$. Therefore, in this special case, *the error term vanishes for all* $\sigma \in G_K$. Summing up our above arguments, we obtain the functional equation

$$\textbf{(6.14)} \qquad \tilde{\chi}_2^z(\sigma) + \tilde{\chi}_2^{1-z}(\sigma) + \rho_z(\sigma)\rho_{1-z}(\sigma) = \frac{1}{24}(\chi(\sigma)^2 - 1) \qquad (\sigma \in G_K).$$

**Question** The above $\ell$-adic functional equation (6.14) suggests a possibility of reducing Galois transformations of

$$\prod_{a=0}^{\ell^n-1}(1 - \zeta_{\ell^n}^a z^{1/\ell^n})^{\frac{a}{\ell^n}}(1 - \zeta_{\ell^n}^a(1-z)^{1/\ell^n})^{\frac{a}{\ell^n}}$$

to simpler invariants $\rho_z$, $\rho_{1-z}$ and $\chi$, in a somewhat purely arithmetic way as in the second proof of Proposition 5.13. It seems to the authors a non-trivial question.

**6.2 $\mathbf{Li_2(z) + Li_2(z/(z-1))}$** We apply a similar argument to the above subsection to $f_1(z) = z$ and $f_2(z) = \frac{z}{z-1}$. In this case, $f_2(\vec{v}) = \overrightarrow{0\infty}$, so we substitute the half anticlockwise rotation from $\overrightarrow{01}$ to $\overrightarrow{0\infty}$ for $\delta_2 \colon \overrightarrow{01} \rightsquigarrow f_2(\vec{v})$, and set $\delta := \delta_2$ which is $[0_1^\infty]$ in the notation of §5.4. Then, $(f_2(x), f_2(y)) = \delta^{-1}(x, y^{-1}x^{-1})\delta$. For (i)$_{\mathbb{C}}$, observing that $\mathrm{gr}^2_\gamma(\iota_\delta \circ f_2)$ sends $[\bar{x}, \bar{y}]$ to $[\bar{x}, -\bar{x} - \bar{y}]$, we find (6.1) also holds in this case. The condition (ii)$_{\mathbb{C}}$ can be checked by

$$(z) \wedge (z-1) + \Big(\frac{z}{z-1}\Big) \wedge \Big(\frac{z}{z-1} - 1\Big) = 0. \tag{6.15}$$

The consequent functional equation (iii)$_{\mathbb{C}}$ reads

$$\mathcal{L}_{\mathbb{C}}^{\varphi_2}(z, \overrightarrow{01}; \gamma) + \mathcal{L}_{\mathbb{C}}^{\varphi_2}\Big(\frac{z}{z-1}, \overrightarrow{10}; f_2(\gamma)\Big) = 0. \tag{6.16}$$

for any path $\gamma \colon \vec{v} \rightsquigarrow z$. Let us apply Proposition 5.11(i). In the same way as in the previous example, $\mathcal{L}_{\mathbb{C}}^{\varphi_2}(z, \overrightarrow{01}; \gamma) = \mathrm{li}_2(z, \gamma)$. For the second term, we calculate $\mathsf{P}_2(\{a_i\}_{i=0}^2, \{b_i\}_{i=0}^2) = a_2 + b_2 + \frac{1}{2}(a_0 b_1 - b_0 a_1)$, where

$$\{a_i\}_{i=0}^2 = \left\{\mathrm{li}_0\left(\frac{z}{z-1}, \delta f_2(\gamma)\right), \mathrm{li}_1\left(\frac{z}{z-1}, \delta f_2(\gamma)\right), \mathrm{li}_2\left(\frac{z}{z-1}, \delta f_2(\gamma)\right)\right\}, \tag{6.17}$$

$$\{b_i\}_{i=0}^2 = \{-\mathrm{li}_0(\overrightarrow{0\infty}, \delta), -\mathrm{li}_1(\overrightarrow{0\infty}, \delta), -\mathrm{li}_2(\overrightarrow{0\infty}, \delta)\} = \Big\{\frac{\pi i}{2\pi i}, 0, 0\Big\}, \tag{6.18}$$

to get

$$\begin{aligned}
&\mathcal{L}_{\mathbb{C}}^{\varphi_2}\Big(\frac{z}{z-1}, \overrightarrow{10}; f_2(\gamma)\Big) \\
&= \mathrm{li}_2\left(\frac{z}{z-1}, \delta f_2(\gamma)\right) - \frac{1}{2}(-\mathrm{li}_0(\overrightarrow{0\infty}, \delta)) \cdot \mathrm{li}_1\left(\frac{z}{z-1}, \delta f_2(\gamma)\right) \\
&= \mathrm{li}_2\left(\frac{z}{z-1}, \delta f_2(\gamma)\right) - \frac{1}{4}\mathrm{li}_1\left(\frac{z}{z-1}, \delta f_2(\gamma)\right) \\
&= \frac{1}{4\pi^2}\left(Li_2\Big(\frac{z}{z-1}\Big) + \frac{1}{2}\log\Big(\frac{z}{z-1}\Big)\log\Big(\frac{1}{1-z}\Big)\right) + \frac{1}{4}\left(\frac{1}{2\pi i}\log\Big(\frac{1}{1-z}\Big)\right).
\end{aligned}$$

Putting these into (6.16) combined with our choice of logarithmic branches $\log(\frac{z}{z-1}) = \log z - \log(1-z) + \pi i$, we obtain a classical functional equation from [Le]:

$$Li_2(z) + Li_2\Big(\frac{z}{z-1}\Big) = -\frac{1}{2}\log^2(1-z). \tag{\textbf{6.19}}$$

We next consider the $\ell$-adic analog. The condition (iii)$_\ell$ of Theorem 5.7 reads in this case:

$$\mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_1)_{\vec{x}}}(z, \overrightarrow{01}; \gamma)(\sigma) + \mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_2)_{\vec{x}}}\Big(\frac{z}{z-1}, \overrightarrow{10}; f_2(\gamma)\Big)(\sigma) = E(\sigma, \gamma). \tag{6.20}$$

The first term on the left-hand side is the same as (6.12). For the second term, noting that $\mathfrak{f}_\sigma^\delta = \delta \cdot \sigma(\delta)^{-1} = x^{\frac{1-\chi(\sigma)}{2}}$, we have

$$\ell i_0(\overrightarrow{0\infty}, \delta, \vec{x}) = \begin{cases} \frac{\chi(\sigma)-1}{2}, & k = 0, \\ 0, & k \geq 1. \end{cases} \tag{6.21}$$

Hence, by Proposition 5.11(ii), it follows that

$$\mathcal{L}_{\mathrm{nv}}^{\varphi_2(f_2)_{\vec{x}}}\Big(\frac{z}{z-1}, \overrightarrow{10}; f_2(\gamma)\Big)(\sigma) = \mathsf{P}_2\left(\Big\{\frac{1-\chi(\sigma)}{2}, 0, 0\Big\}, \Big\{\ell i_i\Big(\frac{z}{z-1}, \delta f_2(\gamma), \vec{x}\Big)\Big\}_{i=0}^{2}\right)$$

$$= \ell i_2\Big(\frac{z}{z-1}, \delta f_2(\gamma), \vec{x}\Big) + \frac{1}{2}\left(\frac{1-\chi(\sigma)}{2}\right)\ell i_1\Big(\frac{z}{z-1}\Big)$$

$$= -\left(\tilde{\chi}_2^{\frac{z}{z-1}}(\sigma) + \frac{1}{2}\rho_{\frac{z}{z-1}}(\sigma)\rho_{\frac{1}{1-z}}(\sigma)\right) + \frac{1}{2}\left(\frac{1-\chi(\sigma)}{2}\right)\rho_{\frac{1}{1-z}}(\sigma).$$

To evaluate the error term on the right-hand side of (6.20), we employ the same splitting of $L(\pi_{X,v})$ as (6.13). We calculate:

$$\begin{aligned} E(\sigma,\gamma) &= \varphi_{2,\vec{x}}(\delta\, f_2([\log(\mathfrak{f}_\sigma^\gamma)^{-1}]_{<2})\, \delta^{-1}) \\ &= \varphi_{2,\vec{x}}(\delta\, f_2(\rho_z(\sigma)\log x + \rho_{1-z}(\sigma)\log y)\, \delta^{-1}) \\ &= \varphi_{2,\vec{x}}\Big(\rho_z(\sigma)\log x + \rho_{1-z}(\sigma)\log(y^{-1}x^{-1})\Big) \\ &= \varphi_{2,\vec{x}}\left(\frac{1}{2}\rho_{1-z}(\sigma)[\log x, \log y]\right) = \frac{1}{2}\rho_{1-z}(\sigma). \end{aligned}$$

Taking care of the choice of paths to fix branches of the involved Kummer characters such as $\rho_{\frac{z}{z-1}} = \rho_z - \rho_{1-z} + \frac{\chi-1}{2}$, $\rho_{\frac{1}{1-z}} = -\rho_{1-z}$, we obtain from (6.20) the following functional equation:

$$\tilde{\chi}_2^{z}(\sigma) + \tilde{\chi}_2^{\frac{z}{z-1}}(\sigma) + \frac{1}{2}\rho_{1-z}(\sigma)^2 - \frac{\chi(\sigma)-1}{2}\rho_{1-z}(\sigma) = \frac{1}{2}\rho_{1-z}(\sigma),$$

or equivalently,

$$\tilde{\chi}_2^{z}(\sigma) + \tilde{\chi}_2^{\frac{z}{z-1}}(\sigma) = \frac{\rho_{1-z}(\sigma)}{2}\Big(\chi(\sigma) - \rho_{1-z}(\sigma)\Big) \quad (\sigma \in G_K). \tag{6.22}$$

Note the $\ell$-integrality, i.e., $\in \mathbb{Z}_\ell$ of the right-hand side above for all $\sigma \in G_K$ even when $\ell = 2$, as should be expected from the definition of $\ell$-adic polylogarithmic character appearing on the left-hand side. This shows that the error term $E(\sigma,\gamma)$ is unavoidable. Concerning the functional equation (6.22), one may also ask a question similar to what was raised just after (6.14).

**6.3 Inversion formula** Here, we consider two automorphisms $f_1(z) = z$ and $f_2(z) = z^{-1}$ of $P_0$. Take $\delta_1 : v = \overrightarrow{01} \rightsquigarrow f_1(v)$ to be the trivial path, and $\delta_2 : v \rightsquigarrow f_1(v) = \overrightarrow{\infty 1}$ to be $\langle 0, 1\rangle [1_0^\infty] \langle \infty, 1\rangle$ in the notation of §5.4. Then, $\delta := \delta_2$ is the same as the path $\delta$ illustrated in loc. cit. Let $n \geq 2$. In the tensor space $(\mathrm{Sym}^{n-2}\mathcal{O}_h^\times) \otimes (\wedge^2\mathcal{O}_h^\times)$ of $\mathcal{O}_h^\times = \mathcal{O}_h^\times(V^{an})$, since

$$\Big(\frac{1}{z}\Big)^{\odot n-2} \otimes \Big(\frac{1}{z} \wedge \frac{1-z}{z}\Big) = (-1)^{n-1} z^{\odot n-2} \otimes (z \wedge (1-z)),$$

we find Theorem 5.7 (ii)$_{\mathbb{C}}$ holds for $c_1 = 1$, $c_2 = (-1)^n$. Consequently, for any path $\gamma\colon \overrightarrow{01} \rightsquigarrow z$ on $P_0$, we have the functional equation (iii)$_{\mathbb{C}}$ in the form

$$\mathcal{L}_{\mathbb{C}}^{\varphi_n}(z, \overrightarrow{01}; \gamma) + (-1)^n \mathcal{L}_{\mathbb{C}}^{\varphi_n}\Big(\frac{1}{z}, \overrightarrow{\infty 1}; f_2(\gamma)\Big) = 0 \qquad (n \geq 2). \tag{6.23}$$

The first term, $\mathcal{L}_{\mathbb{C}}^{\varphi_n}(z, \overrightarrow{01}; \gamma) = \mathrm{li}_n(z, \gamma)$, is already calculated in Proposition 5.2. For the second, applying Proposition 5.11(i) with the chain rule, we wish to compute the BCH-sum

$$\mathcal{L}_{\mathbb{C}}^{\varphi_n}\Big(\frac{1}{z}, \overrightarrow{\infty 1}; f_2(\gamma)\Big) = \mathsf{P}_n(\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n), \tag{6.24}$$

where

$$\begin{aligned}
\{a_i\}_{i=0}^n &= \Big\{\mathrm{li}_i\Big(\frac{1}{z}, \delta f_2(\gamma)\Big)\Big\}_{i=0}^n \\
&= \left\{(-1)^{i+1} \sum_{k=0}^{i-1} \frac{B_k}{k!} \left(\frac{-\log z}{2\pi i}\right)^k \frac{Li_{i-k}(z^{-1}, \delta f_2(\gamma))}{(2\pi i)^{i-k}}\right\}_{i=0}^n, \\
\{b_i\}_{i=0}^n &= \Big\{-\mathrm{li}_i(\overrightarrow{\infty 1}, \delta)\Big\}_{i=0}^n = \left\{0, -B_1, -\frac{B_2}{2!}, \dots, -\frac{B_n}{n!}\right\} \quad \text{by (5.11).}
\end{aligned}$$

It now turns out that we should work inductively on $n$. Let us set

$$\begin{aligned}
&\mathsf{L}_0 := \frac{-\log z}{2\pi i}, \quad \mathsf{L}_1 := \frac{Li_1(z)}{2\pi i} = \frac{-\log(1-z)}{2\pi i}, \quad \mathsf{L}_k := \frac{Li_k(z)}{(2\pi i)^k} \ (k \geq 2); \\
&\bar{\mathsf{L}}_0 := \frac{\log z}{2\pi i}, \quad \bar{\mathsf{L}}_1 := \frac{Li_1(z^{-1})}{2\pi i} = \frac{\log z - \log(z-1)}{2\pi i}, \quad \bar{\mathsf{L}}_k := \frac{Li_k(z^{-1})}{(2\pi i)^k} \ (k \geq 2);
\end{aligned}$$

so that

$$\begin{aligned}
&a_0 = \bar{\mathsf{L}}_0 = -\mathsf{L}_0, \quad a_1 = \bar{\mathsf{L}}_1 = -\mathsf{L}_0 + \mathsf{L}_1 - \frac{1}{2}, \quad b_0 = 0; \\
&a_k = (-1)^{k+1} \sum_{i=0}^{k-1} \frac{B_i}{i!} \mathsf{L}_0^i \bar{\mathsf{L}}_{k-i} \ (k \geq 2), \quad b_k = -\frac{B_k}{k!} \ (k \geq 1).
\end{aligned}$$

Consider then the generating functions:

$$\mathsf{L}(t) := \sum_{i=0}^{\infty} \mathsf{L}_{i+1} t^i, \quad \bar{\mathsf{L}}(t) := \sum_{i=0}^{\infty} \bar{\mathsf{L}}_{i+1} t^i, \quad \mathcal{B}(t) := \frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \frac{B_i}{i!} t^i, \tag{6.25}$$

and define the quantities $\mathsf{D}_i$, $\mathsf{P}_i$ ($i \geq 1$) by

$$\mathsf{D}(t) := \sum_{i=1}^{\infty} \mathsf{D}_i t^i = t\mathcal{B}(\mathsf{L}_0 t)\mathsf{L}(-t),$$

$$\mathsf{P}(t) := \sum_{i=0}^{\infty} \mathsf{P}_i t^i = t\mathcal{B}(-\mathsf{L}_0 t)\bar{\mathsf{L}}(-t) + \mathsf{L}_0 t\mathcal{B}(t) - \mathcal{B}(t)\mathcal{B}(-\mathsf{L}_0 t) + \mathcal{B}(-\mathsf{L}_0 t).$$

Then, from (5.8) and (5.10) follows that this $\mathsf{P}_n$ coincides with $\mathsf{P}_n(\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n)$ for $n \geq 2$, and it turns out that the functional equation (6.23) is reduced to the equation $\mathsf{P}(-t) + \mathsf{D}(t) = 0$. After computations, we obtain

$$\bar{\mathsf{L}}(t) - \mathsf{L}(-t) = \frac{e^{\mathsf{L}_0 t}}{e^{-t} - 1} + t^{-1} = \sum_{n=1}^{\infty} \frac{B_n(-\mathsf{L}_0)}{n!}(-t)^{n-1}.$$

Comparing the coefficients, we get $\bar{\mathsf{L}}_n + (-1)^n \mathsf{L}_n = (-1)^{n-1} B_n(-\mathsf{L}_0)/n!$, i.e., what is called the inversion formula of polylogarithms:

**(6.26)** $$Li_n(z) + (-1)^n Li_n(\frac{1}{z}) = -\frac{(2\pi i)^n}{n!} B_n(\frac{\log z}{2\pi i}) \quad (n \geq 2).$$

Next, we consider the $\ell$-adic version (iii)$_\ell$:

(6.27) $$\mathcal{L}_{\mathrm{nv}}^{\varphi_n(f_1)_{\vec{x}}}(z, \overrightarrow{01}; \gamma)(\sigma) + (-1)^n \mathcal{L}_{\mathrm{nv}}^{\varphi_n(f_2)_{\vec{x}}}(\frac{1}{z}, \overrightarrow{\infty 1}; f_2(\gamma))(\sigma) = E(\sigma, \gamma)$$

for $\sigma \in G_K$, $n \geq 2$ ($z \in K \subset \mathbb{C}$). In the below, we shall occasionally omit $\sigma$ for simplicity. The first term is $\ell i_m(z, \gamma, \vec{x})$ that is expressed by $\ell$-adic polylogarithmic characters as in (5.3). For the second, applying Proposition 5.11(ii), one can write

(6.28) $$\mathcal{L}_{\mathrm{nv}}^{\varphi_n(f_2)_{\vec{x}}}\Big(\frac{1}{z}, \overrightarrow{\infty 1}; f_2(\gamma)\Big) = \mathsf{P}_n(\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n),$$

with

$$\{a_i\}_{i=0}^n = \{-\ell i_i(\overrightarrow{\infty 1}, \delta)\}_{i=0}^n = \left\{0, B_1(\chi - 1), \frac{B_2}{2!}(\chi^2 - 1), \ldots, \frac{B_n}{n!}(\chi^n - 1)\right\},$$

$$\{b_i\}_{i=0}^n = \{\ell i_i\Big(\frac{1}{z}, \delta f_2(\gamma)\Big)\}_{i=0}^n = \left\{(-1)^{i+1} \sum_{k=0}^{i-1} \frac{B_k}{k!}(\rho_z)^k \frac{\tilde{\chi}_{i-k}^{1/z}}{(i-k-1)!}\right\}_{i=0}^n.$$

(In the expression above for $a_i$, we have used the chain rule to find that $\mathfrak{f}_\sigma^\delta = \mathfrak{f}_\sigma(y, x^{-1}y^{-1})y^{\frac{1-\chi(\sigma)}{2}}\mathfrak{f}_\sigma(x, y)$. This differs from $G_\sigma^{-1}$ of the first proof of

Proposition 5.13 only in the sign of $\frac{1-\chi(\sigma)}{2}$.) Let us set

$$\mathsf{L}_0 := \rho_z, \qquad \mathsf{L}_1 := \tilde{\chi}_1^z = \rho_{1-z}, \qquad \mathsf{L}_k := \frac{\tilde{\chi}_k^z}{(k-1)!} \ (k \geq 2);$$

$$\bar{\mathsf{L}}_0 := -\rho_z, \quad \bar{\mathsf{L}}_1 := \tilde{\chi}_1^{1/z} = -\rho_z + \rho_{1-z} + \frac{\chi - 1}{2}, \quad \bar{\mathsf{L}}_k := \frac{\tilde{\chi}_k^{1/z}}{(k-1)!} \ (k \geq 2);$$

so that

$$a_k = \frac{B_k}{k!}(\chi^k - 1) \quad (k \geq 0);$$

$$b_0 = \bar{\mathsf{L}}_0 = -\mathsf{L}_0, \quad b_1 = \bar{\mathsf{L}}_1 = -\mathsf{L}_0 + \mathsf{L}_1 + \frac{\chi - 1}{2};$$

$$b_k = (-1)^{k+1} \sum_{i=0}^{k-1} \frac{B_i}{i!} \mathsf{L}_0^i \bar{\mathsf{L}}_{k-i}.$$

Keeping $\mathsf{L}(t)$, $\bar{\mathsf{L}}(t)$ as in (6.25), introduce the quantities $\mathsf{D}_i$, $\mathsf{P}_i$ $(i \geq 1)$ by

$$\mathsf{D}(t) := \sum_{i=1}^{\infty} \mathsf{D}_i t^i = t\mathcal{B}(\mathsf{L}_0 t)\mathsf{L}(-t),$$

$$\mathsf{P}(t) := \sum_{i=0}^{\infty} \mathsf{P}_i t^i = t\mathcal{B}(-\mathsf{L}_0 t)\bar{\mathsf{L}}(-t) + (\mathcal{B}(\chi t) - \mathcal{B}(t))\mathcal{B}(-\mathsf{L}_0 t).$$

Then, $\mathsf{D}_n = \ell i_n(z, \gamma, \vec{x})$ $(n \geq 1)$, and $\mathsf{P}_n = \mathsf{P}_n(\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n)$ $(n \geq 1)$ as seen from (5.8) and (5.10). Thus, the functional equation (6.27) turns out to be in the form

$$\mathsf{D}_n + (-1)^n \mathsf{P}_n = (-1)^n \mathsf{E}_n \quad (n \geq 2), \tag{6.29}$$

where the error term on the right-hand side is evaluated by Corollary 5.8 as follows:

$$\mathsf{E}_n := \varphi_{n,\vec{x}}(\delta f_2([\log(\mathfrak{f}_\sigma^\gamma)^{-1}]_{<n})\, \delta^{-1}).$$

Observing that $\delta f_2(x)\delta^{-1} = yzy^{-1} = x^{-1}y^{-1}$, $\delta f_2(x)\delta^{-1} = y$, we see from (5.1) that

$$\mathsf{E}_n = \varphi_{n,\vec{x}}(\rho_z \log(x^{-1}y^{-1})) = -\frac{B_{n-1}}{(n-1)!}\rho_z \qquad (n \geq 2). \tag{6.30}$$

If we extend the above expression of $\mathsf{E}_n$ also for $n = 1$, then we still have $\mathsf{D}_1 - \mathsf{P}_1 + \mathsf{E}_1 = 0$. Summing up our discussions, we obtain from (6.29) the functional equation of generating functions

$$\mathsf{D}(t) + \mathsf{P}(-t) = \sum_{n=1}^{\infty} (-1)^n \mathsf{E}_n t^n = \rho_z\, t\, \mathcal{B}(-t),$$

which yields

$$\mathsf{L}(-t) - \bar{\mathsf{L}}(t) = -\left(\frac{-\chi\, t}{e^{-\chi t} - 1}\right) - \left(\frac{e^{\mathsf{L}_0 t}}{e^{-t} - 1}\right).$$

Comparing the coefficients of the above, we get

$$(-1)^{n-1}\mathsf{L}_n - \bar{\mathsf{L}}_n = (-1)^n \left\{\frac{B_n(-\mathsf{L}_0)}{n!} - \frac{\chi^n B_n}{n!}\right\}$$

from which we finally conclude the $\ell$-adic inversion formula:

**(6.31)** $$\tilde{\chi}_n^z(\sigma) + (-1)^n \tilde{\chi}_n^{1/z}(\sigma) = -\frac{1}{n}\{B_n(-\rho_z(\sigma)) - B_n \chi(\sigma)^n\}$$

$$(\sigma \in G_K, n \geq 1).$$

**Remark** In [W9], theorem 2.6, an inversion formula without lower degree terms was obtained. Note however that, in loc. cit., one must have taken certain "suitable" $\ell$-adic paths to determine those $\ell i_m(z)$ and $\ell i_m(\frac{1}{z})$. By comparison, in the functional equations (6.27) and (6.31), the $\ell$-adic polylogarithms or $\ell$-adic polylogarithmic characters on the left-hand sides are taken along the explicit paths composed of $\gamma$, $\delta$ and $f_2(\gamma)$, for which non-trivial lower degree terms must appear as on the right-hand sides.

**Remark** Applying $z = \overrightarrow{10}$ to the above inversion formula (6.31) reproves Proposition 5.13, where our above argument specialized to this case is essentially of the same (geometric) nature as the first proof presented in §5.4. Note also that, putting $n = 2$ and $z = -1$ in (6.31) confirms the formula given in Remark 5.14, where we also apply the formula $\tilde{\chi}_2^{z=-1}(\sigma; x^{-1}p) = \tilde{\chi}_2^{z=-1}(\sigma; p) + \chi(\sigma)\cdot\rho_2(\sigma)$ $(\sigma \in G_{\mathbb{Q}})$ for $p = [0_1^\infty]\langle 0, -1\rangle$ (cf. [NW2]).

**6.4 Abel's equation** In this subsection, we take for $V$ the moduli space $M_{0,5}$ of the isomorphism classes of the projective line with ordered 5 marked points $(\mathbf{P}^1; a_1, \ldots, a_5)$. We consider $V = M_{0,5}$ to be a variety defined over a subfield $K \subset \mathbb{C}$ equipped with a standard tangential base point $\vec{v}$ determined by the $K(t)$-rational point $(\mathbf{P}^1; 0, t^2, t, 1, \infty)$. The topological fundamental group $\pi_1(V(\mathbb{C}), \vec{v})$ is known to be a quotient of pure sphere braid group with 5 strings. We fix a standard generator system $\{x_{ij} \mid i, j = 1, ..., 5\}$ of it as in [Ih1] or [N1] (3.1.4). Regard now $P_0 = \mathbf{P}^1 - \{0, 1, \infty\}$ as the moduli space of the $(\mathbf{P}^1; b_1, \ldots, b_4)$, i.e., of the isomorphism classes of the projective line with ordered 4 marked points, and consider, for each $i = 1, \ldots, 5$, the morphism $f_i \colon V \to P_0$ obtained by forgetting the marked point $a_i$ and leaving the other $a_j$ $(j \neq i)$ as $b_1, \ldots, b_4$ so that the order is preserved. It is easy to check that $f_i(\vec{v}) = \overrightarrow{01}$, hence we can take all connecting paths $\delta_i\colon \overrightarrow{01} \rightsquigarrow f_i(v)$ to be trivial. We refer the reader to [N1] and [N2] for basic properties of these forgetful morphisms with respect to the generator systems of the fundamental groups. For example, one can compute $f_2(x_{24}) = 1$, $f_2(x_{14}) = (xy)^{-1}$, $f_3(x_{15}) = y$ and so on; $f_k(x_{ij})$ is equal to one of the $x, y, (xy)^{-1}$ depending on the choice of $(i, j, k)$ with $1 \leq i, j, k \leq 5$. The graded quotient $\mathrm{gr}^2(\pi_1(V(\mathbb{C})), \vec{v})$ is a 4-dimensional vector space with a basis $[\bar{x}_{12}, \bar{x}_{23}]$, $[\bar{x}_{15}, \bar{x}_{25}]$, $[\bar{x}_{15}, \bar{x}_{35}]$, $[\bar{x}_{25}, \bar{x}_{35}]$. We summarize their images by $\mathrm{gr}^2_\Gamma(f_{i*})$ $(i = 1, \ldots, 5)$ in the following table:

| $\sharp$ | $\mathrm{gr}^2_\Gamma f_{1*}(\sharp)$ | $\mathrm{gr}^2_\Gamma f_{2*}(\sharp)$ | $\mathrm{gr}^2_\Gamma f_{3*}(\sharp)$ | $\mathrm{gr}^2_\Gamma f_{4*}(\sharp)$ | $\mathrm{gr}^2_\Gamma f_{5*}(\sharp)$ |
|---|---|---|---|---|---|
| $[\bar{x}_{12}, \bar{x}_{23}]$ | $[\bar{x}, \bar{y}]$ | $-[\bar{x}, \bar{y}]$ | 0 | 0 | 0 |
| $[\bar{x}_{15}, \bar{x}_{25}]$ | 0 | $-[\bar{x}, \bar{y}]$ | $[\bar{x}, \bar{y}]$ | 0 | 0 |
| $[\bar{x}_{15}, \bar{x}_{35}]$ | 0 | $[\bar{x}, \bar{y}]$ | 0 | $-[\bar{x}, \bar{y}]$ | 0 |
| $[\bar{x}_{25}, \bar{x}_{35}]$ | 0 | $-[\bar{x}, \bar{y}]$ | 0 | 0 | $[\bar{x}, \bar{y}]$ |

Consequently, we see that $\sum_{i=1}^{5}(-1)^{i-1}\mathrm{gr}^2_\Gamma(f_{i*}) = 0$ as a homomorphism of $\mathrm{gr}^2_\Gamma\pi_1(V(\mathbb{C}), v)$ to $\mathrm{gr}^2_\Gamma\pi_1(P_0(\mathbb{C}), \overrightarrow{01})) = \mathbb{C}\cdot[\bar{x}, \bar{y}]$.

Now, let us apply Theorem 5.7, and compute the functional equations $(\mathrm{iii})_\mathbb{C}$ and $(\mathrm{iii})_\ell$. Pick a point $z \in V(K)$ representing $(\mathbf{P}^1; 0, st, s, 1, \infty)$ with $s = \frac{\xi}{1-\eta}$, $t = \frac{\eta}{1-\xi}$. $(\xi, \eta \in K - \{0, 1\})$. Then, the images of $z$ by the above morphisms $f_1, \ldots, f_5$ are calculated as

$$\begin{aligned}
f_1(z) &= \left[(\mathbf{P}^1; 0, \frac{s(1-t)}{1-st}, 1, \infty)\right] = \xi;\\
f_2(z) &= \left[(\mathbf{P}^1; 0, s, 1, \infty)\right] = \frac{\xi}{1-\eta};\\
f_3(z) &= \left[(\mathbf{P}^1; 0, st, 1, \infty)\right] = \frac{\xi\eta}{(1-\xi)(1-\eta)};\\
f_4(z) &= \left[(\mathbf{P}^1; 0, t, 1, \infty)\right] = \frac{\eta}{1-\xi};\\
f_5(z) &= \left[(\mathbf{P}^1; 0, \frac{t(1-s)}{1-st}, 1, \infty)\right] = \eta.
\end{aligned}$$

Therefore, $(\mathrm{iii})_\mathbb{C}$ leads to

$$\begin{aligned}
\mathrm{li}_2(\xi, f_1(\gamma)) - \mathrm{li}_2\left(\frac{\xi}{1-\eta}, f_2(\gamma)\right) + \mathrm{li}_2\left(\frac{\xi\eta}{(1-\xi)(1-\eta)}, f_3(\gamma)\right)&\\
- \mathrm{li}_2\left(\frac{\eta}{1-\xi}, f_4(\gamma)\right) + \mathrm{li}_2(\eta, f_5(\gamma)) = 0.&
\end{aligned}$$

Applying (6.7) to each term above, we obtain what is called Abel's equation:

**(6.32)**
$$Li_2\Big(\frac{\xi\eta}{(1-\xi)(1-\eta)}\Big) = Li_2\Big(\frac{\xi}{1-\eta}\Big) + Li_2\Big(\frac{\eta}{1-\xi}\Big) - Li_2(\xi) - Li_2(\eta) - \log(1-\xi)\log(1-\eta).$$

Next, we consider the $\ell$-adic version. We shall state it as a theorem.

**Theorem 6.1** (**Abel's equation for $\ell$-adic polylogarithms**) *With notation as above, we have*

$$\ell i_2(\xi, f_1(\gamma), \vec{x})(\sigma) - \ell i_2\left(\frac{\xi}{1-\eta}, f_2(\gamma), \vec{x}\right)(\sigma) + \ell i_2\left(\frac{\xi\eta}{(1-\xi)(1-\eta)}, f_3(\gamma), \vec{x}\right)(\sigma) - \ell i_2\left(\frac{\eta}{1-\xi}, f_4(\gamma), \vec{x}\right)(\sigma) + \ell i_2(\eta, f_5(\gamma), \vec{x})(\sigma) = 0 \qquad (\sigma \in G_K).$$

**Remark 6.2** This functional equation seems to be nicer than the one proved in [W5], theorem 11.1.14, for $\sigma \in G_{K(\mu_{\ell^\infty})}$, because in the present approach we have no lower degree terms even for $\sigma \in G_K$.

*Proof* Condition $(\mathrm{iii})_\ell$ reads:

(6.33)
$$\ell i_2(\xi, f_1(\gamma), \vec{x})(\sigma) - \ell i_2\left(\frac{\xi}{1-\eta}, f_2(\gamma), \vec{x}\right)(\sigma) + \ell i_2\left(\frac{\xi\eta}{(1-\xi)(1-\eta)}, f_3(\gamma), \vec{x}\right)(\sigma) - \ell i_2\left(\frac{\eta}{1-\xi}, f_4(\gamma), \vec{x}\right)(\sigma) + \ell i_2(\eta, f_5(\gamma), \vec{x})(\sigma) = E(\sigma, \gamma)$$

for all $\sigma \in G_K$. To estimate the error term $E(\sigma,\gamma)$ of (6.33), set $\mathcal{S} = \{(1,2), (2,3), (1,5), (2,5), (3,5)\}$ so that the $\bar{x}_{ij}$ ($(i,j) \in \mathcal{S}$) form a basis of $\pi_1(V(\mathbb{C}))^{\mathrm{ab}}$ and fix a splitting of the $\ell$-adic Lie algebra $L(\pi_{V,v}) = L_{<2} \oplus \Gamma^2 L(\pi_{V,v})$ such that $L_{<2} = \sum_{(i,j)\in\mathcal{S}} \mathbb{Q}_\ell X_{ij}$ where $X_{ij} = \log x_{ij}$. Write

$$[\log(\mathfrak{f}^z_\sigma)^{-1}]_{<2} = \sum_{(i,j)\in\mathcal{S}} C_{ij}(\sigma) X_{ij}.$$

Then, by Corollary 5.8, we have

(6.34)
$$E(\sigma,\gamma) = \sum_{k=1}^{5} (-1)^{i-1} \varphi_{2,\vec{x}}(f_k([\log(\mathfrak{f}^z_\sigma)^{-1}]_{<2})).$$

Noting that $f_k(X_{ij})$ ($k = 1, \dots, 5$, $(i, j) \in \mathcal{S}$) are summarized as

| $f_k(X_{ij})$ | $X_{12}$ | $X_{23}$ | $X_{15}$ | $X_{25}$ | $X_{35}$ |
|---|---|---|---|---|---|
| $f_1$ | 0 | $X$ | 0 | $Y$ | $\log(y^{-1}x^{-1})$ |
| $f_2$ | 0 | 0 | $Y$ | 0 | $\log(y^{-1}x^{-1})$ |
| $f_3$ | $X$ | 0 | $Y$ | $\log(y^{-1}x^{-1})$ | 0 |
| $f_4$ | $X$ | $Y$ | $Y$ | $\log(y^{-1}x^{-1})$ | $X$ |
| $f_5$ | $X$ | $Y$ | 0 | 0 | 0 |

we find that the right-hand side of (6.34) applied to $\varphi_{2,\vec{x}} \circ f_k$ raises non-vanishing terms from $C_{ij}(\sigma)$ only when the degree 2 term of $f_k(X_{ij})$ survives, i.e., $f_k(X_{ij}) = \log(y^{-1}x^{-1}) = -X-Y-\frac{1}{2}[X, Y]+\cdots$, in which case $-\frac{1}{2}C_{ij}(\sigma)$ occurs. Summing up, we obtain

$$E(\sigma, \gamma) = -\frac{1}{2}C_{35}(\sigma) + \frac{1}{2}C_{35}(\sigma) - \frac{1}{2}C_{25}(\sigma) + \frac{1}{2}C_{25}(\sigma) = 0,$$

namely, the error term vanishes on the right-hand side of (6.33). □

**Remark 6.3** In the above discussion, it is, in fact, not difficult to determine the individual coefficient characters $C_{ij} \colon G_K \to \mathbb{Q}_\ell$ as Kummer cocycles along roots of certain values (and paths from $\overrightarrow{01}$) depending on $(\xi, \eta)$. This can be done only by observing Galois actions on the image of those paths in the abelianized fundamental groups after projections $f_k \colon V \to P_0$. We leave such enjoyable calculations to interested readers.

Finally, applying (6.12) allows us to interpret the left-hand side of the above theorem in terms of $\tilde{\chi}_2^z$ and $\rho_z\rho_{1-z}$. By simple calculations, we deduce the following Abel's equation for ℓ-adic polylogarithmic characters of degree 2:

$$\textbf{(6.35)} \quad -\tilde{\chi}_2^{\xi}(\sigma) + \tilde{\chi}_2^{\frac{\xi}{1-\eta}}(\sigma) - \tilde{\chi}_2^{\frac{\xi\eta}{(1-\xi)(1-\eta)}}(\sigma) + \tilde{\chi}_2^{\frac{\eta}{1-\xi}}(\sigma) - \tilde{\chi}_2^{\eta}(\sigma) = \rho_{1-\xi}(\sigma)\rho_{1-\eta}(\sigma),$$

which holds for all $\sigma \in G_K$.

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