Lie algebras of Galois representations on fundamental groups

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Abstract

In this paper we are studying Lie algebras associated with Galois representations on the fundamental groups of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{2^n})$ and $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{3^n})$.

0 Introduction

The Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_n))$ acts on the étale fundamental group $\pi_1^{et}(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_n\}; \overrightarrow{01})$. One of the most interesting problems is to describe the image of $Gal(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_n))$ in the group of automorphisms of $\pi_1^{et}(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_n\}; \overrightarrow{01})$. To simplify the situation one usually considers pro-l quotient of π_1 and then an infinitesimal version of the problem (see [1] and [6]).

Let $Lie(X, Y_0, \dots, Y_{n-1})$ be a free Lie algebra over $\mathbb{Q}_l$ on $n+1$ free generators $X, Y_0, \dots, Y_{n-1}$. We equipped it with the Ihara bracket $\{\,,\}$ and we denote the resulting Lie algebra by $Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}}$.

In the infinitesimal version of the problem we get a representation of the Lie algebra $LieGal(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_n))$ (whatever it means) into the Lie algebra $Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}}$.

So as the first step to understand the Galois action on π_1, we are studying the Lie algebra $Lie(X, Y_0, \dots, Y_{n-1})_{\{\ \}}$. The results of section 2 are very elementary and very likely well known. We do not even know how to describe the abelianization of $Lie(X, Y_0, \dots, Y_{n-1})_{\{\ \}}$.

In sections 1 and 3 are collected some general facts about Galois actions on fundamental groups of the projective line minus a finite number of points taken from previous papers of the author.

In the last two sections we are studying the infinitesimal version of Galois actions on fundamental groups of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{2^n})$ and $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{3^n})$. In the case of 2^n we extend slightly our previous result (see [11, Corollary 15.6.3.]). We show that the Lie subalgebra of $LieGal(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_{l^\infty}))$ in $Lie(X, Y_0, \dots, Y_{2^n-1})_{\{\ ,\}}$ genated by all elements in degree greater than 1 and 3/4 of generators in degree 1 is free.

We prove also the analogous result for 3^n.

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1 Galois action on π_1

In this section we review some results and constructions from our previous papers (see [9], [10], [11], [12] and [13])..

Let K be a number field. Let $a_1, \dots, a_n \in K$. Let us set $V := \mathbb{P}^1_K \setminus \{a_1, \dots, a_n, \infty\}$. Let v and z be K-points of V or tangential K-points of V. Let l be a fixed prime. We denote by $\pi_1(V_{\bar{K}}; v)$ the maximal pro-l quotient of the étale fundamental group of $V_{\bar{K}}$ based at v and by $\pi(V_{\bar{K}}; z, v)$ the $\pi_1(V_{\bar{K}}; v)$-torsor of l-adic paths from v to z. Let v_i be a tangential K-point on $V_{\bar{K}}$ at a_i for $i = 1, \dots, n$. Let $s_i \in \pi_1(V_{\bar{K}}; v_i)$ be a generator of the inertia group of a place over a_i and let $\gamma_i \in \pi(V_{\bar{K}}; v_i, v)$.

We set

$$x_i := \gamma_i^{-1} \cdot s_i \cdot \gamma_i$$

for $i = 1, \dots, n$. The elements $x_1, \dots, x_n$ are free generators of $\pi_1(V_{\bar{K}}; v)$.

Let γ be a path from v to z. For any $\sigma \in G_K$ we set

$$\mathfrak{f}_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1(V_{\bar{K}}; v)$$

.

Proposition 1.1. The action of G_K on $\pi_1(V_{\bar{K}}; v)$ is given by the formulas

$$\sigma(x_i) = \mathfrak{f}_{\gamma_i}(\sigma)^{-1} \cdot x_i^{\chi(\sigma)} \cdot \mathfrak{f}_{\gamma_i}(\sigma)$$

for $i = 1, \dots, n$.

Let $\mathbb{Q}_l\{\{X_1,\dots,X_n\}\}$ be a $\mathbb{Q}_l$-algebra of formal power series on non commuting variables $X_1,\dots,X_n$. We define a continuous multiplicative embedding

$$k:\pi_1(V_{\bar{K}};v)\to \mathbb{Q}_l\{\{X_1,\dots,X_n\}\}$$

setting $k(x_i)=\exp(X_i)$ for $i=1,\dots,n$.

Let set

$$\Lambda_\gamma(\sigma):=k(\mathfrak{f}_\gamma(\sigma)).$$

The action of G_K on $\pi_1(V_{\bar{K}};v)$ induces the action of G_K on $\mathbb{Q}_l\{\{X_1,\dots,X_n\}\}$, hence we get the representation

$$\varphi_v:G_K\to \mathrm{Aut}\mathbb{Q}_l\{\{X_1,\dots,X_n\}\}$$

such that

$$\varphi_v(X_i)=\Lambda_{\gamma_i}(\sigma)^{-1}\cdot(\chi(\sigma)X_i)\cdot\Lambda_{\gamma_i}(\sigma)$$

for $i=1,\dots,n$.

Let $t_\gamma:\pi(V_{\bar{K}};z,v)\to\pi_1(V_{\bar{K}};v)$ be given by $t_\gamma(\delta)=\gamma^{-1}\cdot\delta$. Composing t_γ with k we get an embedding of $\pi(V_{\bar{K}};z,v)$ into $\mathbb{Q}_l\{\{X_1,\dots,X_n\}\}$. Hence the action of G_K on $\pi(V_{\bar{K}};z,v)$ induces the representation

$$\psi_\gamma:G_K\to \mathrm{GL}(\mathbb{Q}_l\{\{X_1,\dots,X_n\}\})$$

such that for any $w\in\mathbb{Q}_l\{\{X_1,\dots,X_n\}\}$

$$\psi_\gamma(\sigma)(w)=\Lambda_\gamma(\sigma)\cdot(\varphi_v(\sigma)(w)).$$

Let $\{\mathfrak{l}\mid l\}$ be the set of finite places of K lying over the prime ideal (l) of $\mathbb{Z}$.

The representations φ_v and ψ_γ are weighted Tate representations. If the pair (V,v) (resp. the triple (V,z,v)) has good reduction outside a finite set S of finite places of K then the representation φ_v (resp. ψ_γ) is unramified outside $S\cup\{\mathfrak{l}\mid l\}_K$, hence it factors through the weighted Tate $\mathbb{Q}_l$-completion $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ of $\pi_1(\mathrm{Spec}\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};\mathrm{Spec}\bar{K})$ (see [4] and [5]).

Let $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ be the associated graded Lie algebra with respect to the weight filtration of the affine prounipotent proalgebraic group

$$\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l):=ker(\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)\to\mathbb{G}_m).$$

The representations φ_v and ψ_γ induce morphisms of graded Lie algebras

$$\Phi_v:L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)\to\mathrm{Der}(\mathbb{Q}_l\{\{X_1,\dots,X_n\}\})$$

and

$$\Psi_{z,v}:L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)\to\mathrm{End}(\mathbb{Q}_l\{\{X_1,\dots,X_n\})$$

respectively.

In degree 1 the Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ has more generators than the corresponding Lie algebra of the motivic fundamental group of mixed Tate motives over $\mathrm{Spec}\mathcal{O}_{K,S}$. These additional generators are mapped to zero by morphisms Φ_v and $\Psi_{z,v}$ (see [13, Lemma 3.1.1 and Theorem 3.1]). Hence we can define the quotient Lie algebra $L_l(\mathcal{O}_{K,S})$, which has the correct number of generators and through which Φ_v and $\Psi_{z,v}$ factor. It can be also done in a more general setting using the notion of cristiline representations (see [4]).

Finally we recall the definition of l-adic polylogarithms from [10].

Let γ be a path on $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}$ from $\overrightarrow{01}$ to a K-point of $\mathbb{P}^1_K \setminus \{0,1,\infty\}$ or to a tangential point defined over K. After the standard embedding of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \overrightarrow{01})$ into $\mathbb{Q}_l\{\{X,Y\}\}$ we get a power series

$$\Lambda_\gamma(\sigma) \in \mathbb{Q}_l\{\{X,Y\}\}.$$

Let I_2 be an ideal of $\mathbb{Q}_l\{\{X,Y\}\}$ generated by monomials with two or more Y's. The l-adic polylogarithms $l(z)$ and $l_n(z)$ are defined by the following formula

$$\log \Lambda_\gamma(\sigma) \equiv l(z)_\gamma(\sigma)X + \sum_{n=1}^{\infty} l_n(z)_\gamma(\sigma)[Y, X^{(n-1)}] \mod I_2.$$

The distribution relations satisfied by l-adic polylogarithms are shown in [10]. They will be used in the last two sections.

2 Lie algebras

We denote by $Lie(X, Y_0, \ldots, Y_{n-1})$ the free Lie algebra over a field K on free generators $X, Y_0, \ldots, Y_{n-1}$. The Lie bracket we denote by $[\,,]$.

Let $A = A(X, Y_0, \ldots, Y_{n-1}) \in Lie(X, Y_0, \ldots, Y_{n-1})$. We define a derivation

$$D_A : Lie(X, Y_0, \ldots, Y_{n-1}) \to Lie(X, Y_0, \ldots, Y_{n-1})$$

by the formulas

$$D_A(X) = 0 \text{ and } D_A(Y_i) = [Y_i, A(X, Y_i, Y_{i+1}, \ldots, Y_{i+n-1})]$$

for $i = 0, 1, \ldots, n-1$. The sum $a + b$ is calculated modulo n. Observe that

$$D_A \circ D_B - D_B \circ D_A = D_{[A,B]+D_A(B)-D_B(A)} \tag{2.1}$$

We denote by $\mathrm{Der}(Lie(X, Y_0, \ldots, Y_{n-1}))$ the Lie algebra of all derivations of $Lie(X, Y_0, \ldots, Y_{n-1})$. It follows from (2.1) that

$$\mathrm{Der}^*_{\mathbb{Z}/n}(Lie(X, Y_0, \ldots, Y_{n-1})) := \{D_A \in \mathrm{Der}(Lie(X, Y_0, \ldots, Y_{n-1})) \mid A \in Lie(X, Y_0, \ldots, Y_{n-1})\}$$

is a Lie subalgebra of $\mathrm{Der}(Lie(X, Y_0, \ldots, Y_{n-1}))$.

Let $\langle a \rangle$ be a one-dimensional vector subspace of $Lie(X, Y_0, \ldots, Y_{n-1})$ generated by a. The map

$$Lie(X, Y_0, \ldots, Y_{n-1})/\langle Y_0 \rangle \to \mathrm{Der}^*_{\mathbb{Z}/n}(Lie(X, Y_0, \ldots, Y_{n-1})), \quad A \mapsto D_A$$

is an isomorphism of vector spaces.

We define a new bracket $\{\,,\}$, called the Ihara bracket (see [7]), on $Lie(X, Y_0, \ldots, Y_{n-1})/\langle Y_0\rangle$ by the formula

$$\{A, B\} := [A, B] + D_A(B) - D_B(A). \tag{2.2}$$

It follows from (2.1) that the bracket $\{\,,\}$ satisfies the Jacobi identity. Hence the vector space $Lie(X, Y_0, \ldots, Y_{n-1})/\langle Y_0\rangle$ equipped with $\{,\}$ is a Lie algebra, which we shall denote by $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$. Observe that the one-dimensional vector subspace $\langle X\rangle$ is a Lie ideal of $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$. Hence

$$Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}} = \langle X\rangle \oplus (Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}})/\langle X\rangle$$

as Lie algebras.

If p is a prime number greater than 3 then the Lie algebra $(Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}})/\langle X\rangle$ is not free (see [13, Proposition 8.1]).

The main method to show that a family of elements of the Lie algebra $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$ generate a free Lie subalgebra of $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$ is to show that the Lie bracket $\{\,,\}$ on these elements modulo some vector subspace reduces to the standard Lie bracket $[\,,]$ of $Lie(X, Y_0, \ldots, Y_{n-1})$ (see [2] and [3]). Hence first we shall study some useful Lie ideals and Lie subalgebras of $Lie(X, Y_0, \ldots, Y_{n-1})$ and $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$.

We denote by I_r the Lie ideal of $Lie(X, Y_0, \ldots, Y_{n-1})$ generated by Lie brackets in generators $X, Y_0, \ldots, Y_{n-1}$, which contain at least r elements (possible with repetitions) of the set $\{Y_0, \ldots, Y_{n-1}\}$.

It is clear that $I_{r+1} \subset I_r$. The filtration $\{I_r\}_{r\in\mathbb{N}}$ of $Lie(X, Y_0, \ldots, Y_{n-1})$ is called the depth filtration. Observe that

$$Lie(X, Y_0, \ldots, Y_{n-1})/I_2 \approx \langle X\rangle \oplus \bigoplus_{i=0}^{n-1}\bigoplus_{k=0}^{\infty}\langle [Y_i, X^{(k)}]\rangle \tag{2.3}$$

as vector spaces. Observe that in $Lie(X, Y_0, \ldots, Y_{n-1})/I_2$, $[Y_i, X^{(k)}]$ and $[Y_j, X^{(l)}]$ commute and $[[Y_i, X^{(k)}], X] = [Y_i, X^{(k+1)}]$.

It is clear that I_r is also a Lie ideal of $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$. The quotient Lie algebra

$$Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}/I_2 \approx \langle X\rangle \oplus \bigoplus_{i=0}^{n-1}\bigoplus_{k=0}^{\infty}\langle [Y_i, X^{(k)}]\rangle/\langle Y_0\rangle$$

is commutative. We do not know if the natural surjection

$$Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}/\Gamma^2 Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}} \to Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}/I_2$$

is an isomorphism.

Let S be a subset of $\{0, 1, \ldots, n-1\}$. Let $r \geq 2$. We denote by $I_r(S)$ the Lie ideal of $Lie(X, Y_0, \ldots, Y_{n-1})$ generated by Lie brackets in generators

$X, Y_0, \dots, Y_{n-1}$, which contain at least r elements (possible with repetitions) of the set $\{Y_0, Y_1, \dots, Y_{n-1}\}$ and at least one of these elements is Y_s with $s \in S$ and at least one of these elements is Y_t with $t \notin S$.

It is clear that $I_{r+1}(S) \subset I_r(S)$ for any $r \geq 2$. Hence we have a filtration $\{I_r(S)\}_{r\geq 2}$ of $Lie(X, Y_0, \dots, Y_{n-1})$ by Lie ideals. Notice that $I_r(S)$ is not a Lie ideal of $Lie(X, Y_0, \dots, Y_{n-1})_{\{\ \}}$.

Let $A(S)$ be a Lie subalgebra of $Lie(X, Y_0, \dots, Y_{n-1})$ generated by elements X and Y_s with $s \in S$. We set

$$I_r(A(S)) := I_r \cap A(S).$$

We identify the set $\{0, 1, \dots, n-1\}$ with $\mathbb{Z}/n$. If $S \subset \{0, 1, \dots, n-1\}$ then the subsets $S + S := \{a + b \in \mathbb{Z}/n \mid a, b \in S\}$ and $-S := \{-a \in \mathbb{Z}/n \mid a \in S\}$ are well defined.

Lemma 2.4. Let S be a subset of $\{0, 1, \dots, n-1\}$ such that $(S + S) \cap S = \emptyset$. Let $r, r_1 \geq 2$ and $p, p_1 \geq 1$. Let $w \in I_r(S)$, $w_1 \in I_{r_1}(S)$ and $a \in I_p(A(S))$, $a_1 \in I_{p_1}(A(S))$. Then

$$[w, w_1] \in I_{r+r_1}(S),\ \ D_w(w_1) \in I_{r+r_1}(S),\ \ [a, w] \in I_{r+p}(S),\ \ D_a(w) \in I_{r+p}(S),$$

$$D_w(a) \in I_{r+p}(S),\ \ D_a(a_1) \in I_{p+p_1}(S).$$

Proof. It is clear that $[w, w_1]$ and $D_w(w_1)$ belong to $I_{r+r_1}(S)$ as well as that $[a, w]$ and $D_a(w)$ belong to $I_{r+p}(S)$.

Let $s, s_1 \in S$. Observe that $D_{Y_{s_1}}(Y_s) = [Y_s, Y_{s+s_1}]$. The element $s \in S$ and $s + s_1 \notin S$ by the assumption that $(S + S) \cap S = \emptyset$. Hence it follows that $D_w(a) \in I_{r+p}(S)$ and $D_a(a_1) \in I_{p+p_1}(S)$. □

Corollary 2.5. The assumptions are the same as in Lemma 2.4. Then

$$\{a, w\} = [a, w] + D_a(w) - D_w(a) \in I_{r+p}(S),$$

$$\{w, w_1\} = [w, w_1] + D_w(w_1) - D_{w_1}(w) \in I_{r+r_1}(S)$$

and

$$\{a, a_1\} \equiv [a, a_1] \bmod I_{p+p_1}(S).$$

Let $V(S)$ be a vector subspace of $Lie(X, Y_0, \dots, Y_{n-1})$ generated by Lie brackets in $X, Y_0, \dots, Y_{n-1}$ which contain at least one Y_s with $s \in S$. $V(S)$ is clearly a Lie ideal of $Lie(X, Y_0, \dots, Y_{n-1})$.

Proposition 2.6. Let S be a subset of $\{0, 1, \dots, n-1\}$ such that $(S+S)\cap S = \emptyset$. Then

i) $V(S)$ is a Lie subalgebra of $Lie(X, Y_0, \dots, Y_{n-1})_{\{\ \}}$.

ii) For any $r \geq 2$, $I_r(S) \subset V(S)$.

iii) $I_r(S)$ is a Lie ideal of $V(S)$ considered as a Lie subalgebra of $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$ and $\{I_r(S), V(S)\} \subset I_{r+1}(S)$.

Proof. Let $a = a(Y_s...)$ and $b = b(Y_{s_1}...)$ belong to $V(S)$. Then $\{a, b\} = [a, b] + D_a(b) - D_b(a) \in V(S)$, hence $V(S)$ is a Lie subalgebra of $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$. It is clear that $I_r(S) \subset V(S)$. If $a \in V(S)$ and $w \in I_r(S)$ then $[a, w] \in I_{r+1}(S)$, $D_a(w) \in I_{r+1}(S)$. The assumption $(S+S) \cap S = \emptyset$ implies that $D_w(a) \in I_{r+1}(S)$. Hence $\{a, w\} \in I_{r+1}(S) \subset I_r(S)$. $\square$

Let $\mathcal{L}$ be a Lie algebra over a field K. Let $\mathcal{Z} = \{z_i \in \mathcal{L} \mid i \in \mathbb{N}\}$ be a linearly independent subset of $\mathcal{L}$. We denote by $\mathcal{L}(\mathcal{Z})$ a Lie subalgebra of $\mathcal{L}$ generated by the subset $\mathcal{Z}$.

Let $Lie(Z_i \mid i \in \mathbb{N})$ be a free Lie algebra over K on symbols Z_i, $i \in \mathbb{N}$. We denote by $Hallbasis(Z_i \mid i \in \mathbb{N})$ the set of basic Lie elements of $Lie(Z_i \mid i \in \mathbb{N})$ formed from the sequence $Z_1, Z_2, \ldots, Z_n, \ldots$ following the rules described in [8] on pages 322-327. The set $Hallbasis(Z_i \mid i \in \mathbb{N})$ is a basis of the vector space $Lie(Z_i \mid i \in \mathbb{N})$ (see [8, Theorem 5.8.]).

Let $\mathcal{P} : Lie(Z_i \mid i \in \mathbb{N}) \to \mathcal{L}$ be a morphism of Lie algebras given by $\mathcal{P}(Z_i) = z_i$ for $i \in \mathbb{N}$. Observe that the image of $\mathcal{P}$ is the Lie algebra $\mathcal{L}(\mathcal{Z})$. We denote by $\mathcal{HB}(\mathcal{Z})_{\mathcal{L}}$ the image of the set $Hallbasis(Z_i \mid i \in \mathbb{N})$ by the morphism $\mathcal{P}$.

Lemma 2.7. The Lie subalgebra $\mathcal{L}(\mathcal{Z})$ of $\mathcal{L}$ is free, freely generated by the subset $\mathcal{Z}$ of $\mathcal{L}$ if and only if the set $\mathcal{HB}(\mathcal{Z})_{\mathcal{L}}$ is linearly independent over K.

Proof. If the Lie algebra $\mathcal{L}(\mathcal{Z})$ is free, freely generated by the subset $\mathcal{Z}$ of $\mathcal{L}$ then it follows from [8, Theorem 5.8.] that the subset $\mathcal{HB}(\mathcal{Z})_{\mathcal{L}}$ of $\mathcal{L}$ is linearly independent.

If the subset $\mathcal{HB}(\mathcal{Z})_{\mathcal{L}}$ of $\mathcal{L}$ is linearly independent then the morphism of Lie algebras $\mathcal{P}Lie(Z_i \mid i \in \mathbb{N}) \to \mathcal{L}(\mathcal{Z})$ is an isomorphism. Hence the Lie algebra $\mathcal{L}(\mathcal{Z})$ is free, freely generated by $\mathcal{Z}$. $\square$

In the next proposition we indicate how to construct various free Lie subalgebras of the Lie algebra $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$.

Proposition 2.8. Let $S \subset \{0, 1, \ldots, n-1\}$ be such that $(S + S) \cap S = \emptyset$. Let $z_s^k \in Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$ for $s \in S$ and $k = 1, 2, \ldots$ be such that

$$z_s^k \equiv [Y_s, X^{(k-1)}] \mod I_2. \tag{2.8.1}$$

Then the elements z_s^k for $s \in S$ and $k \in \mathbb{N}$ generate freely a free Lie subalgebra of $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$.

Proof. Let us set $y_s^k := [Y_s, X^{(k-1)}]$. Let $\mathcal{Z} := \{z_s^k \mid s \in S, k \in \mathbb{N}\}$ and $\mathcal{Y} := \{y_s^k \mid s \in S, k \in \mathbb{N}\}$. Observe that the subset $\mathcal{Y}$ of $Lie(X, Y_0, \ldots, Y_{n-1})$ is linearly independent. It follows from the congruences (2.8.1) that the subset $\mathcal{Z}$ is also linearly independent.

The Lie subalgebra of $Lie(X, Y_0, \ldots, Y_{n-1})$ generated by $\mathcal{Y}$ is free, freely generated by $\mathcal{Y}$. Hence Lemma 2.7 implies that the subset $\mathcal{HB}(\mathcal{Y})_{Lie(X,Y_0,\ldots,Y_{n-1})}$ of $Lie(X, Y_0, \ldots, Y_{n-1})$ is linearly independent.

It follows from the congruences (2.8.1) that for any arrangements of brackets of length m

$$\{..\{z_{s_1}^{k_1}, z_{s_2}^{k_2}\}.., z_{s_m}^{k_m}\} \equiv \{..\{y_{s_1}^{k_1}, y_{s_2}^{k_2}\}.., y_{s_m}^{k_m}\} \mod I_{m+1}. \tag{2.8.2}$$

Observe that

$$\{y_{s_1}^{k_1}, y_{s_2}^{k_2}\} \equiv [y_{s_1}^{k_1}, y_{s_2}^{k_2}] \mod I_2(S).$$

It follows by induction from Corollary 2.5 that for any arrangements of brackets of length m

$$\{..\{y_{s_1}^{k_1}, y_{s_2}^{k_2}\}.., y_{s_m}^{k_m}\} \equiv [..[y_{s_1}^{k_1}, y_{s_2}^{k_2}].., y_{s_m}^{k_m}] \mod I_m(S). \tag{2.8.3}$$

The set $\mathcal{HB}(\mathcal{Y})_{Lie(X,Y_0,\ldots,Y_{n-1})}$ is linearly independent. Hence it follows from the congruences (2.8.3) that the set $\mathcal{HB}(\mathcal{Y})_{Lie(X,Y_0,\ldots,Y_{n-1})_{\{\ \}}}$ is linearly independent. Therefore it follows from the congruences (2.8.2) that the set $\mathcal{HB}(\mathcal{Z})_{Lie(X,Y_0,\ldots,Y_{n-1})_{\{\ \}}}$ is linearly independent.

It follows from Lemma 2.7 that the elements of $\mathcal{Z}$ generate freely a free Lie subalgebra of $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}$. □

3 Galois action on the fundamental group of the projective line minus 0, ∞ and the nth roots of 1.

From now on let $V := \mathbb{P}^1_{\mathbb{Q}(\mu_n)} \setminus (\{0, \infty\} \cup \mu_n)$. The generators $x, y_0, \ldots, y_{n-1}$ of $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$ we choose as in [11]. Let p be the standard path from $\overrightarrow{01}$ to $\overrightarrow{10}$ on $V_{\bar{\mathbb{Q}}}$. The Galois group $G_{\mathbb{Q}(\mu_n)}$ acts on $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$. The action is described in the next proposition.

Proposition 3.1. (see [11, Proposition 15.17]). Let $\sigma \in G_{\mathbb{Q}(\mu_n)}$. We have

$$\sigma(x) = x^{\chi(\sigma)},\ \sigma(y_0) = \mathfrak{f}_p(\sigma)(x, y_0, \ldots, y_{n-1})^{-1} \cdot y_0^{\chi(\sigma)} \cdot \mathfrak{f}_p(\sigma)(x, y_0, \ldots, y_{n-1}),$$

$$\sigma(y_k) = x^{\frac{-(\chi(\sigma)-1)k}{n}} \cdot \mathfrak{f}_p(\sigma)(x, y_k, \ldots, y_{n-1}, x^{-1}y_0x, \ldots, x^{-1}y_{k-1}x)^{-1} \cdot y_k^{\chi(\sigma)}$$
$$\cdot \mathfrak{f}_p(\sigma)(x, y_k, \ldots, y_{n-1}, x^{-1}y_0x, \ldots, x^{-1}y_{k-1}x) \cdot x^{\frac{(\chi(\sigma)-1)k}{n}},$$

As usual we embed $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$ into the $\mathbb{Q}_l$-algebra $\mathbb{Q}_l\{\{X, Y_0, \ldots, Y_{n-1}\}\}$ by the continuous multiplicative map k such that $k(x) = e^X$ and $k(y_i) = e^{Y_i}$.

The pair $(V, \overrightarrow{01})$ has good reduction outside the prime divisors of n. Hence we get a morphism of graded Lie algebras

$$\Phi_{V,\overrightarrow{01}} : L_l(\mathbb{Z}[\mu_n][\frac{1}{n}]) \to Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}.$$

Let us set $\xi_n := e^{\frac{2\pi\sqrt{-1}}{n}}$. The next result follows immediately from [11, Lemma 15.3.1].

Proposition 3.2. Let $k > 1$ and let $\sigma \in L_l(\mathbb{Z}[\mu_n][\frac{1}{n}])_k$. Then

$$\Phi_{V,\overrightarrow{01}}(\sigma) \equiv \sum_{i=0}^{n-1} l_k(\xi_n^{-i})(\sigma)[Y_i, X^{(k-1)}] \mod I_2.$$

Let $\sigma \in L_l(\mathbb{Z}[\mu_n][\frac{1}{n}])_1$. Then $\Phi_{V,\overrightarrow{01}}(\sigma) = \sum_{i=1}^{n-1} l(1 - \xi_n^{-i})(\sigma)Y_i$.

4 Projective line minus 0, ∞ and the 3^nth roots of 1.

Let $V := \mathbb{P}^1_{\mathbb{Q}(\mu_{3^n})} \setminus (\{0, \infty\} \cup \mu_{3^n})$. The action of $G_{\mathbb{Q}(\mu_{3^n})}$ on $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$ induces the morphism of graded Lie algebras

$$\Phi_{V,\overrightarrow{01}} : L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}]) \to Lie(X, Y_0, \ldots, Y_{3^n-1})_{\{,\}}.$$

The 3-units $(1-\xi_{3^n}^i)$ for $0 < i < \frac{3^n}{2}$ and $(i, 3) = 1$ generate freely $(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])^\times \otimes \mathbb{Q}$. Hence the elements dual to the Kummer characters $l(1 - \xi_{3^n}^i)$ for $0 < i < \frac{3^n}{2}$ and $(i, 3) = 1$ generate freely $L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])_1$. Let us set

$$\sigma_i^{(1)} := l(1 - \xi_{3^n}^i)^\diamond.$$

For further applications we introduce the following convention

$$\sigma_{i+3^n}^{(1)} = \sigma_i^{(1)} \text{ and } \sigma_{3^n-i}^{(1)} = \sigma_i^{(1)}.$$

There are the following relations between 3-units. For any $0 < i < \frac{3^n}{2}$, $(i, 3) = 1$ and $k = n, n-1, \ldots, 2$ we have

$$(1 - \xi_{3^k}^i) \cdot (1 - \xi_{3^k}^{i+3^{k-1}}) \cdot (1 - \xi_{3^k}^{i+2\cdot 3^{k-1}}) = (1 - \xi_{3^{k-1}}^i). \quad (4.0)$$

and

$$(1 - \xi_{3^k}^i) = -\xi_{3^k}^i (1 - \xi_{3^k}^{-i}). \quad (4.0)$$

Lemma 4.1. We have

$$\Phi_{V,\vec{01}}(\sigma_i^{(1)}) = Y_i + Y_{3^n-i} + \sum_{k=1}^{n-1}(Y_{3^k i} + Y_{3^n-3^k i})$$

for $0 < i < \frac{3^n}{2}$ and $(i,3) = 1$.

Proof. The lemma follows from Proposition 3.2 and the equalities (4.0). $\square$

Let $k \geq 2$. Then we have

$$3^{k-1} \cdot (l_k(\xi_{3^j}^i) + l_k(\xi_{3^j}^{i+3^{j-1}}) + l_k(\xi_{3^j}^{i+2\cdot 3^{j-1}})) = l_k(\xi_{3^{j-1}}^i) \tag{4.2}$$

for $0 < i < \frac{3^n}{2}$, $(i,3) = 1$ and $j = n, n-1, \ldots, 2$. Hence it follows from the distribution relation $3^{n(k-1)}(\sum_{i=0}^{3^n-1} l_k(\xi_{3^n}^i)) = l_k(1)$ that

$$3^{n(k-1)}(\sum_{i=1}^{3^n-1}\sum_{(3,i)=1} d_i l_k(\xi_{3^n}^i)) = (1 - 3^{n(k-1)})l_k(1) \tag{4.3}$$

for certain $d_i \in \mathbb{Z}$.

In degree $k > 1$ the graded Lie algebra $L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])$ has 3^{n-1} free generators. Let us denote by $\sigma_i^{(k)}$ these generators. We choose them to be dual to l-adic polylogarithms $l_k(\xi_{3^n}^i)$, with $0 < i < \frac{3^n}{2}$ and $(i,3) = 1$, in the sense that $l_k(\xi_{3^n}^j)(\sigma_i^{(k)}) = \delta_i^j$.

In the next lemma we will consider an element x which is congruent modulo I_2 to some y, whose coefficients belong to the ring $\mathbb{Z}_{(3)}$. Hence we can reduce them further modulo 3. This procedure we shall denote by mod I_2 mod 3.

Lemma 4.4. Let $k \geq 2$. We have

$$\Phi_{V,\vec{01}}(\sigma_i^{(k)}) \equiv -[Y_i, X^{(k-1)}] + [Y_{3^n-i}, X^{(k-1)}] \mod I_2 \mod 3$$

for k even and

$$\Phi_{V,\vec{01}}(\sigma_i^{(k)}) \equiv [Y_i, X^{(k-1)}] + [Y_{3^n-i}, X^{(k-1)}] \mod I_2 \mod 3$$

for k odd.

Proof. The proof follows from Proposition 3.2, the equalities (4.2) and (4.3) and the definition of the generators $\sigma_i^{(k)}$. $\square$

Unfortunately we are not able to show that the morphism

$$\Phi_{V,\vec{01}} : L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}]) \to Lie(X, Y_0, \ldots, Y_{3^n-1})_{\{,\}}$$

is injective. We shall define a certain Lie subalgebra of $L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])$ and we shall show that the map $\Phi_{V,\vec{01}}$ restricted to this subalgebra is injective.

Definition 4.5. Let $\mathcal{L}_{3^n}$ be a Lie algebra of $L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])$ generated by differences

$$\sigma_i^{(1)} - \sigma_{i+2\cdot 3^{n-1}}^{(1)},\ \sigma_{i+3^{n-1}}^{(1)} - \sigma_{i+2\cdot 3^{n-1}}^{(1)}$$

for $0 < i < \frac{3^n}{2\cdot 3}$ and $(i,3) = 1$ and by elements $\sigma_j^{(k)}$ for $k > 1$, $0 < j < \frac{3^n}{2}$ and $(j,3) = 1$.

Notice that $\mathcal{L}_{3^n}$ is a free Lie algebra, freely generated by the elements indicated in the definition.

Theorem 4.6. The morphism of graded Lie algebras

$$\Phi_{V,\vec{01}} : L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}]) \to Lie(X, Y_0, \dots, Y_{3^n-1})_{\{,\}}$$

restricted to $\mathcal{L}_{3^n}$ is injective.

Proof. Let L_{3^n} be a Lie subalgebra of $Lie(X, Y_0, \dots, Y_{3^n-1})_{\{,\}}$ generated by elements $z_i^{(1)} := Y_i + Y_{3^n-i} - Y_{i+2\cdot 3^{n-1}} - Y_{3^n-(i+2\cdot 3^{n-1})}$, $z_{i+3^{n-1}}^{(1)} := Y_{i+3^{n-1}} + Y_{3^n-(i+3^{(n-1)})} - Y_{i+2\cdot 3^{n-1}} - Y_{3^n-(i+2\cdot 3^{n-1})}$ with $0 < i < \frac{3^n}{2\cdot 3}$ and $(i,3) = 1$ and by elements

$$z_i^{(k)} := (-1)^{k-1}[Y_i, X^{(k-1)}] + [Y_{3^n-i}, X^{(k-1)}]$$

for $k > 1$ and $0 < i < \frac{3^n}{2}$ and $(i,3) = 1$. Let us denote by $\mathcal{Z}$ the set of these elements.

Observe that if $(i,3) = 1$ then of the two numbers i and $3^n - i$, one is congruent to 1 modulo 3 and the other, to 2 modulo 3.

Let $k > 1$. We shall write

$$z_i^{(k)} = y_i^{(k)} + d_i^{(k)},$$

where $y_i^{(k)} = (-1)^{k-1}[Y_i, X^{(k-1)}]$ and $d_i^{(k)} = [Y_{3^n-i}, X^{(k-1)}]$ if $i \equiv 1$ (3) and vice versa if $i \equiv 2$ (3). Similarly we decompose

$$z_i^{(1)} = y_i^{(1)} + d_i^{(1)},$$

where in $y_i^{(1)}$ appear Y_α's with $\alpha \equiv 1$ (3) and in $d_i^{(1)}$, Y_β's with $\beta \equiv 2$ (3). Let $\mathcal{Y}$ be the set of all elements $y_i^{(k)}$.

Let $I(0,2)$ be a Lie ideal of $Lie(X, Y_0, \dots, Y_{3^n-1})$ generated by Y_t's with $t \equiv 2$ (3) and by Y_0.

We shall show by induction the following two statements:
A(r): for any $r \geq 2$ and any Lie bracket of length r and any r elements of $\mathcal{Z}$, $z_1 = y_1 + d_1, \dots, z_r = y_r + d_r$ we have

(4.6.1) $$\{..\{z_1, z_2\}.., z_r\} \equiv [..[y_1, y_2].., y_r] \mod I(0,2),$$

B(r): a Lie bracket only with Y_α's such that $\alpha \equiv 0$ (3) does not appear in the decomposition of $\{..\{z_1, z_2\}.., z_r\}$.

For any two elements of the set $\mathcal{Z}$, $z = y + d$ and $z_1 = y_1 + d_1$ we have

$$\{z, z_1\} \equiv [y, y_1] \mod I(0, 2).$$

Observe also that a Lie bracket only with Y_α's such that $\alpha \equiv 0 \bmod 3$ does not appear in the decomposition of $\{z, z_1\}$ in the Hall base of $Lie(X, Y_0, \ldots, Y_{3^n-1})$. Hence **A(2)** and **B(2)** are true.

Let us assume that the statements are proved for m. Let $1 \leq r \leq m$ and $1 \leq s \leq m$. For r we denote by Z the left hand side of (4.6.1) and by Y the right side. Therefore we can write $Z = Y + D$ with $D \in I(0, 2)$. Similarly for s we can write $Z_1 = Y_1 + D_1$. Then

(4.6.2)

$$\{Z, Z_1\} = [Y+D, Y_1+D_1]+D_Y(Y_1)+D_Y(D_1)+D_D(Y_1)+D_D(D_1)-D_{Y_1+D_1}(Y+D).$$

Observe that $D_Y(Y_1) \in I(0, 2)$, $D_Y(D_1) \in I(0, 2)$ and $D_D(D_1) \in I(0, 2)$. The assumption **B(r)** implies that $D_D(Y_1) \in I(0, 2)$ Hence

$$\{Z, Z_1\} \equiv [Y, Y_1] \mod I(0, 2).$$

Therefore we have proved **A(m+1)**. From the form of all terms in (4.6.2) it is clear that **B(m+1)** is true.

The subset $\mathcal{Y}$ of $Lie(X, Y_0, \ldots, Y_{3^n-1})$ is linearly independent hence the Lie algebra $Lie(X, Y_0, \ldots, Y_{3^n-1})(\mathcal{Y})$ is free, freely generated by $\mathcal{Y}$. Therefore the set $\mathcal{HB}(\mathcal{Y})_{Lie(X,Y_0,\ldots,Y_{3^n-1})}$ is linearly independent. It follows from the congruences (4.6.1) that the set $\mathcal{HB}(\mathcal{Z})_{Lie(X,Y_0,\ldots,Y_{3^n-1})_{\{\,,\}}}$ is also linearly independent.

Observe that all this is true over any field K hence it is true over the field $\mathbb{F}_3 = \mathbb{Z}/3$.

Let us set $\zeta_i^{(k)} := \Phi_{V,\vec{01}}(\sigma_i^{(k)})$ for $k > 1$. For $k = 1$ we set $\zeta_i^{(1)} := \Phi_{V,\vec{01}}(\sigma_i^{(1)} - \sigma_{i+2\cdot 3^{n-1}}^{(1)}) = z_i^{(1)}$ and $\zeta_{i+3^{n-1}}^{(1)} := \Phi_{V,\vec{01}}(\sigma_{i+3^{n-1}}^{(1)} - \sigma_{i+2\cdot 3^{n-1}}^{(1)}) = z_{i+3^{n-1}}^{(1)}$ for $0 < i < \frac{3^n}{2\cdot 3}$ and $(i, 3) = 1$. Lemma 4.4 implies

$$\zeta_i^{(k)} \equiv z_i^{(k)} \mod I_2 \mod 3$$

for $k > 1$. We have also

$$\zeta_i^{(1)} = z_i^{(1)}.$$

Let Σ be the set of all $\zeta_i^{(k)}$. Let $\zeta_1, \ldots, \zeta_r \in \Sigma$. Let $z_1, \ldots, z_r \in \mathcal{Z}$ be such that $\zeta_i \equiv z_i \bmod I_2 \bmod 3$. Then for any Lie bracket of length r

(4.6.3) $$\{..\{\zeta_1, \zeta_2\}.., \zeta_r\} \equiv \{..\{z_1, z_2\}.., z_r\} \mod I_2 \mod 3.$$

The set $\mathcal{HB}(\mathcal{Z})_{Lie(X,Y_0,\ldots,Y_{3^n-1})_{\{\,,\}}}$ is linearly independent in the Lie algebra over $\mathbb{Z}/3$, hence also over $\mathbb{Q}$ and $\mathbb{Q}_l$. Hence it follows from the congruences (4.6.3) that the set $\mathcal{HB}(\Sigma)_{Lie(X,Y_0,\ldots,Y_{3^n-1})_{\{\,,\}}}$ is linearly independent. Lemma 2.7 implies that the set Σ generates freely a free Lie subalgebra. Hence the morphism $\Phi_{V,\vec{01}}$ restricted to $\mathcal{L}_{3^n}$ is injective. □

5 Projective line minus 0, ∞ and the 2^nth roots of 1.

Let $V := \mathbb{P}^1_{\mathbb{Q}(\mu_{2^n})} \setminus (\{0,\infty\} \cup \mu_{2^n})$. The action of $G_{\mathbb{Q}(\mu_{2^n})}$ on $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$ induces the morphism of graded Lie algebras

$$\Phi_{V,\overrightarrow{01}} : L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}]) \to Lie(X, Y_0, \ldots, Y_{2^n-1})_{\{,\}}.$$

The Lie algebra $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])$ has 2^{n-2} free generators in each degree. Generators in degree 1 are constructed in the following way.

The 2-units $(1-\xi^i_{2^n})$ for $0 < i < 2^{n-1}$ and $(i,2) = 1$ generate freely $(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])^\times \otimes \mathbb{Q}$. Hence the elements dual to the Kummer characters $l(1-\xi^i_{2^n})$ for $0 < i < 2^{n-1}$ and $(i,2) = 1$ generate freely $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])_1$. Let us set

$$\sigma_i^{(1)} := l(1-\xi^i_{2^n})^\diamond$$

for $0 < i < 2^{n-1}$ and $(i,2) = 1$. We shall also use the convention

$$\sigma_i^{(1)} = \sigma^{(1)}_{2^n-i} = \sigma^{(1)}_{2^n+i}.$$

Observe that we have the following equalities

$$(5.1) \qquad (1-\xi^1_4)\cdot(1-\xi^3_4) = 2, \;\; (1-\xi^i_{2^n}) = -\xi^i_{2^n}\cdot(1-\xi^{-i}_{2^n})$$

and

$$(5.2) \qquad (1-\xi^i_{2^k})\cdot(1-\xi^{i+2^{k-1}}_{2^k}) = (1-\xi^i_{2^{k-1}})$$

for $0 < i < 2^{n-1}$, $(i,2) = 1$ and $k = n, n-1, \ldots, 3$. Proposition 3.1 and the equalities (5.1) and (5.2) imply that

$$(5.3) \qquad \Phi_{V,\overrightarrow{01}}(\sigma_i^{(1)}) = Y_i + Y_{2^n-i} + \sum_{a=1}^{n-2}(Y_{2^a i} + Y_{2^n-2^a i}) + 2Y_{2^{n-1}}.$$

Generators of $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])$ in degree $k > 1$ we denote by $\sigma_i^{(k)}$ and we choose them dual to l-adic polylogarithms $l_k(\xi^j_{2^n})$, $0 < j < 2^{n-1}$ and $(j,2) = 1$, in the sense that

$$l_k(\xi^j_{2^n})(\sigma_i^{(k)}) = \delta^j_i.$$

The inversion relations

$$l_k(\xi^j_{2^n}) + (-1)^k l_k(\xi^{-j}_{2^n}) = 0$$

and the distribution relations

$$(2^n)^{k-1}\cdot\sum_{i=0}^{2^n-1} l_k(\xi^i_{2^n}) = l_k(1)$$

and

$$2^{k-1}(l_k(\xi_{2^j}^i)+l_k(\xi_{2^j}^{i+2^{j-1}}))=l_k(\xi_{2^{j-1}}^i)$$

for $j=n,n-1,\dots,2$ and $(i,2)=1$ imply that

$$\Phi_{V,\overrightarrow{01}}(\sigma_i^{(k)})\equiv[Y_i,X^{(k-1)}]+[Y_{2^n-i},X^{(k-1)}] \mod I_2 \mod 2. \tag{5.4}$$

Definition 5.5 Let $\mathcal{L}_{2^n}$ be a Lie subalgebra of $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])$ generated by differences

$$\sigma_i^{(1)}-\sigma_{-i+2^{n-1}}^{(1)},\ \ \sigma_{i+2^{n-2}}^{(1)}-\sigma_{-i+2^{n-1}}^{(1)},\ \ \sigma_{-i+2^{n-2}}^{(1)}-\sigma_{-i+2^{n-1}}^{(1)}$$

for $0<i<2^{n-3}$ and $(i,2)=1$ and by the elements $\sigma_i^{(k)}$ for $k>1$ and $0<i<2^{n-1}$ and $(i,2)=1$.

(These four elements $\sigma_i^{(1)}$, $\sigma_{-i+2^{n-2}}^{(1)}$, $\sigma_{i+2^{n-2}}^{(1)}$ and $\sigma_{-i+2^{n-1}}^{(1)}$ are such that in their images by $\Phi_{V,\overrightarrow{01}}$ appear exactly the same Y_t's with $t\equiv 0$ modulo 4.)

Theorem 5.6. The morphism of graded Lie algebras

$$\Phi_{V,\overrightarrow{01}}:L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])\to Lie(X,Y_0,\dots,Y_{2^n-1})_{\{,\}}.$$

restricted to $\mathcal{L}_{2^n}$ is injective.

Proof. Let L_{2^n} be a Lie subalgebra of $Lie(X,Y_0,\dots,Y_{2^n-1})_{\{,\}}$ generated by elements

$$z_i^{(1)}:=Y_i+Y_{2^n-i}+Y_{2i}+Y_{2^n-2i}-(Y_{2^{n-1}-i}+Y_{2^n-2^{n-1}+i}+Y_{2^n-2i}+Y_{2i}),$$

$$z_{-i+2^{n-2}}^{(1)}:=Y_{2^{n-2}-i}+Y_{2^n-2^{n-2}+i}+Y_{2^{n-1}-2i}+Y_{2^n-2^{n-1}+2i}-(Y_{2^{n-1}-i}+Y_{2^n-2^{n-1}+i}+Y_{2^n-2i}+Y_{2i}),$$

$$z_{i+2^{n-2}}^{(1)}:=Y_{i+2^{n-2}}+Y_{2^n-2^{n-2}-i}+Y_{2i+2^{n-1}}+Y_{2^n-2^{n-1}-2i}-(Y_{2^{n-1}-i}+Y_{2^n-2^{n-1}+i}+Y_{2^n-2i}+Y_{2i}),$$

for $0<i<2^{n-3}$ and i odd and by elements

$$z_i^{(k)}:=[Y_i,X^{(k-1)}]+[Y_{2^n-i},X^{(k-1)}]$$

for $k>1$ and $0<i<2^{n-1}$ and i odd. Let us denote by $\mathcal{Z}$ the set of all these elements.

Observe that if i is odd then of the two numbers i and 2^n-i, one is congruent to 1 modulo 4 and the other, to 3 modulo 4.

Let us write

$$z_j^{(k)}=y_j^{(k)}+d_j^{(k)},$$

where in $y_j^{(k)}$ appear Y_α's with $\alpha\equiv 1$ modulo 4 and in $d_j^{(1)}$ appear only Y_β's with $\beta\equiv 3$ modulo 4. Let $\mathcal{Y}$ be the set of all elements $y_j^{(k)}$.

Let $I(0,2,3)$ be a Lie ideal of $Lie(X, Y_0, \dots, Y_{2^n-1})$ generated by Y_t's with $t \equiv \epsilon$ modulo 4, where $\epsilon \in \{0,2,3\}$.

One proves by induction the following two statements:

A(r): for any $r \geq 2$ and any Lie bracket of length r and any r elements of $\mathcal{Z}$, $z_1 = y_1 + d_1, \dots, z_r = y_r + d_r$ we have

$$\{..\{z_1, z_2\}.., z_r\} \equiv [..[y_1, y_2].., y_r] \mod I(0,2,3), \tag{5.6.1}$$

B(r): a Lie bracket only with Y_α's such that $\alpha \equiv 0$ modulo 4 does not appear in the decomposition of $\{..\{z_1, z_2\}.., z_r\}$.

One checks that **A(2)** and **B(2)** are true. The rest of the proof goes in the same way as the proof of Theorem 4.6. □

References

[1] P.Deligne, Le groupe fondamental de la droite projective moins trois points, *in* Galois Groups over Q (ed. Y.Ihara, K.Ribet and J.-P. Serre), *Mathematical Sciences Research Institute Publications*, **16** (1989), pp. 79-297.

[2] P. Deligne, Lecture on the conference in Schloss Ringberg, 1998.

[3] P. Deligne, Le Groupe fondamental de $\mathbb{G}_m \setminus \mu_N$, pour $N = 2, 3, 4, 6$ ou 8, http://www.math.ias.edu/people/faculty/deligne/preprints.

[4] R.Hain, M.Matsumoto, Weighted completion of Galois groups and Galois actions on the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, *in* Galois Groups and Fundamental Groups (ed. L. Schneps), *Mathematical Sciences Research Institute Publications* **41** (2003), pp. 183-216.

[5] R.Hain, M.Matsumoto, Tannakian Fundamental Groups Associated to Galois Groups, Compositio Mathematica **139**, No. 2, (2003), pp. 119-167.

[6] Y.Ihara, Profinite braid groups, Galois representations and complex multiplications, Annals of Math. 123 (1986), pp. 43-106.

[7] Y.Ihara, The Galois representations arising from $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and Tate twists of even degree, *in* Galois Groups over Q (ed. Y.Ihara, K.Ribet and J.-P. Serre), *Mathematical Sciences Research Institute Publications*, **16** (1989), pp. 299-313.

[8] W.Magnus, A.Karras, D.Solitar, Combinatorial Group Theory, Interscience Publishers, 1966.

[9] Z. Wojtkowiak, On ℓ-adic iterated integrals, I Analog of Zagier Conjecture, Nagoya Math. Journal, Vol. 176 (2004), 113-158.

[10] Z. Wojtkowiak, On ℓ-adic iterated integrals, II Functional equations and ℓ-adic polylogarithms, Nagoya Math. Journal, Vol. 177 (2005), 117-153.

[11] Z. Wojtkowiak, On ℓ-adic iterated integrals, III Galois actions on fundamental groups, Nagoya Math. Journal, Vol. 178 (2005), pp. 1-36.

[12] Z. Wojtkowiak, On ℓ-adic iterated integrals, IV Ramifications and generators of Galois actions on fundamental groups and on torsors of paths, Math. Journal of Okayama University, 51 (2009), pp. 47-69.

[13] Z. Wojtkowiak, Periods of mixed Tate motives, examples, l-adic side, Max-Planck-Institut für Mathematik, Preprint Series 2010(80), accepted in the Proceedings of Summer Schools 2008 and 2009 in the Galatasaray University, Istanbul.

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