A POLYLOGARITHMIC MEASURE ASSOCIATED WITH A PATH ON $\mathbb{P}^1 \setminus \{0,1,\infty\}$ AND A P-ADIC HURWITZ ZETA FUNCTION

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Contents

Abstract. With every path on $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}$ there is associated a measure on $\mathbb{Z}_p$. The group $\mathbb{Z}_p^\times$ acts on measures. We consider two measures. One measure is associated to a path from $\overrightarrow{01}$ to a root of unity ξ of order prime to p. Another measure is associated to a path from $\overrightarrow{01}$ to ξ^{-1} and next it is acted by $-1 \in \mathbb{Z}_p^\times$. We show that the sum of these measures can be defined in a very elementary way. Integrating against this sum of measures we get p-adic Hurwitz zeta functions constructed previously by Shiratani.

0. Introduction

Let K be a number field, let $z \in \mathbb{P}^1(K) \setminus \{0,1,\infty\}$ and let γ be a path on $\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}$ from $\overrightarrow{01}$ to z, i.e. an isomorphism of the corresponding fiber functors. Let p be a fixed prime number. The Galois group G_K acts on

$$\pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, \overrightarrow{01})$$

– the pro-p étale fundamental group. Let $\mathbb{Q}_p\{\{X,Y\}\}$ be the $\mathbb{Q}_p$-algebra of non-commutative formal power series in two non-commuting variables X and Y. Let

$$E : \pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, \overrightarrow{01}) \to \mathbb{Q}_p\{\{X,Y\}\}$$

be the continuous multiplicative embedding given by $E(x) = \exp X$ and $E(y) = \exp Y$, where x and y are standard generators of $\pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, \overrightarrow{01})$. For any $\sigma \in G_K$ we define

$$\mathfrak{f}_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, \overrightarrow{01})$$

and

$$\Lambda_\gamma(\sigma) := E(\mathfrak{f}_\gamma(\sigma)) \in \mathbb{Q}_p\{\{X,Y\}\} \ .$$

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In the special case of the path π from $\overrightarrow{01}$ to $\overrightarrow{10}$, the element $\mathfrak{f}_\pi(\sigma)$ was studied by Ihara and his students (see [4] and more other papers), Deligne (see [1]), Grothendieck. The coefficients of the power series $\Lambda_\pi(\sigma)$ are analogues of the multizeta numbers studied already by Euler. For an arbitrary path γ the coefficients of the power series $\Lambda_\gamma(\sigma)$ are analogues of values of iterated integrals evaluated at z.

Observe that

$$\Lambda_\gamma(\sigma) \equiv 1 + l_\gamma(z)(\sigma)X \ \text{ modulo } \ I^2 + (Y)$$

for a certain $l_\gamma(z)(\sigma) \in \mathbb{Z}_p$, where I is the augmentation ideal of $\mathbb{Q}_p\{\{X, Y\}\}$ and (Y) is the principal ideal generated by Y. Let us set

$$\Delta_\gamma(\sigma) := \exp(-l_\gamma(z)(\sigma)X) \cdot \Lambda_\gamma(\sigma) \ .$$

One possible way to calculate (some) coefficients of the power series $\Lambda_\pi(\sigma)$ and some other power series $\Lambda_\gamma(\sigma)$ is to use symmetries of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}$, i.e. the so called Drinfeld-Ihara relations (see [3] and [5]). For example in [11], we have calculated even polylogarithmic coefficients of the power series $\Lambda_\pi(\sigma)$ using the symmetries of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}$.

In [8] the authors have constructed a measure on $\mathbb{Z}_p$ for any path γ and expressed the k-th polylogarithmic coefficient of the power series $\log\Delta_\gamma(\sigma)$ as integrals of the polynomial x^{k-1} against this measure recovering the old result of O. Gabber (see [2]). Let us denote this measure by $K(z)_\gamma$.

Now we shall describe the main result of this note. Let m be a positive integer not divisible by p. Let us set

$$\xi_m = \exp(\frac{2\pi\sqrt{-1}}{m}) \ .$$

Let $0 < i < m$. Further we chose paths β_i (resp. β_{m-i}) on $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ from $\overrightarrow{01}$ to ξ_m^i (resp. ξ_m^{m-i}) such that $l_{\beta_i}(\xi_m^i) = 0$ and $l_{\beta_{m-i}}(\xi_m^{m-i}) = 0$.

In [12] using the symmetry $\mathfrak{z} \mapsto 1/\mathfrak{z}$ of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ we have shown that the polylogarithmic coefficient in degree k of the formal power series

$$\log\Lambda_{\beta_{m-i}}(\sigma) + (-1)^k \log\Lambda_{\beta_i}(\sigma) \tag{1}$$

is equal $\frac{B_k(\frac{i}{m})}{k!}(1 - \chi^k(\sigma))$, where $B_k(X)$ is the k-th Bernoulli polynomial and $\chi : G_{\mathbb{Q}(\mu_m)} \to \mathbb{Z}_p^\times$ is the cyclotomic character (see [12, Theorem 10.2.]). In this paper we shall calculate the same polylogarithmic coefficients using the measure

$$K(\xi_m^{m-i})_{\beta_{m-i}} + \iota(K(\xi_m^i)_{\beta_i}) \, ,$$

where ι is the complex conjugation acting on measures. To calculate these measures we use the symmetry $\mathfrak{z} \to 1/\mathfrak{z}$ of the tower of coverings

$$\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0, \infty\} \cup \mu_{p^n}) \to \mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}, \ \mathfrak{z} \mapsto \mathfrak{z}^{p^n}$$

of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}$. However in contrast with the calculations in [12] we need to work only with terms in degree 1. We show that the measure $K(\xi_m^{m-i})_{\beta_{m-i}} + \iota(K(\xi_m^i)_{\beta_i})$ is the sum of the Bernoulli measure $E_{1,\chi}$ (see [6, the formula E.1 on page 38]) and the measure we denote by $\mu_\chi(\frac{i}{m})$. The definition of the measure $\mu_\chi(\frac{i}{m})$ is very elementary and perhaps it is well known. From this it follows immediately the formula for the k-th polylogarithmic coefficient of the power series (1). The measure we got, allows to get the p-adic Hurwitz zeta functions as Mellin transform in the

same way as the p-adic L-functions are the Mellin transforms of the measure $\psi E_{1,c}$, where ψ is a character on $\mathbb{Z}_p^\times$ (see [6, Chapter 4]).

1. An example of a measure on $\mathbb{Z}_p$

This section can be seen as an attempt to construct a measure on $\mathbb{Z}_p$ which to a subset $a+p^n\mathbb{Z}_p$ associates $1/p^n$. We found the measure in question studying Galois actions on torsors of paths (see section 3). The measure is elementary and we think that it should be known.

If $a \in \mathbb{Z}_p$ and $a = \sum_{i=0}^{\infty} \alpha_i p^i$ with $0 \leq \alpha_i \leq p-1$ then we set

$$v_n(a) := \sum_{i=0}^{n} \alpha_i p^i \ \text{ and } \ t_{n+1}(a) := \frac{a - v_n(a)}{p^{n+1}}\,.$$

Let us fix a positive integer $m > 1$. For $k \in \mathbb{Q}^\times$, $k = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $(b, m) = 1$ we define

$$[k]_m \ \in \mathbb{N}$$

by the following two conditions

$$0 \leq [k]_m < m \ \text{ and } \ b[k]_m \equiv a \ \text{ modulo } m\,.$$

Let us assume that p does not divide m. Let i be such that $0 < i < m$. Observe that

$$[p^{-r}[ip^{-n}]_m]_m = [ip^{-(n+r)}]_m\,. \tag{2}$$

We define a sequence of integers

$$(k_r(i))_{r\in\mathbb{N}}$$

by the equalities

$$p[ip^{-r}]_m = [ip^{-(r-1)}]_m + k_{r-1}(i)m\,. \tag{3}$$

Observe that

$$0 < \frac{[ip^{-(r-1)}]_m}{m} < 1 \ \text{ and } \ 0 < \frac{p[ip^{-r}]_m}{m} < p\,.$$

Hence it follows that

$$0 \leq k_r(i) \leq p-1$$

for all $r \geq 0$. Applying successively the formula (3) we get

$$p^n[ip^{-n}]_m = i + \big(\sum_{\alpha=0}^{n-1} k_\alpha(i)p^\alpha\big)m\,. \tag{4}$$

It follows from (4) that

$$\frac{-i}{m} = \sum_{\alpha=0}^{\infty} k_\alpha(i)p^\alpha$$

and

$$\frac{i}{m} = 1 + \sum_{\alpha=0}^{\infty}(p-1-k_\alpha(i))p^\alpha\,. \tag{5}$$

Another consequence of (4) is the equality

$$t_n(-\frac{i}{m}) = \frac{-[ip^{-n}]_m}{m}\,.$$

For any positive integer a such that $0 \leq a < p^n$ we set

$$\delta_n(a) := \begin{cases} -1 & \text{if } a \geq 1 + \sum_{\alpha=0}^{n-1}(p-1-k_\alpha(i))p^\alpha, \\ 0 & \text{if } a < 1 + \sum_{\alpha=0}^{n-1}(p-1-k_\alpha(i))p^\alpha \,. \end{cases}$$

Definition-Proposition 1.1. The function from the open-closed subsets of $\mathbb{Z}_p$ to $\mathbb{Z}_p$ defined by the formula

$$\mu(\frac{i}{m})(a + p^n\mathbb{Z}_p) := \frac{[ip^{-n}]_m}{m} + \delta_n(a)$$

for $0 \leq a < p^n$ is a measure.

Proof. Let $0 \leq a < p^n$. We have

$$\sum_{b=0}^{p-1} \mu(\frac{i}{m})(a + bp^n + p^{n+1}\mathbb{Z}_p) = \sum_{b=0}^{p-1}(\frac{[ip^{-(n+1)}]_m}{m} + \delta_{n+1}(a + bp^n)) =$$

$$\frac{p[ip^{-(n+1)}]_m}{m} + \sum_{b=0}^{p-1}\delta_{n+1}(a + bp^n) = \frac{[ip^{-n}]_m}{m} + k_n(i) + \sum_{b=0}^{p-1}\delta_{n+1}(a + bp^n)$$

by the equality(3). Observe that

$$\sum_{b=0}^{p-1}\delta_{n+1}(a + bp^n) := \begin{cases} -k_n(i) - 1 & \text{if } a \geq 1 + \sum_{\alpha=0}^{n-1}(p-1-k_\alpha(i))p^\alpha, \\ -k_n(i) & \text{if } a < 1 + \sum_{\alpha=0}^{n-1}(p-1-k_\alpha(i))p^\alpha \,. \end{cases}$$

Hence finally we get $\sum_{b=0}^{p-1}\mu(\frac{i}{m})(a + bp + p^{n+1}\mathbb{Z}_p) = \frac{[ip^{-n}]_m}{m} + \delta_n(a) = \mu(\frac{i}{m})(a + p^n\mathbb{Z}_p)$. □

Proposition 1.2. For $k \geq 1$ we have

i)

$$\int_{\mathbb{Z}_p} x^{k-1} d\mu(\frac{i}{m})(x) = \frac{1}{k}\big(B_k(\frac{i}{m}) - B_k\big)\,,$$

ii)

$$\int_{\mathbb{Z}_p^\times} x^{k-1} d\mu(\frac{i}{m})(x) = \frac{1}{k}\big(B_k(\frac{i}{m}) - B_k\big) - \frac{p^{k-1}}{k}\big(B_k(\frac{[ip^{-1}]_m}{m}) - B_k\big)\,.$$

Proof. First we shall prove the formula i). Let us calculate the Riemann sum

$$\sum_{\alpha=0}^{p^n-1}\alpha^{k-1}\mu(\frac{i}{m})(\alpha + p^n\mathbb{Z}_p) = \sum_{\alpha=0}^{p^n-1}\alpha^{k-1}\big(\frac{[ip^{-n}]_m}{m} + \delta_n(\alpha)\big) =$$

$$\frac{[ip^{-n}]_m}{m}\sum_{\alpha=0}^{p^n-1}\alpha^{k-1} - \sum_{\alpha=0}^{p^n-1}\alpha^{k-1} + \sum_{\alpha=0}^{v_{n-1}(\frac{i}{m})-1}\alpha^{k-1}\,.$$

Observe that

$$\sum_{\alpha=0}^{v_{n-1}(\frac{i}{m})-1}\alpha^{k-1} = \frac{1}{k}\big(B_k(v_{n-1}(\frac{i}{m})) - B_k\big)$$

and it tends to $\frac{1}{k}\big(B_k(\frac{i}{m}) - B_k\big)$ if n tends to ∞. Hence the formula i) of the proposition follows because $\sum_{\alpha=0}^{p^n-1}\alpha^{k-1}$ tends to 0 if n tends to ∞.

Observe that

$$\int_{\mathbb{Z}_p^\times} x^{k-1} d\mu(\frac{i}{m})(x) = \int_{\mathbb{Z}_p} x^{k-1} d\mu(\frac{i}{m})(x) - \int_{p\mathbb{Z}_p} x^{k-1} d\mu(\frac{i}{m})(x)\,.$$

We shall calculate Riemann sums for the integral $\int_{p\mathbb{Z}_p} x^{k-1} d\mu(\frac{i}{m})(x)$. We have

$$\sum_{\alpha=0}^{p^n-1} (p\alpha)^{k-1} \mu(p\alpha + p^{n+1}\mathbb{Z}_p) = \sum_{\alpha=0}^{p^n-1} p^{k-1}\alpha^{k-1} \frac{[ip^{-(n+1)}]_m}{m} + \sum_{\alpha=0}^{p^n-1} p^{k-1}\alpha^{k-1}\delta_{n+1}(p\alpha)\,.$$

The first sum tend to 0 if n tends to ∞. Observe that

$$\sum_{\alpha=0}^{p^n-1} p^{k-1}\alpha^{k-1}\delta_{n+1}(p\alpha) = \sum_{0<\alpha<p^n,\, p\alpha \geq v_n(\frac{i}{m})} p^{k-1}\alpha^{k-1}(-1) =$$

$$-\sum_{\alpha=0}^{p^n-1} p^{k-1}\alpha^{k-1} + \sum_{0<\alpha<p^n,\, p\alpha < v_n(\frac{i}{m})} p^{k-1}\alpha^{k-1}\,.$$

Let $0 \leq \beta_0 < p$ be such that $v_n(\frac{i}{m}) \equiv \beta_0$ modulo p. Then

$$v_{n-1}(\frac{[ip^{-1}]_m}{m}) = \begin{cases} 1 + \frac{1}{p}(v_n(\frac{i}{m}) - \beta_0) & \text{if } \beta_0 \neq 0, \\ \frac{1}{p} v_n(\frac{i}{m}) & \text{if } \beta_0 = 0\,. \end{cases}$$

Hence it follows that

$$\sum_{0<\alpha<p^n,\, p\alpha<v_n(\frac{i}{m})} p^{k-1}\alpha^{k-1} = p^{k-1} \sum_{\alpha=0}^{v_{n-1}(\frac{[ip^{-1}]_m}{m})-1} \alpha^{k-1}.$$

If n tends to ∞ the last sum tends to $p^{k-1}\frac{1}{k}\big(B_k(\frac{[ip^{-1}]_m}{m}) - B_k\big)$. Hence the proof of the formula ii) is finished. □

If $c \in \mathbb{Z}_p^\times \setminus \mu_{p-1}$ we define

$$\mu_c(\frac{i}{m}) := \mu(\frac{i}{m}) - c\mu(\frac{i}{m}) \circ c^{-1}\,. \tag{6}$$

Then we have

$$\frac{1}{1-c^k} \int_{\mathbb{Z}_p} x^{k-1} d\mu_c(\frac{i}{m})(x) = \frac{1}{k}\big(B_k(\frac{i}{m}) - B_k\big)\,. \tag{7}$$

Corollary 1.3. Let $P : \mathbb{Z}_p[[\mathbb{Z}_p]] \to \mathbb{Z}_p[[T]]$ be the Iwasawa isomorphism given by $P(1) = 1 + T$. Then

$$P(\mu(\frac{i}{m}))(T) = \frac{(1+T)^{\frac{i}{m}} - 1}{T}$$

and

$$P(\mu_c(\frac{i}{m})) = \frac{(1+T)^{\frac{i}{m}} - 1}{T} - \frac{c\big((1+T)^{c\frac{i}{m}} - 1\big)}{(1+T)^c - 1}\,.$$

Proof. The power series $P(\mu(\frac{i}{m}))(\exp X - 1)$ is equal $\sum_{k=0}^\infty \big(\int_{\mathbb{Z}_p} x^k d\mu(\frac{i}{m})(x)\big) X^k$. Hence by the point i) of Proposition 1.2 it is equal

$$\sum_{k=0}^\infty \frac{1}{(k+1)!}\big(B_{k+1}(\frac{i}{m}) - B_{k+1}\big) X^k\,.$$

It follows from the definition of the Bernoulli numbers and the Bernoulli polynomials that this power series is equal $\frac{\exp\frac{i}{m}X-1}{\exp X-1}$. Replacing $\exp X$ by $1+T$ we get the power series $P(\mu(\frac{i}{m}))(T)$. □

We denote by

$$\omega : \mathbb{Z}_p^\times \to \mu_{p-1} \subset \mathbb{Z}_p^\times$$

the Teichmüller character. For $x \in \mathbb{Z}_p^\times$ we set

$$[x] := x\omega(x)^{-1}\,.$$

Let us define

$$\tilde{H}_p(1-s,\omega^b,\frac{i}{m}) := \int_{\mathbb{Z}_p^\times} [x]^s x^{-1}\omega(x)^b d\mu(\frac{i}{m})(x)\,.$$

Proposition 1.4. Let $k \equiv b$ modulo $p-1$. Then

$$\tilde{H}_p(1-k,\omega^b,\frac{i}{m}) = \frac{1}{k}\big(B_k(\frac{i}{m}) - B_k\big) - \frac{p^{k-1}}{k}\big(B_k(\frac{[ip^{-1}]_m}{m}) - B_k\big)\,.$$

Proof. We have

$$\tilde{H}_p(1-k,\omega^b,\frac{i}{m}) = \int_{\mathbb{Z}_p^\times} [x]^k x^{-1}\omega(x)^b d\mu(\frac{i}{m})(x) = \int_{\mathbb{Z}_p^\times} x^{k-1} d\mu(\frac{i}{m})(x)\,.$$

Hence the proposition follows from the formula ii) of Proposition 1.2. □

Remark 1.5. A function closely related to our function $\tilde{H}_p(1-s,\omega^b,\frac{i}{m})$ appears in a paper of Shiratani (see [10, Theorem 1, case $p \nmid f$]).

2. Action of the complex conjugation on measures

We define an action of $\mathbb{Z}_p^\times$ on the group ring $\mathbb{Z}_p[\mathbb{Z}_p]$ by the formula

$$\alpha(\sum_{i=1}^n a_i(x_i)) = \alpha\sum_{i=1}^n a_i(\alpha^{-1}x_i)$$

and we extend by continuity to the action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}_p[[\mathbb{Z}_p]]$. The action of $-1 \in \mathbb{Z}_p^\times$ we denote by ι. Then

$$\mathbb{Z}_p[[\mathbb{Z}_p]] = \mathbb{Z}_p[[\mathbb{Z}_p]]^+ \oplus \mathbb{Z}_p[[\mathbb{Z}_p]]^-\,,$$

where ι acts on $\mathbb{Z}_p[[\mathbb{Z}_p]]^+$ (resp. on $\mathbb{Z}_p[[\mathbb{Z}_p]]^-$) as the identity (resp. as the multiplication by -1). For any $\mu \in \mathbb{Z}_p[[\mathbb{Z}_p]]$ we have the decomposition

$$\mu = \mu^+ + \mu^-\,,$$

where $\mu^+ = \frac{1}{2}(\mu+\iota(\mu)) \in \mathbb{Z}_p[[\mathbb{Z}_p]]^+$ and $\mu^- = \frac{1}{2}(\mu-\iota(\mu)) \in \mathbb{Z}_p[[\mathbb{Z}_p]]^-$. Observe that

$$\int_{\mathbb{Z}_p} x^{k-1} d\iota(\mu) = (-1)^k \int_{\mathbb{Z}_p} x^{k-1} d\mu\,. \tag{8}$$

Hence it follows

$$\int_{\mathbb{Z}_p} x^{k-1} d\mu^+ := \begin{cases} 0 & \text{for } k \text{ odd,} \\ \int_{\mathbb{Z}_p} x^{k-1} d\mu & \text{for } k \text{ even} \end{cases} \tag{9}$$

and

$$\int_{\mathbb{Z}_p} x^{k-1} d\mu^- := \begin{cases} \int_{\mathbb{Z}_p} x^{k-1} d\mu & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even}. \end{cases} \tag{10}$$

In [12, Proposition 10.5] we have shown that

$$\int_{\mathbb{Z}_p} x^{k-1} d(K(\xi_m^{-i}) + K(\xi_m^i)) = \frac{1}{k} B_k(\frac{i}{m})(1-\chi^k) \text{ for } k \text{ even} \tag{11}$$

and

$$\int_{\mathbb{Z}_p} x^{k-1} d(K(\xi_m^{-i}) - K(\xi_m^i)) = \frac{1}{k} B_k(\frac{i}{m})(1-\chi^k) \text{ for } k \text{ odd}. \tag{12}$$

Hence it follows from (9) and (10) that

$$\int_{\mathbb{Z}_p} x^{k-1} d\big((K(\xi_m^{-i}) + K(\xi_m^i))^+ + (K(\xi_m^{-i}) - K(\xi_m^i))^-\big) = \frac{1}{k} B_k(\frac{i}{m})(1-\chi^k) \text{ for } k \geq 1\,. \tag{13}$$

Observe that

$$(K(\xi_m^{-i}) + K(\xi_m^i))^+ + (K(\xi_m^{-i}) - K(\xi_m^i))^- = K(\xi_m^{-i}) + \iota(K(\xi_m^i))\,.$$

Hence we get

$$\int_{\mathbb{Z}_p} x^{k-1} d(K(\xi_m^{-i}) + \iota(K(\xi_m^i))) = \frac{1}{k} B_k(\frac{i}{m})(1-\chi^k) \text{ for } k \geq 1\,. \tag{14}$$

The proof of the formulas (11) and (12) given in [12] is based on the symmetry $\mathfrak{z} \mapsto 1/\mathfrak{z}$ of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ and the study of the polylogarithmic coefficients (at YX^{k-1}) of the power series $\Lambda_{\beta_i}(\sigma)$ and $\Lambda_{\beta_{m-i}}(\sigma)$. Recently, H. Nakamura (see [7]) got these formulas using directly the inversion formula from [9, section 6.3].

In this paper we calculate explicitely the measure $K(\xi_m^{-i}) + \iota(K(\xi_m^i))$. We use also the symmetry $\mathfrak{z} \mapsto 1/\mathfrak{z}$ of the tower of coverings

$$\mathbb{P}1_{\bar{\mathbb{Q}}} \setminus (\{0, \infty\} \cup \mu_{p^n}) \to \mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\},\ \mathfrak{z} \mapsto \mathfrak{z}^{p^n}$$

but only in degree 1.

The third possible method to calculate the measure $K(\xi_m^{-i}) + \iota(K(\xi_m^i))$ is to use the explicit formula for measures $K(z)$ (see [8, Proposition 3]). Compare the three different proofs of Proposition 5.13 in [9]. Two proofs are given in [9] and the third one in [12] (the second proof of Lemma 4.1.)

3. Measures associated with roots of unity

We set

$$\xi_r := \exp(\frac{2\pi\sqrt{-1}}{r})$$

for a natural number r. Let us set

$$V_n := \mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0, \infty\} \cup \mu_{p^n}).$$

We recall that $\pi_1(V_n, \overrightarrow{01})$ - pro-p étale fundamental group - is free on generators x_n - loop around 0 - and $y_{n,i}$ - loops around $\xi_{p^n}^i$ for $0 \leq i < p^n$.

For each $0 < i < m$, let α_i be a path on $V_0 = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ from $\overrightarrow{01}$ to ξ_m^i which is the composition of an arc from $\overrightarrow{01}$ to $\overrightarrow{0\xi_m^i}$ in an infinitesimal neighbourhood of 0 followed by the canonical path (straight line) from $\overrightarrow{0\xi_m^i}$ to ξ_m^i.

Let us set

$$\beta_i := \alpha_i \cdot x^{-\frac{i}{m}} .$$

Observe that $l(\xi_m^i)_{\beta_i} = 0$. If we regard the path α_i as the path on V_n then we denote it by

$${}_n\alpha_i .$$

Then

$${}_n\beta_i := {}_n\alpha_i \cdot x_n^{-\frac{i}{m}}$$

is also a path on V_n. Let

$$\tilde{\beta}_i^n \ \ (\text{resp. } \tilde{\alpha}_i^n)$$

be the lifting of β_i (resp. α_i) to V_n starting from $\overrightarrow{01}$. Let $0 \le j < p^n$. We denote by s_n^j a lifting of x_0^j to V_n starting from $\overrightarrow{01}$. Observe that s_n^j is a path on V_n from $\overrightarrow{01}$ to $\overrightarrow{0\xi_{p^n}^j}$

Lemma 3.1. We have

$$\tilde{\beta}_i^n = {}_n\beta_{[ip^{-n}]_m} = {}_n\alpha_{[ip^{-n}]_m} \cdot x_n^{-\frac{[ip^{-n}]_m}{m}} .$$

Proof. Observe that the lifting of $x^{-\frac{i}{m}}$ to V_n is equal $s_n^{v_{n-1}(-\frac{i}{m})} \cdot x_n^{t_n(-\frac{i}{m})}$. The lifting of α_i to V_n is a path (an arc) from $\overrightarrow{01}$ to $\overrightarrow{w} := \overrightarrow{0\xi_{p^n m}^i}$ in the positive sense composed with the canonical path from $\overrightarrow{w}$ to $\xi_{p^n m}^i$. Hence the lifting of β_i is the composition of $s_n^{v_{n-1}(-\frac{i}{m})} \cdot x_n^{t_n(-\frac{i}{m})}$ with the lifting of α_i multiplied by $\xi_{p^n}^{v_{n-1}(-\frac{i}{m})}$.

We have

$$\xi_{p^n}^{v_{n-1}(-\frac{i}{m})} \xi_{p^n m}^i = \xi_{p^n m}^{m v_{n-1}(-\frac{i}{m})+i} .$$

Observe that $0 \le v_{n-1}(-\frac{i}{m}) \cdot m + i < p^n m$ and that p^n divides $v_{n-1}(-\frac{i}{m}) \cdot m + i$. Moreover we have $\frac{v_{n-1}(-\frac{i}{m}) \cdot m + i}{p^n} \cdot p^n \equiv i$ modulo m. Hence it follows that $\frac{v_{n-1}(-\frac{i}{m}) \cdot m + i}{p^n} = [ip^{-n}]_m$. Therefore we get

$$-\frac{[ip^{-n}]_m}{m} = -\frac{1}{p^n}(v_{n-1}(-\frac{i}{m}) + \frac{i}{m}) = t_n(-\frac{i}{m}).$$

Hence it follows that the lifting of β_i is ${}_n\alpha_{[ip^{-n}]_m} \cdot x_n^{-\frac{[ip^{-n}]_m}{m}}$. □

To simplify the notation we set

$$r_n = [ip^{-n}]_m \ \text{ and } \ v_{n-1} = v_{n-1}(-\frac{i}{m}) .$$

Then we have

$$\tilde{\beta}_i^n = {}_n\alpha_{r_n} \cdot x_n^{-\frac{r_n}{m}} \ \text{ and } \ \tilde{\beta}_{m-i}^n = {}_n\alpha_{m-r_n} \cdot x_n^{\frac{r_n}{m}-1}$$

Let $h : V_n \to V_n$ be given $\mathfrak{z} \to 1/\mathfrak{z}$. Let p_n be the canonical path from $\overrightarrow{01}$ to $\overrightarrow{10}$ on V_n, t_n a path from $\overrightarrow{10}$ to $\overrightarrow{1\infty}$ (half circle in the positive sense in an infinitesimal neighbourhood of 1) and $q_n = h(p_n)$. We set

$$\Gamma_n := q_n \cdot t_n \cdot p_n \, .$$

Lemma 3.2. We have

$$\tilde{\beta}^n_{m-i} = h(\tilde{\beta}^n_i) \cdot \Gamma_n \cdot z_n^{\frac{r_n}{m}} \cdot x_n \cdot y_{n,-1} \cdot \ldots y_{n,-v_{n-1}} \cdot x_n^{\frac{r_n}{m}-1}$$

in $\pi_1(V_n, \overrightarrow{01})$.

Proof. One checks that ${}_n\alpha_{m-r_n} = h({}_n\alpha_{r_n}) \cdot \Gamma_n \cdot x_n \cdot y_{n,-1} \cdot \ldots y_{n,-v_{n-1}}$. The formula of the lemma follows from Lemma 3.1. □

Lemma 3.3. Let $\sigma \in G_{\mathbb{Q}(\mu_m)}$. Then writting additively we have

$$\mathfrak{f}_{\Gamma_n}(\sigma) \equiv \sum_{k=0}^{p^n-1} E^{(n)}_{1,\chi(\sigma)}(k) y_{n,k} \ \text{ modulo } \ (\pi_1(V_n, \overrightarrow{01}), \pi_1(V_n, \overrightarrow{01})) \, .$$

Proof. See the proof of Lemma 4.1 in [12] or the second proof of Proposition 5.13 in [9]. □

It follows from Lemma 3.2 that

$$\mathfrak{f}_{\tilde{\beta}^n_{m-i}}(\sigma) \equiv \Gamma_n^{-1} h(\mathfrak{f}_{\tilde{\beta}^n_i}(\sigma)) \Gamma_n \cdot \mathfrak{f}_{\Gamma_n}(\sigma) \cdot$$

$$\big(z_n^{\frac{r_n}{m}} \cdot x_n \cdot y_{n,-1} \cdot \ldots y_{n,-v_{n-1}} \cdot x_n^{\frac{r_n}{m}-1}\big)^{-1} \cdot \sigma\big(z_n^{\frac{r_n}{m}} \cdot x_n \cdot y_{n,-1} \cdot \ldots y_{n,-v_{n-1}} \cdot x_n^{\frac{r_n}{m}-1}\big)$$

modulo $(\pi_1(V_n, \overrightarrow{01}), \pi_1(V_n, \overrightarrow{01}))$. Hence writting the result additively we get

$$\sum_{k=0}^{p^n-1} K^{(n)}(\xi_m^{-i})(\sigma)(k) y_{n,k} \equiv \sum_{k=0}^{p^n-1} K^{(n)}(\xi_m^{i})(\sigma)(k) y_{n,-k} + \sum_{k=0}^{p^n-1} E^{(n)}_{1,\chi(\sigma)}(k) y_{n,k} +$$

$$\sum_{k=0}^{p^n-1} (1-\chi(\sigma)) \frac{[ip^{-n}]_m}{m} y_{n,k} - \sum_{j=1}^{v_{n-1}(-\frac{i}{m})} y_{n,-j} + \chi(\sigma) \sum_{j=1}^{v_{n-1}(-\frac{i}{m})} y_{n,-[j\chi(\sigma)]_{p^n}}$$

modulo $(\pi_1(V_n, \overrightarrow{01}), \pi_1(V_n, \overrightarrow{01}))$. Observe that $v_{n-1}(\frac{i}{m}) = p^n - v_{n-1}(-\frac{i}{m})$, Hence the last two sums we can rewrite in the form

$$\sum_{j=v_{n-1}(\frac{i}{m})}^{p^n-1} y_{n,j} + \chi(\sigma) \sum_{j=v_{n-1}(\frac{i}{m})}^{p^n-1} y_{n,[j\chi(\sigma)]_{p^n}} \, .$$

Comparing coefficients at $y_{n,k}$ we get for $0 \le k < p^n$

$$K^{(n)}(\xi_m^{-i})(\sigma)(k) - K^{(n)}(\xi_m^{i})(\sigma)(-k) = \tag{15}$$

$$E^{(n)}_{1,\chi(\sigma)}(k) + \frac{[ip^{-n}]_m}{m} + \delta_n(k) - \chi(\sigma)\frac{[ip^{-n}]_m}{m} + \chi(\sigma)\delta_n([\chi(\sigma)^{-1}k]_{p^n}) =$$

$$E^{(n)}_{1,\chi(\sigma)}(k) + \mu_{\chi(\sigma)}(\frac{i}{m})(k)$$

by the definition of the measure $\mu_{\chi(\sigma)}(\frac{i}{m})$.

Theorem 3.5. Let m be a positive integer not divisible by p and let $0 < i < m$. Then we have

$$K(\xi_m^{-i})(\sigma) + \iota(K(\xi_m^i)(\sigma)) = E_{1,\chi(\sigma)} + \mu_{\chi(\sigma)}(\frac{i}{m})\,.$$

Proof. The theorem follows from the formula (15). □

Corollary 3.6. Let $\sigma \in G_{\mathbb{Q}(\mu_m)}$ be such that $\chi(\sigma)^{p-1} \neq 1$. Then we have

i)

$$\frac{1}{1-\chi(\sigma)^k}\int_{\mathbb{Z}_p} x^{k-1} d\big(K(\xi_m^{-i})(\sigma) + \iota(K(\xi_m^i)(\sigma))\big) = \frac{B_k(\frac{i}{m})}{k}\,,$$

ii)

$$P(K(\xi_m^{-i})(\sigma) + \iota(K(\xi_m^i)(\sigma)))(T) = \frac{(1+T)^{\frac{i}{m}}}{T} - \frac{\chi(\sigma)(1+T)^{\chi(\sigma)\frac{i}{m}}}{(1+T)^{\chi(\sigma)}-1}\,.$$

Proof. The point i) of the corollary follows from Theorem 3.5 and the formula (7). The point ii) follows immediately from Corollary 1.8 and the equality $P(E_{1,\chi(\sigma)})(T) = \frac{1}{T} - \frac{\chi(\sigma)}{(1+T)^{\chi(\sigma)}-1}$. □

Now we define

$$L^\beta(1-s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) :=$$
$$\frac{1}{1-\omega(\chi(\sigma))^\beta[\chi(\sigma)]^s}\int_{\mathbb{Z}_p^\times} [x]^s x^{-1}\omega(x)^\beta d\big((K(\xi_m^{-i}) + \iota(K(\xi_m^i)))\big)(\sigma)\,.$$

Theorem 3.7. Let $\sigma \in G_{\mathbb{Q}(\mu_m)}$ be such that $\chi(\sigma)^{p-1} \neq 1$.

i) Let $k \equiv \beta$ modulo $(p-1)$. Then

$$L^\beta(1-k, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) = \frac{1}{k}B_k(\frac{i}{m}) - p^{k-1}\frac{1}{k}B_k(\frac{[ip^{-1}]_m}{m})\,.$$

ii) Let $\sigma, \sigma_1 \in G_{\mathbb{Q}(\mu_m)}$ be such that $\chi(\sigma)^{p-1} \neq 1$ and $\chi(\sigma_1)^{p-1} \neq 1$.Then

$$L^\beta(1-s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) = L^\beta(1-s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma_1)\,,$$

i.e. the function $L^\beta(1-s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma)$ does not depend on σ.

Proof. For $k \equiv \beta$ modulo $p-1$ we have

$$L^\beta(1-k, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) = \frac{1}{1-\chi(\sigma)^k}\int_{\mathbb{Z}_p^\times} x^{k-1} d(\mu_{\chi(\sigma)}(\frac{i}{m}) + E_{1,\chi(\sigma)})$$

by Theorem 3.5. It follows from [6, Theorem 2.3] that $\frac{1}{\chi(\sigma)^k-1}\int_{\mathbb{Z}_p} x^{k-1} dE_{1,\chi(\sigma)} = -\frac{1}{k}B_k$. The "periodicity"property $E^{(n)}_{1,\chi(\sigma)}(i) = E^{(n+1)}_{1,\chi(\sigma)}(pi)$ of the measure $E_{1,\chi(\sigma)}$ implies that

$$\frac{1}{1-\chi(\sigma)^k}\int_{\mathbb{Z}_p^\times} x^{k-1} dE_{1,\chi(\sigma)} = (1-p^{k-1})\frac{1}{k}B_k\,. \tag{16}$$

Integrating the function x^{k-1} against the measure $\mu_{\chi(\sigma)}(\frac{i}{m})$ we get

$$\frac{1}{\chi(\sigma)^k-1}\int_{\mathbb{Z}_p^\times} x^{k-1} d\mu_{\chi(\sigma)}(\frac{i}{m})(x) =$$

$$\frac{1}{\chi(\sigma)^k-1}\Big(\int_{\mathbb{Z}_p^\times} x^{k-1}d\mu(\frac{i}{m})(x)-\int_{\mathbb{Z}_p^\times} x^{k-1}d(\chi(\sigma)\mu(\frac{i}{m})\circ\chi(\sigma)^{-1})(x)\Big).$$

Observe that $\int_{\mathbb{Z}_p^\times} x^{k-1}d(\chi(\sigma)\mu(\frac{i}{m})\circ\chi(\sigma)^{-1})(x)=\chi(\sigma)^k\int_{\mathbb{Z}_p^\times} y^{k-1}d\mu(\frac{i}{m})(y)$ if we set $\chi(\sigma)y=x$. It follows from Proposition 1.9 that

(17)
$$\frac{1}{\chi(\sigma)^k-1}\int_{\mathbb{Z}_p^\times} x^{k-1}d\mu_{\chi(\sigma)}(\frac{i}{m})=\frac{1}{k}\big(B_k(\frac{i}{m})-B_k\big)-p^{k-1}\frac{1}{k}\big(B_k(\frac{[ip^{-1}]_m}{m})-B_k\big).$$

After the addition of (16) and (17) we get the point i) of the theorem.

Concerning the point ii) observe that the functions $L^\beta(1-s,(\xi_m^{-i})+\iota(\xi_m^i);\sigma)$ and $L^\beta(1-s,(\xi_m^{-i})+\iota(\xi_m^i);\sigma_1)$ coincide for $k\equiv\beta$ modulo $(p-1)$. Hence these functions are equal because they are equal on a dense subset of $\mathbb{Z}_p$. □

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