Chapter 14

On ℓ-adic iterated integrals V: linear independence, properties of ℓ-adic polylogarithms, ℓ-adic sheaves

Zdzisław Wojtkowiak

Abstract In a series of papers we have introduced and studied ℓ-adic polylogarithms and ℓ-adic iterated integrals which are analogues of the classical complex polylogarithms and iterated integrals in ℓ-adic Galois realizations.

In this note we shall show that in the generic case ℓ-adic iterated integrals are linearly independent over $\mathbb{Q}_\ell$. In particular they are non trivial. This result can be viewed as analogous of the statement that the classical iterated integrals from 0 to z of sequences of one forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ are linearly independent over $\mathbb{Q}$.

We also study ramification properties of ℓ-adic polylogarithms and the minimal quotient subgroup of the absolute Galois group G_K of a number field K on which ℓ-adic polylogarithms are defined. In the final sections of the paper we study ℓ-adic sheaves and their relations with ℓ-adic polylogarithms. We show that if an ℓ-adic sheaf has the same monodromy representation as the classical complex polylogarithms then the action of G_K in stalks is given by ℓ-adic polylogarithms.

Key words: Galois group, polylogarithms, fundamental group

14.1 Introduction

In this paper we study properties of ℓ-adic iterated integrals and ℓ-adic polylogarithms introduced in [Wo04] and [Wo05a]. We describe briefly the main results of the paper, though in the introduction we do not present them in full generality.

Let K be a number field with algebraic closure $\bar{K}$. Throughout this paper we fix an embedding $\bar{K} \subset \mathbb{C}$. Let $z \in K \setminus \{0,1\}$ or let z be a tangential point of

---

Zdzisław Wojtkowiak
Université de Nice-Sophia Antipolis, Département de Mathématiques, Laboratoire Jean Alexandre Dieudonné, U.R.A. au C.N.R.S., No 168, Parc Valrose - B.P.N° 71, 06108 Nice Cedex 2, France
e-mail: wojtkow@unice.fr

$$\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}$$

defined over K, and let γ be an ℓ-adic path from $\overrightarrow{01}$ to z on $\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}$. For any $\sigma \in G_K = \mathrm{Gal}(\bar{K}/K)$ we set

$$f_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \ \in \ \pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, \overrightarrow{01})_{\text{pro}-\ell}.$$

Here and later our convention of composing a path α from y to z with a path β from x to y will be that $\alpha \cdot \beta$ is defined as a path from x to z.

Let V be an algebraic variety defined over K and let v be a K-point or a tangential point defined over K. By the comparison homomorphism

$$\pi_1(V(\mathbb{C}), v) \to \pi_1^{\text{ét}}(V_{\bar{K}}, v)_{\text{pro}-\ell}$$

any element of $\pi_1(V(\mathbb{C}), v)$ determines canonically an element of $\pi_1^{\text{ét}}(V_{\bar{K}}, v)_{\text{pro}-\ell}$, and we shall use the same notation for an element of $\pi_1(V(\mathbb{C}), v)$ and its image. In particular, we have the comparison homomorphism

$$\pi_1(U, \overrightarrow{01}) \to \pi_1^{\text{ét}}(\mathrm{Spec}\,\bar{K}((z)), \overrightarrow{01})_{\text{pro}-\ell},$$

where $U \subset \mathbb{C} \setminus \{0\}$ is a punctured infinitesimal neighbourhood of 0 and $\mathrm{Spec}\,\bar{K}((z))$ is an algebraic infinitesimal punctured neighbourhood of 0 in $\mathbb{P}^1_{\bar{K}}$. Hence a loop around 0 in $\mathbb{C} \setminus \{0\}$ determines canonically an element of $\pi_1^{\text{ét}}(\mathrm{Spec}\,\bar{K}((z)), \overrightarrow{01})$. Similarly we have the comparison map from the torsor of paths from v to z on $V(\mathbb{C})$ to the torsor of ℓ-adic paths from v to z on $V_{\bar{K}}$.

Informally, we define ℓ-adic iterated integrals from $\overrightarrow{01}$ to z as functions

$$l_b(z) = l_b(z)_\gamma \ : \ G_K \to \mathbb{Q}_\ell$$

given by coefficients of $f_\gamma(\ )$ indexed by elements b in a Hall basis $\mathscr{B}$ of the free Lie algebra $\mathrm{Lie}(X,Y)$ on two generators X and Y. Let $\mathscr{B}_n$ be the set of elements of degree n in $\mathscr{B}$. Let

$$H_n \subset G_{K(\mu_{\ell^\infty})}$$

be the subgroup of $G_{K(\mu_{\ell^\infty})}$ defined by the condition that all $l_b(z)$ and $l_b(\overrightarrow{10})$ vanish on H_n for all $b \in \bigcup_{i<n} \mathscr{B}_i$.

Our first result concerns linear independence of ℓ-adic iterated integrals.

Theorem 1. *Assume that $z \in K \setminus \{0,1\}$ is not a root of any equation of the form $z^p \cdot (1-z)^q = 1$, where p and q are integers such that $p^2+q^2 > 0$. Then the functions*

$$l_b(z) \ : \ H_n \to \mathbb{Q}_\ell$$

for $b \in \mathscr{B}_n$ are linearly independent over $\mathbb{Q}_\ell$.

Our next results concerns ℓ-adic polylogarithms. Hence we recall here their definition (see [Wo05a, Definition 11.0.1.]). Let x and y be the standard generators of $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, \overrightarrow{01})_{\text{pro}-\ell}$ (see for example [Wo05a, Picture 1 on page 126]). Let $\mathbb{Q}_\ell\{\{X,Y\}\}$ be the $\mathbb{Q}_\ell$-algebra of non-commutative formal power series in non-commutative variables X and Y. Let

$$E \;:\; \pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, \overrightarrow{01})_{\text{pro}-\ell} \to \mathbb{Q}_\ell\{\{X,Y\}\}$$

be a continuous multiplicative embedding of $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, \overrightarrow{01})_{\text{pro}-\ell}$ into the $\mathbb{Q}_\ell$-algebra of non-commutative formal power series $\mathbb{Q}_\ell\{\{X,Y\}\}$ given by

$$E(x) = \exp(X),$$

$$E(y) = \exp(Y).$$

The ℓ-adic polylogarithms $l_n(z)$ and the ℓ-adic logarithm $l(z)$ are defined as functions on $\sigma \in G_K$ by the coefficients of the following expansion

$$\log E(f_\gamma(\sigma)) = l(z)(\sigma)X + \sum_{n=1}^{\infty} l_n(z)(\sigma)YX^{n-1} + \ldots,$$

where only relevent terms on the right hand side are written. The ℓ-adic polylogarithms $l_n(z)$ and $l(z)$ depend on a choice of a path γ from $\overrightarrow{01}$ to z. If we want to indicate the dependence on a path γ we shall write $l_n(z)_\gamma$ and $l(z)_\gamma$. The function

$$l(z) : G_K \to \mathbb{Q}_\ell$$

takes its values in $\mathbb{Z}_\ell$ and agrees with the Kummer character $\kappa(z)$ associated to z (see [Wo05b, Proposition 14.1.0.]).

Our second result concerns the minimal quotient of G_K, on which the ℓ-adic polylogarithms $l_n(z)$ are defined and their ramification properties. For $z \in K \setminus \{0,1\}$ we consider the fields $K(\mu_{\ell^\infty})$ and $K(\mu_{\ell^\infty}, z^{1/\ell^\infty})$. Let

$$M(K(\mu_{\ell^\infty}, z^{1/\ell^\infty}))^{ab}_{\ell,1-z}$$

be the maximal pro-ℓ abelian extension of $K(\mu_{\ell^\infty}, z^{1/\ell^\infty})$ that is unramified outside ℓ and $1-z$.

Theorem 2. *Assume that $z \in K \setminus \{0,1\}$ is not a root of any equation of the form $z^p \cdot (1-z)^q = 1$, where p and q are integers such that $p^2+q^2 > 0$. Then we have:*

(1) The ℓ-adic polylogarithm

$$l_n(z) : G_K \to \mathbb{Q}_\ell$$

factors through the group

$$\text{Gal}(M(K(\mu_{\ell^\infty}, z^{1/\ell^\infty}))^{ab}_{\ell,1-z}/K).$$

(2) The ℓ-adic polylogarithm $l_n(z)$ ramifies only at prime factors of the fractional ideals

$$(\ell),\ (z),\ (1-z).$$

(3) The ℓ-adic polylogarithm $l_n(z)$ determines a non-trivial element in the group

$$\mathrm{Hom}\big(\mathrm{Gal}(M(K(\mu_{\ell^\infty},z^{1/\ell^\infty}))^{ab}_{\ell,1-z}/K(\mu_{\ell^\infty},z^{1/\ell^\infty})),\mathbb{Q}_\ell\big).$$

Our third result connects ℓ-adic polylogarithms to non-abelian Iwasawa theory though we are not sure if our terminology of non-abelian Iwasawa theory is not an exaggeration, since the result is quite elementary. Let us set

$$\mathscr{G}=\mathrm{Gal}\big(M(K(\mu_{\ell^\infty},z^{1/\ell^\infty}))^{ab}_{\ell,1-z}/K(\mu_{\ell^\infty},z^{1/\ell^\infty})\big)$$

and

$$\varPhi=\mathrm{Gal}(K(\mu_{\ell^\infty},z^{1/\ell^\infty})/K).$$

The Galois group $\mathscr{G}$ is a $\varPhi$-module, hence it is also a $\mathbb{Z}_\ell[[\varPhi]]$-module. Therefore $\mathrm{Hom}(\mathscr{G},\mathbb{Q}_\ell)$ is also a $\mathbb{Z}_\ell[[\varPhi]]$-module. If $f\in\mathrm{Hom}(\mathscr{G},\mathbb{Q}_\ell)$ and $\mu\in\mathbb{Z}_\ell[[\varPhi]]$ then we denote by f^μ the element f acted (multiplied) by μ.

Let $\chi:G_K\to\mathbb{Z}_\ell^\times$ denote the ℓ-adic cyclotomic character. Observe that χ and $l(z)$ are continuous functions on $\varPhi$, hence we can integrate them against the measure μ.

Theorem 3. *Let z belong to $K\setminus\{0,1\}$. Then we have*

$$(l_m(z))^\mu=\Big(\int_\varPhi\chi^m(x)d\mu\Big)l_m(z)+\sum_{k=1}^{m-1}\Big(\int_\varPhi\frac{(-l(z)(x))^k}{k!}\chi^{m-k}(x)d\mu\Big)l_{m-k}(z)$$

for any $\mu\in\mathbb{Z}_\ell[[\varPhi]]$.

In the final sections of the paper we study ℓ-adic sheaves. We shall show that if an ℓ-adic sheaf has the same monodromy representation as the classical complex polylogarithm then the Galois action in stalks is given by the ℓ-adic polylogarithms.

We say a few words about our terminology and our notation. The functions $l_n(z)$, $l(z)$ and $l_b(z)$ appear exactly at the same place in our studies as the classical complex polylogarithms, the logarithm and iterated integrals when calculating sections of the universal prounipotent connection on $\mathbb{P}^1(\mathbb{C})$ minus a finite number of points. Moreover the equation $l(xy)=l(x)+l(y)$, the fact that $l(z):G_K\to\mathbb{Z}_\ell(1)$ is a cocycle, functional equations of $l_n(z)$ and $l_b(z)$, the fact that $l_n(\xi)$ is a cocycle for ξ a root of unity, the precise value of $l_{2n}(\overrightarrow{10})$ are proved using geometry (see [Wo05a], [Wo05b] and [Wo09]). Geometry is also used to calculate them explicitely (see [NW02] for ℓ-adic polylogarithms and our work in progress for arbitrary ℓ-adic iterated integrals).

14.1.1 Notation

The tensor product $(-)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ on $\mathbb{Z}_\ell$-modules we shall denote by $(-)\otimes\mathbb{Q}_\ell$.

14.2 The projective line minus 0, ∞ and n-th roots of unity

In this section we recall some elementary results concerning Galois actions on fundamental groups in the special case of

$$\mathbb{P}^1_{\mathbb{Q}(\mu_n)} \setminus (\{0,\infty\} \cup \mu_n),$$

see [Wo05b] and [DW04]. Let us fix a rational prime ℓ. Let K be a number field containing the group μ_n of n-th roots of unity. We abbreviate

$$V = \mathbb{P}^1_K \setminus (\{0,\infty\} \cup \mu_n)$$

and denote by

$$\pi_1(V_{\bar{K}}, \overrightarrow{01})$$

the pro-ℓ completion of the étale fundamental group of $V_{\bar{K}}$ based at $\overrightarrow{01}$.

First we describe how to choose generators of $\pi_1(V_{\bar{K}}, \overrightarrow{01})$. We fix the primitive n-th root of unity

$$\xi := \exp(\frac{2\pi i}{n})$$

using the fixed embedding $\bar{K} \subset \mathbb{C}$. Let β be the standard path from $\overrightarrow{01}$ to $\overrightarrow{10}$. Let x be a loop around 0 in the counterclockwise direction based at $\overrightarrow{01}$ in an infinitesimal neighbourhood of 0 and such that the integral along x of the one-form dz/z is $2\pi i$. Let y_0' be a loop around 1 based at $\overrightarrow{10}$ in an infinitesimal neighbourhood of 1 defined in the analogous way. Let s_k be a path from $\overrightarrow{01}$ to $\overrightarrow{0\xi^k}$ in an infinitesimal neibourhood of 0 as in [Wo05b, Picture 2, page 20]. Let $r_k : V \to V$ be the automorphism given by

$$r_k(z) = \xi^k \cdot z,$$

and set

$$y_k = \begin{cases} \beta^{-1} \cdot y_0' \cdot \beta & \text{for } k = 0, \\ s_k^{-1} \cdot r_{k,*}(y_0) \cdot s_k & \text{for } 0 < k < n. \end{cases}$$

Then the elements

$$x, y_0, y_1, \ldots, y_{n-1}$$

are free generators of $\pi_1(V_{\bar{K}}, \overrightarrow{01})$. Observe that

$$s_k^{-1} \cdot r_{k,*}(y_j) \cdot s_k = \begin{cases} y_{j+k} & \text{for } j+k < n, \\ x^{-1} \cdot y_{j+k} \cdot x & \text{for } j+k \geq n. \end{cases} \tag{14.1}$$

Let z be either a K-rational point $z \in V(K)$ or a tangential point defined over K. Let γ be an ℓ-adic path from $\overrightarrow{01}$ to z on $V_{\bar{K}}$. For every $\sigma \in G_K$, the element

$$f_\gamma(\sigma) = \gamma^{-1} \cdot \sigma(\gamma)$$

is a pro-ℓ word in the generators of $\pi_1(V_{\bar{K}},\overrightarrow{01})$, hence we shall write

$$f_\gamma(\sigma) = f_\gamma(\sigma)(x,y_0,\dots,y_{n-1}).$$

Observe that $r_{k,*}(\gamma)\cdot s_k$ is a path from $\overrightarrow{01}$ to $\xi^k z$ and by (14.1)

$$f_{r_{k,*}(\gamma)\cdot s_k}(\sigma) = s_k^{-1}\cdot r_{k,*}(\gamma^{-1})\cdot\sigma(r_{k,*}(\gamma)\cdot s_k) = s_k^{-1}\cdot r_{k,*}(f_\gamma(\sigma))\cdot s_k\cdot f_{s_k}(\sigma)$$

$$= f_\gamma(\sigma)(x,y_k,y_{k+1},\dots,y_{n-1},x^{-1}y_0x,\dots,x^{-1}y_{k-1}x)\cdot x^{\frac{k(\chi(\sigma)-1)}{n}}. \tag{14.2}$$

Let

$$E\ :\ \pi_1(V_{\bar{K}},\overrightarrow{01})\to\mathbb{Q}_\ell\{\{X,Y_0,\dots,Y_{n-1}\}\}$$

be the continuous multiplicative embedding of $\pi_1(V_{\bar{K}},\overrightarrow{01})$ into the $\mathbb{Q}_\ell$-algebra of non-commutative formal power series $\mathbb{Q}_\ell\{\{X,Y_0,\dots,Y_{n-1}\}\}$ given by

$$E(x) = \exp(X),$$

$$E(y_j) = \exp(Y_j)$$

for $0\le j<n$. Let

$$\pi_1(V_{\bar{K}};z,\overrightarrow{01})$$

be the right $\pi_1(V_{\bar{K}},\overrightarrow{01})$-torsor of ℓ-adic paths from $\overrightarrow{01}$ to z. The map

$$\delta\to\gamma^{-1}\cdot\delta$$

defines a bijection as right torsors

$$t_\gamma\ :\ \pi_1(V_{\bar{K}};z,\overrightarrow{01})\to\pi_1(V_{\bar{K}},\overrightarrow{01}).$$

Composing t_γ with the embedding E we get an embedding

$$E_\gamma\ :\ \pi_1(V_{\bar{K}};z,\overrightarrow{01})\to\mathbb{Q}_\ell\{\{X,Y_0,\dots,Y_{n-1}\}\}.$$

The Galois group G_K acts on $\pi_1(V_{\bar{K}},\overrightarrow{01})$ and on $\pi_1(V_{\bar{K}};z,\overrightarrow{01})$ compatible with the torsor structure. Hence we get two Galois representations

$$\varphi_{\overrightarrow{01}}\ :\ G_K\to\mathrm{Aut}(\mathbb{Q}_\ell\{\{X,Y_0,\dots,Y_{n-1}\}\}),$$

$$\psi_\gamma\ :\ G_K\to\mathrm{GL}(\mathbb{Q}_\ell\{\{X,Y_0,\dots,Y_{n-1}\}\})$$

deduced from the action of G_K on $\pi_1(V_{\bar{K}},\overrightarrow{01})$ via the embedding E and on the torsor of paths $\pi_1(V_{\bar{K}};z,\overrightarrow{01})$ via the embedding E_γ respectively (see [Wo04, section 4] and also [Wo07, section 1]).

Before going farther we fix the following notation. The set of **Lie polynomials** in $\mathbb{Q}_\ell\{\{X,Y_0,\dots,Y_{n-1}\}\}$ we denote by

$$\mathrm{Lie}(X, Y_0, \dots, Y_{n-1}).$$

It is a free Lie algebra on $n+1$ generators $X, Y_0, \dots, Y_{n-1}$. The set of **formal Lie power series** in $\mathbb{Q}_\ell\{\{X, Y_0, \dots, Y_{n-1}\}\}$ we denote by

$$\mathrm{L}(X, Y_0, \dots, Y_{n-1}).$$

We denote by

$$I_2$$

the closed Lie ideal of $\mathrm{L}(X, Y_0, \dots, Y_{n-1})$ generated by Lie brackets with two or more Y's. We shall use the following inductively defined short hand notation

$$[Y_k, X^{(m)}] = \begin{cases} Y_k & \text{if } m = 0 \\ [[Y_k, X^{(m-1)}], X] & \text{for } m > 0. \end{cases}$$

In an algebra the operator of the left (resp. right) multiplication by a we denote by L_a (resp. R_a).

We recall the definition of **ℓ-adic iterated integrals** from [Wo04]. Let $\mathscr{B}$ be a Hall base of the free Lie algebra $\mathrm{Lie}(X, Y_0, \dots, Y_{n-1})$ on $n+1$ free generators $X, Y_0, \dots, Y_{n-1}$ and let $\mathscr{B}_m$ be the set of elements of degree m in $\mathscr{B}$. For $b \in \mathscr{B}$ we define ℓ-adic iterated integrals

$$l_b(z) \; : \; G_{K(\mu_{\ell^\infty})} \to \mathbb{Q}_\ell$$

as follows. For $\sigma \in G_{K(\mu_{\ell^\infty})}$ the expression $(\log \psi_\gamma(\sigma))(1)$ is a Lie element, hence

$$(\log \psi_\gamma(\sigma))(1) = \sum_{b \in \mathscr{B}} l_b(z)(\sigma) \cdot b.$$

More naively, for $\sigma \in G_K$ we define functions

$$li_b(z) \; : \; G_K \to \mathbb{Q}_\ell$$

by the equality

$$\log \Lambda_\gamma(\sigma) = \sum_{b \in \mathscr{B}} li_b(z)(\sigma) \cdot b, \tag{14.3}$$

where

$$\Lambda_\gamma(\sigma) := E(f_\gamma(\sigma)). \tag{14.4}$$

If $n = 1$ and $Y = Y_0$ then the formula from the introduction defining ℓ-adic polylogarithms has the form

$$\log \Lambda_\gamma(\sigma) \equiv l(z)(\sigma) X + \sum_{n=1}^{\infty} l_n(z)(\sigma)[Y, X^{(n-1)}] \mod I_2.$$

With the representations $\varphi_{\overrightarrow{01}}$ and ψ_γ there are associated the filtrations

$$\{G_m = G_m(V, \overrightarrow{01})\}_{m \in N}$$

$$\{H_m = H_m(V; z, \overrightarrow{01})\}_{m \in N}$$

of G_K (see [Wo04], Section 3, pp. 122-124). We recall that

$$H_m = \left\{ \sigma \in G_{K(\mu_{\ell^\infty})} \;\middle|\; \begin{array}{c} l_b(z)(\sigma) = 0 \text{ and } l_b(\xi^k)(\sigma) = 0 \\ \text{for } 0 \le k < n \text{ and for all } b \in \bigcup_{i<m} \mathscr{B}_i \end{array} \right\}.$$

If $b \in \mathscr{B}_m$ and $\sigma \in H_m$ then $l_b(z)(\sigma) = li_b(z)(\sigma)$.

Let L be a Lie algebra. The Lie ideals of the lower central series are defined recursively by

$$\Gamma^m L = \begin{cases} L & \text{if } m = 1 \\ [\Gamma^{m-1} L, L] & \text{for } m > 1. \end{cases}$$

Proposition 4. *For* $\sigma \in H_m(V; z, \overrightarrow{01})$ *we have*

$$(\log \psi_\gamma(\sigma))(1) \equiv \log \Lambda_\gamma(\sigma) \equiv \Lambda_\gamma(\sigma) - 1 \mod \Gamma^{m+1} \mathrm{L}(X, Y_0, \ldots, Y_{n-1}). \tag{14.5}$$

Proof. It follows from [Wo04, Lemma 1.0.2.] that

$$\psi_\gamma(\sigma) = L_{\Lambda_\gamma(\sigma)} \circ \varphi_{\overrightarrow{01}}(\sigma).$$

After taking logarithm and applying the Baker-Campbell-Hausdorff formula we get the first congruence of the proposition. The second congruence is clear. □

Let us set

$$\gamma_k := r_{k,*}(\gamma) \cdot s_k. \tag{14.6}$$

Our next result is a consequence of the formula (14.2).

Proposition 5. *Let* $\sigma \in H_m(V; z, \overrightarrow{01})$*. Then*

$$\log(\Lambda_{\gamma_k}(\sigma)(X, Y_0, \ldots, Y_{n-1})) \equiv \log(\Lambda_\gamma(\sigma)(X, Y_k, \ldots, Y_{n-1}, Y_0, \ldots, Y_{k-1}))$$

modulo $\Gamma^{m+1} \mathrm{L}(X, Y_0, \ldots, Y_{n-1})$.

Proof. The proof is the same as the proof of Lemma 15.2.1 in [Wo05b]. □

Corollary 6. *(1) Let* $m > 1$ *and let* $\sigma \in H_m(V; z, \overrightarrow{01})$*. Then we have*

$$\log(\Lambda_\gamma(\sigma)(X, Y_0, \ldots, Y_{n-1})) \equiv \sum_{k=0}^{n-1} l_m(\xi^{-k} z)(\sigma) [Y_k, X^{(m-1)}]$$

modulo $\Gamma^{m+1} \mathrm{L}(X, Y_0, \ldots, Y_{n-1}) + I_2$.

(2) Let $\sigma \in G_{K(\mu_{\ell^\infty})}$*. Then we have*

$$\log(\Lambda_\gamma(\sigma)(X, Y_0, \ldots, Y_{n-1})) \equiv l(z)(\sigma) X + \sum_{k=0}^{n-1} l(1 - \xi^{-k} z)(\sigma) Y_k$$

modulo $\Gamma^2\mathrm{L}(X,Y_0,\dots,Y_{n-1})$.

Proof. The corollary follows from the very definition of ℓ-adic polylogarithms and from Proposition 5. □

Now we shall define **polylogarithmic quotients** of the representations $\varphi_{\overrightarrow{01}}$ and ψ_γ. Let $\mathscr{I}$ be a closed ideal of $\mathbb{Q}_\ell\{\{X,Y_0,\dots,Y_{n-1}\}\}$ generated by monomials with any two Y's and by monomials Y_kX for $0\leq k\leq n-1$. We set

$$\mathrm{Pol}(X,Y_0,\dots,Y_{n-1}):=\mathbb{Q}_\ell\{\{X,Y_0,\dots,Y_{n-1}\}\}/\mathscr{I}.$$

Observe that the classes

$$X^i \text{ and } X^iY_k$$

with $i=0,1,\dots$ and $0\leq k\leq n-1$ form a topological base of $\mathrm{Pol}(X,Y_0,\dots,Y_{n-1})$. We denote by

$$\Omega_\gamma(\sigma)\in\mathrm{Pol}(X,Y_0,\dots,Y_{n-1})$$

the image of the power series $\Lambda_\gamma(\sigma)\in\mathbb{Q}_\ell\{\{X,Y_0,\dots,Y_{n-1}\}\}$.

Proposition 7. *(1) The representation $\varphi_{\overrightarrow{01}}$ induces a representation*

$$\bar{\varphi}_{\overrightarrow{01}}\ :\ G_K\to\mathrm{Aut}(\mathrm{Pol}(X,Y_0,\dots,Y_{n-1})).$$

given by

$$\begin{aligned}\bar{\varphi}_{\overrightarrow{01}}(\sigma)(X)&=\chi(\sigma)X\\ \bar{\varphi}_{\overrightarrow{01}}(\sigma)(Y_k)&=\chi(\sigma)Y_k+\sum_{i=1}^{\infty}\frac{(-1)^i}{i!}\chi(\sigma)\big(\frac{k}{n}(\chi(\sigma)-1)\big)^iX^iY_k\end{aligned}$$

for $k=0,1,\dots,n-1$.

(2) The representation ψ_γ induces a representation

$$\bar{\psi}_\gamma\ :\ G_K\to\mathrm{GL}(\mathrm{Pol}(X,Y_0,\dots,Y_{n-1}))$$

given by the formula

$$\bar{\psi}_\gamma(\sigma)=L_{\Omega_\gamma(\sigma)}\circ\bar{\varphi}_{\overrightarrow{01}}(\sigma).$$

(3) For $n=1$ we have

$$\log(\Omega_\gamma(\sigma))=l(z)_\gamma(\sigma)X+\sum_{i=1}^{\infty}(-1)^{i-1}l_i(z)_\gamma(\sigma)X^{i-1}Y_0.$$

Proof. (1) It follows from [Wo05b], Proposition 15.1.7 that $\varphi_{\overrightarrow{01}}(\mathscr{I})\subset\mathscr{I}$. Hence $\varphi_{\overrightarrow{01}}$ induces a representation on the quotient space. The explicit formulae also follow from [Wo05b], Proposition 15.1.7.

(2) We recall that $\psi_\gamma(\sigma)=L_{\Lambda_\gamma(\sigma)}\circ\varphi_{\overrightarrow{01}}(\sigma)$ (see [Wo04], Section 4), hence the existence of $\bar{\psi}_\gamma$ and the explicit formula. Assertion (3) follows from the definition of ℓ-adic polylogarithms. □

For $\alpha \in \mathbb{Q}_\ell^\times$ we denote by $\tau(\alpha)$ the automorphism of the $\mathbb{Q}_\ell$-algebra $\mathrm{Pol}(X,Y)$ such that

$$\tau(\alpha)(X) = \alpha \cdot X$$

$$\tau(\alpha)(Y) = \alpha \cdot Y$$

and continuous with respect to the topology defined by the powers of the augmentation ideal. For $n = 1$ we have a very simple description of $\bar{\varphi}_{\overrightarrow{01}}$.

Corollary 8. *If* $n = 1$ *then* $\bar{\varphi}_{\overrightarrow{01}}(\sigma) = \tau(\chi(\sigma))$.

14.3 Linear independence of ℓ-adic iterated integrals

In this section we shall prove linear independence of ℓ-adic polylogarithms in a generic situation. We use the notation of Section 14.2.

If $a_1, \ldots, a_k$ belong to $K^\times$ we denote by

$$\langle a_1, \ldots, a_k \rangle = \langle a_i \mid 1 \le i \le n \rangle$$

the subgroup of $K^\times$ generated by $a_1, \ldots, a_k$.

Theorem 9. *Suppose that* $z \in K$ *is not a root of any equation*

$$z^p \cdot \prod_{k=0}^{n-1} (z - \xi^k)^{q_k} = 1,$$

where p *and* q_k *are integers not all equal zero. Suppose that*

$$\langle z, 1 - \xi^{-k} z \mid 0 \le k \le n-1 \rangle \cap \langle 1 - \xi^{-k} \mid 1 \le k \le n-1 \rangle \subset \mu_n.$$

Then the homomorphisms

$$l_b(z) \; : \; H_m(V; z, \overrightarrow{01}) / H_{m+1}(V; z, \overrightarrow{01}) \to \mathbb{Q}_\ell$$

for $b \in \mathscr{B}_m$ *are linearly independent over* $\mathbb{Q}_\ell$.

Proof. It follows from the formula

$$\psi_\gamma(\sigma) = L_{\Lambda_\gamma(\sigma)} \circ \varphi_{\overrightarrow{01}}(\sigma)$$

(see [Wo04, page 131]) that the morphism

$$\psi_\gamma \; : \; G_K \to \mathrm{GL}(\mathbb{Q}_\ell\{\{X, Y_0, \ldots, Y_{n-1}\}\})$$

induces the morphism of associated graded Lie algebras

$$\Psi_{z,\overrightarrow{01}}\ :\ \bigoplus_{m=1}^{\infty}\frac{H_m(V;z,\overrightarrow{01})}{H_{m+1}(V;z,\overrightarrow{01})}\otimes\mathbb{Q}_\ell\to \mathrm{Lie}(X,Y_0,\dots,Y_{n-1})\rtimes \mathrm{Der}\big(\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})\big).$$

It follows from [Wo05b, Lemma 15.2.5.] that the image of $\Psi_{z,\overrightarrow{01}}$ is contained in

$$\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})\rtimes \mathrm{Der}^*_{\mathbb{Z}/n}\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})$$

(see [Wo05b, Definition 15.2.4.] for the definition of the right hand factor). The Lie algebra of special derivations

$$\mathrm{Der}^*_{\mathbb{Z}/n}\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})$$

is isomorphic as a vector space to $\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})$ divided by a vector subspace generated by Y_0. The Lie bracket of the Lie algebra of special derivations induces a new bracket denoted by $\{\ \}$ on $\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})$. The obtained Lie algebra we denote by

$$\mathrm{Lie}(X,Y_0,\dots,Y_{n-1})_{\{\ \}}$$

(see [Wo05b, page 24 and Lemma 15.2.8.]). To simplify the notation let us set

$$\widetilde{\mathrm{Lie}}_n := \mathrm{Lie}(X,Y_0,\dots,Y_{n-1})\rtimes \mathrm{Lie}(X,Y_0,\dots,Y_{n-1})_{\{\ \}}$$

Hence, finally, the morphism $\psi_\gamma\ :\ G_K\to \mathrm{GL}(\mathbb{Q}_\ell\{\{X,Y_0,\dots,Y_{n-1}\}\})$ induces the morphism of graded Lie algebras

$$\Psi_{z,\overrightarrow{01}}\ :\ \bigoplus_{m=1}^{\infty}\frac{H_m(V;z,\overrightarrow{01})}{H_{m+1}(V;z,\overrightarrow{01})}\otimes\mathbb{Q}_\ell\to \widetilde{\mathrm{Lie}}_n.$$

For $\sigma\in H_m(V;z,\overrightarrow{01})$ the morphism $\Psi_{z,\overrightarrow{01}}$ is given by the formula

$$\Big[\Psi_{z,\overrightarrow{01}}(\sigma)\Big]_{\deg m}=\Big(\log\Lambda_\gamma(\sigma),\log\Lambda_\beta(\sigma)\Big)\quad \mathrm{mod}\quad \Gamma^{m+1}\big(\widetilde{\mathrm{Lie}}_n\big),$$

(see [Wo07, section 1, page 194]). Hence, it follows from Corollary 6 (2) that the morphism $\Psi_{z,\overrightarrow{01}}$ in degree 1 is given by

$$\Big[\Psi_{z,\overrightarrow{01}}(\sigma)\Big]_{\deg 1}=\Big(l(z)(\sigma)X+\sum_{k=0}^{n-1}l(1-\xi^{-k}z)(\sigma)Y_k,\sum_{k=1}^{n-1}l(1-\xi^{-k})(\sigma)Y_k\Big).$$

The elements z and $1-\xi^{-k}z$, for $0\le k<n$ are linearly independent in $K^\times\otimes\mathbb{Q}$, and the intersection

$$\langle 1-\xi^{-k}\ \mid\ 1\le k\le n-1\rangle\otimes\mathbb{Q}\cap\langle z,1-\xi^{-k}z\ \mid\ 0\le k\le n-1\rangle\otimes\mathbb{Q}$$

is trivial in $K^\times\otimes\mathbb{Q}$. By Kummer theory we find $\tau\in H_1=K(\mu_{\ell^\infty})$ and $\sigma_k\in H_1$ for $0\le k<n$ such that

$$\Big[\Psi_{z,\overrightarrow{01}}(\tau)\Big]_{\deg 1} = (X,0) \quad \text{and} \quad \Big[\Psi_{z,\overrightarrow{01}}(\sigma_k)\Big]_{\deg 1} = (Y_k,0)$$

for $0 \le k < n$. We conclude that the image of $\Psi_{z,\overrightarrow{01}}$ contains the first factor of $\widetilde{\mathrm{Lie}}_n$.

For $m > 1$ and for $\sigma \in H_m(V;z,\overrightarrow{01})$ we have

$$\log \Lambda_\gamma(\sigma) \equiv \sum_{b \in \mathscr{B}_m} l_b(z)(\sigma) b \mod \Gamma^{m+1}\mathrm{L}(X,Y_0,\dots,Y_{n-1}).$$

Hence, it follows that the functions

$$l_b(z) \,:\, H_m(V_K;z,\overrightarrow{01}) \to \mathbb{Q}_\ell$$

are linearly independent over $\mathbb{Q}_\ell$. □

Theorem 1 of Section 14.1 follows immediately from Theorem 9.

Corollary 10. *The ℓ-adic polylogarithms*

$$l_m(\xi^k z) \,:\, H_m(V;z,\overrightarrow{01})/H_{m+1}(V;z,\overrightarrow{01}) \to \mathbb{Q}_\ell$$

for $k = 0,1,\dots,n-1$ are linearly independent over $\mathbb{Q}_\ell$.

Proof. Corollary 10 follows immediately from Theorem 9 and Corollary 6 (1) of Section 14.2. □

Remark 11. Theorem 9 is an analogue of the statement, as far as we know unproven, that the iterated integrals indexed by elements of $\mathscr{B}_m$ as in [Wo91] of sequences of length m of 1-forms $\frac{dz}{z}$ and $\frac{dz}{z-\xi^k}$ for $0 \le k \le n-1$ along a fixed path γ from $\overrightarrow{01}$ to a z satisfying the assumption of Theorem 9, are linearly independent over $\mathbb{Q}$.

14.4 Ramification properties of ℓ-adic polylogarithms

Let as above K be a number field and z either a rational or a tangential point of

$$\mathbb{P}^1 \setminus \{0,1,\infty\}$$

defined over K. Let γ be an ℓ-adic path on $\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}$ from $\overrightarrow{01}$ to z.

For an algebraic extension L/K and $z \in K$, we denote the maximal pro-ℓ (resp. maximal abelian pro-ℓ) extension of L that is unramified outside ℓ and all prime factors of the fractional ideal (z) by

$$M(L)_{\ell,z} \quad (\text{resp. } M(L)^{ab}_{\ell,z}).$$

The triple $(\mathbb{P}^1_K \setminus \{0,1,\infty\}, z, \overrightarrow{01})$ has good reduction outside the prime ideals which are factors of the fractional ideals (z) or $(1-z)$. Therefore the action of G_K

on the torsor of ℓ-adic paths

$$\pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}; z, \overrightarrow{01})$$

from $\overrightarrow{01}$ to z factors through

$$\mathrm{Gal}(M(K(\mu_{\ell^\infty}))_{\ell,z(1-z)}/K).$$

Hence the ℓ-adic polylogarithm $l_m(z)_\gamma \ : \ G_K \to \mathbb{Q}_\ell$ factors as a map

$$l_m(z)_\gamma \ \ : \ \mathrm{Gal}\big(M(K(\mu_{\ell^\infty}))_{\ell,z(1-z)}/K\big) \to \mathbb{Q}_\ell.$$

Let us consider the tower of fields $K \subseteq K(\mu^{\ell^\infty}) \subseteq K(\mu_{\ell^\infty}, z^{1/\ell^\infty})$.

Proposition 12. *The ℓ-adic polylogarithm $l_m(z)_\gamma$ factors through*

$$\mathrm{Gal}\big(M(K(\mu_{\ell^\infty}, z^{1/\ell^\infty}))^{ab}_{\ell,1-z}/K\big).$$

Proof. We consider the polylogarithmic quotient $\bar{\psi}_\gamma : G_K \to \mathrm{GL}(\mathrm{Pol}(X,Y))$ as in Proposition 7 and restrict to

$$G_{K(\mu_{\ell^\infty}, z^{1/\ell^\infty})} \subset G_K.$$

By Proposition 7 (1) we have $\bar{\varphi}_{\overrightarrow{01}}(\sigma) = \mathrm{id}$ for $\sigma \in G_{K(\mu_{\ell^\infty})}$, so that

$$\bar{\psi}_\gamma(\sigma) = L_{\Omega_\gamma(\sigma)} \in \mathrm{GL}(\mathrm{Pol}(X,Y))$$

by Proposition 7 (2). By Proposition 7 (3) for $\sigma \in G_{K(\mu_{\ell^\infty}, z^{1/\ell^\infty})}$ we have

$$\log(\Omega_\gamma(\sigma)) = \sum_{n=1}^{\infty} (-1)^{n-1} l_n(z)_\gamma(\sigma) X^{n-1} Y$$

and

$$\Omega_\gamma(\sigma) = \exp\Big(\sum_{n=1}^{\infty} (-1)^{n-1} l_n(z)_\gamma(\sigma) X^{n-1} Y\Big) = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} l_n(z)_\gamma(\sigma) X^{n-1} Y.$$

Observe that the subgroup in $\mathrm{GL}(\mathrm{Pol}(X,Y))$ of automorphisms of the form L_Ω, where $\Omega = 1 + \sum_{n=1}^{\infty} c_n X^{n-1} Y$ is abelian. Hence we deduce a factorization

$$\bar{\psi}_\gamma \ : \ \mathrm{Gal}(M(K(\mu_{\ell^\infty}, z^{1/\ell^\infty}))^{ab}_{\ell,z(1-z)}/K) \to \mathrm{GL}(\mathrm{Pol}(X,Y)).$$

Therefore the ℓ-adic polylogarithm $l_m(z)_\gamma$ factors through the Galois group

$$\mathrm{Gal}(M(K(\mu_{\ell^\infty}, z^{1/\ell^\infty}))^{ab}_{\ell,z(1-z)}/K).$$

The functions $l_m(z)_\gamma$ are given explicitely by Kummer characters associated to

$$\prod_{i=0}^{\ell^n-1}(1-\xi_{\ell^n}^i z^{1/\ell^n})^{i^{m-1}/\ell^n},$$

see [NW02]. In fact, the $l_m(z)_\gamma$ factor through $\mathrm{Gal}(M(K(\mu_{\ell^\infty},z^{1/\ell^\infty}))^{ab}_{\ell,1-z}/K)$, since $1-\xi^i_{\ell^n}z^{1/\ell^n}\equiv 1$ modulo any prime dividing a prime factor that occurs in (z). □

Corollary 13. *The ℓ-adic polylogarithm $l_m(z)_\gamma$ restricted to the Galois group*

$$\mathrm{Gal}\big(M(K(\mu_{\ell^\infty},z^{1/\ell^\infty}))^{ab}_{\ell,1-z}/K(\mu_{\ell^\infty},z^{1/\ell^\infty})\big)$$

is a homomorphism.

Proof. In the proof of Proposition 12 we have already seen that the representation $\bar{\psi}_\gamma$ restricted to $G_{K(\mu_{\ell^\infty},z^{1/\ell^\infty})}$ is abelian. □

Combining Proposition 12, Corollary 10 and 13 we get Theorem 2 of Section 14.1.

14.5 Iwasawa action on ℓ-adic polylogarithms

We keep the notation of Section 14.4. Let us consider the tower of fields

$$K\subseteq K(\mu_{\ell^\infty})\subseteq K(\mu_{\ell^\infty},z^{1/\ell^\infty})\subseteq M(K(\mu_{\ell^\infty},z^{1/\ell^\infty}))^{ab}_{\ell,1-z}$$

with Galois groups

$$\begin{aligned}\mathscr{G}&=\mathrm{Gal}\big(M(K(\mu_{\ell^\infty},z^{1/\ell^\infty}))^{ab}_{\ell,1-z}/K(\mu_{\ell^\infty},z^{1/\ell^\infty})\big)\\ \mathbb{Z}_\ell(1)&=\mathrm{Gal}(K(\mu_{\ell^\infty},z^{1/\ell^\infty})/K(\mu_{\ell^\infty}))\\ \Gamma&=\mathrm{Gal}(K(\mu_{\ell^\infty})/K)\end{aligned}$$

where $\mathrm{Gal}\big(K(\mu_{\ell^\infty},z^{1/\ell^\infty})/K(\mu_{\ell^\infty})\big)=\mathbb{Z}_\ell(1)$ as a Γ-module. We set

$$\Phi=\mathrm{Gal}\big(K(\mu_{\ell^\infty},z^{1/\ell^\infty})/K\big)=\mathbb{Z}_\ell(1)\rtimes\Gamma$$

and want to understand the action of Φ and of $\mathbb{Z}_\ell[[\Phi]]$ on $\mathscr{G}$. By Corollary 13, the ℓ-adic polylogarithms $l_n(z)_\gamma$ induce elements of $\mathrm{Hom}(\mathscr{G},\mathbb{Q}_\ell)$. As our first step to understand $\mathscr{G}$ we shall study the $\mathbb{Z}_\ell[[\Phi]]$-module generated by $l_n(z)_\gamma$ in $\mathrm{Hom}(\mathscr{G},\mathbb{Q}_\ell)$.

We recall that Φ acts on $\mathscr{G}$ on the left in the following way. Let $\sigma\in\Phi$ and $\tau\in\mathscr{G}$. Let $\tilde{\sigma}\in\mathrm{Gal}\big(M(K(\mu_{\ell^\infty},z^{1/\ell^\infty}))^{ab}_{\ell,1-z}/K\big)$ be a lifting of σ. Then the formula

$${}^{\sigma}\tau:=\tilde{\sigma}\cdot\tau\cdot\tilde{\sigma}^{-1}$$

defines the action of Φ on $\mathscr{G}$. The right action of Φ on $\mathrm{Hom}(\mathscr{G},\mathbb{Q}_\ell)$ is given by

$$f^{\sigma}(\tau)=f(\tilde{\sigma}\cdot\tau\cdot\tilde{\sigma}^{-1}).$$

Lemma 14. *For any* $\alpha, \tau \in G_K$ *we have in* $\mathbb{Q}_\ell\{\{X,Y\}\}$

$$\Lambda_\gamma(\alpha \cdot \tau \cdot \alpha^{-1}) = \Lambda_\gamma(\alpha) \cdot \varphi_{\overrightarrow{01}}(\alpha)(\Lambda_\gamma(\tau)) \cdot \varphi_{\overrightarrow{01}}(\alpha \cdot \tau \cdot \alpha^{-1})(\Lambda_\gamma(\alpha)^{-1})$$

Proof. This follows from [Wo04], Proposition 1.0.7 and Corollary 1.0.8. □

We define the product $\star$ on the Lie algebra of formal Lie power series $L(X,Y)$ by the Baker-Campbell-Hausdorff formula

$$A \star B := \log(e^A \cdot e^B).$$

Proposition 15. *The action of* $\sigma \in \Phi$ *on* $l_m(z)_\gamma \in \mathrm{Hom}(\mathscr{G}, \mathbb{Q}_\ell)$ *is given by*

$$(l_m(z)_\gamma)^\sigma = \chi(\sigma)^m \cdot l_m(z)_\gamma + \sum_{k=1}^{m-1} \frac{(-l(z)_\gamma(\sigma))^k}{k!} \cdot \chi(\sigma)^{m-k} \cdot l_{m-k}(z)_\gamma.$$

Proof. Let $\tau \in \mathscr{G}$ and let $\bar{\sigma}$ and $\bar{\tau}$ be liftings of σ and τ to $\mathrm{Gal}(\bar{K}/K)$. It follows from Lemma 14 that

$$\log\Lambda_\gamma(\bar{\sigma} \cdot \bar{\tau} \cdot \bar{\sigma}^{-1}) = \log\Lambda_\gamma(\bar{\sigma}) \star \varphi_{\overrightarrow{01}}(\bar{\sigma})(\log\Lambda_\gamma(\bar{\tau})) \star \big(\varphi_{\overrightarrow{01}}(\bar{\sigma} \cdot \bar{\tau} \cdot \bar{\sigma}^{-1})(-\log\Lambda_\gamma(\bar{\sigma}))\big).$$

Hence we get modulo I_2 that

$$\sum_{n=1}^{\infty} l_n(z)(^{\sigma}\tau)[Y,X^{(n-1)}] \equiv \Big(l(z)(\bar{\sigma})X + \sum_{n=1}^{\infty} l_n(z)(\bar{\sigma})[Y,X^{(n-1)}]\Big) \star \Big(\chi(\bar{\sigma})l(z)(\tau)X + \sum_{n=1}^{\infty} \chi(\bar{\sigma})^n \cdot l_n(z)(\tau)[Y,X^{(n-1)}]\Big) \star \Big(-l(z)(\bar{\sigma})X - \sum_{n=1}^{\infty} l_n(z)(\bar{\sigma})[Y,X^{(n-1)}]\Big).$$

Observe that $l(z)(\bar{\sigma})$ and $\chi(\bar{\sigma})$ depend only on σ. Hence we replace them by $l(z)(\sigma)$ and $\chi(\sigma)$. We get the formula of the proposition by calculating the right hand side of the congruence and comparing coefficients at $[Y,X^{(n-1)}]$. □

Corollary 16. *Let* $\mu \in \mathbb{Z}_\ell[[\Phi]]$. *Then* $(l_m(z)_\gamma)^\mu$ *equals*

$$\Big(\int_\Phi \chi(x)^m d\mu(x)\Big) \cdot l_m(z)_\gamma + \sum_{k=1}^{m-1} \Big(\int_\Phi \frac{(-l(z)_\gamma(x))^k}{k!} \cdot \chi(x)^{m-k} d\mu(x)\Big) \cdot l_{m-k}(z)_\gamma.$$

Proof. This generalization of Proposition 15 is straightforward. □

Observe that we have just proved Theorem 3 from Section 14.1.

14.6 Profinite sheaves

The ℓ-adic polylogarithms and ℓ-adic iterated integrals studied in [Wo04], [Wo05a], [Wo05b] and in [NW02] arise from actions of Galois groups on ℓ-adic paths

$$\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} - \{a_1, \dots, a_n\}; z, v).$$

On the other side in [BD94], [BL94] and in various other papers motivic polylogarithmic sheaves are studied. Their ℓ-adic realizations are inverse systems of locally constant sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules in étale topology. Each stalk is equipped with a Galois representation. The relation between parallel transport and the Galois representations in stalks is given by the formula

$$\sigma_t \circ p_* = \sigma(p)_* \circ \sigma_s, \tag{14.7}$$

where p_* (resp. $\sigma(p)_*$) is the parallel transport along the path p (resp. $\sigma(p)$) from s to t, σ_s (resp. σ_t) is the action of $\sigma \in G_K$ in the stalk over s (resp. over t) and $\sigma(p)$ is the image of p by σ in the torsor of paths from s to t.

The formula (14.7) is fundamental to relate ℓ-adic polylogarithms introduced in [Wo05a] with polylogarithmic sheaves, which we discuss next.

Let S be a smooth quasi-projective, geometrically connected algebraic variety over K. We denote by $S_{\text{ét}}$ the étale site associated to S. We denote by $\Pi_1(S_{\text{ét}})$ the fundamental groupoid on $S_{\text{ét}}$ (see [SGA1] Exp V).

Definition 17. A **locally constant sheaf** of finite sets on $S_{\text{ét}}$ is a functor

$$\Pi_1(S_{\text{ét}}) \to \Big(\text{ category of finite sets }\Big).$$

Let $\bar{a} : \operatorname{Spec}(\bar{K}) \to S$ be a geometric point of S with values in $\bar{K}$. The category $\Pi_1(S_{\text{ét}})$ is equivalent to the category with one object and automorphism group $\pi_1^{\text{ét}}(S, \bar{a})$. Hence a locally constant sheaf $\mathscr{F}$ of finite sets on $S_{\text{ét}}$ is a finite discrete set $\mathscr{F}_{\bar{a}}$, the stalk of $\mathscr{F}$ in $\bar{a}$, equipped with a continuous action of $\pi_1^{\text{ét}}(S, \bar{a})$.

Let S have a K-point $s : \operatorname{Spec}(K) \to S$ and let $\bar{s} : \operatorname{Spec}(\bar{K}) \to S_{\bar{K}}$, and by abuse of notation also $\bar{s} : \operatorname{Spec}(\bar{K}) \to S$, be the corresponding geometric point of $S_{\bar{K}}$ or S. The structure map $\operatorname{pr} : S \to \operatorname{Spec}(K)$ induces the projection

$$\operatorname{pr}_* \ : \ \pi_1^{\text{ét}}(S, \bar{s}) \to \pi_1(\operatorname{Spec}(K), \operatorname{Spec}(\bar{K})) = G_K$$

that is canonically split by

$$s_* \ : \ G_K \to \pi_1^{\text{ét}}(S, \bar{s}).$$

Therefore we have a semidirect product

$$\pi_1^{\text{ét}}(S, \bar{s}) = \pi_1^{\text{ét}}(S_{\bar{K}}, \bar{s}) \rtimes G_K,$$

and for a locally constant sheaf $\mathscr{F}$ both groups $\pi_1^{\text{ét}}(S_{\bar{K}}, \bar{s})$ and G_K act on $\mathscr{F}_{\bar{s}}$. If s and t are two K-points of S then the relation between actions of G_K on $\mathscr{F}_{\bar{s}}$ and $\mathscr{F}_{\bar{t}}$ is given by the formula (14.7).

Definition 18. A **profinite sheaf** on S is a projective system of locally constant sheaves of finite sets on $S_{\text{ét}}$.

The category of finite sets we can replace by the category of finite groups, finite abelian groups, finite ℓ-groups, finite ℓ-sets, finite $\mathbb{Z}_\ell$-modules, finite $\mathbb{Z}_\ell$-algebras and so on. Then we speak about a profinite sheaf of groups, abelian groups, and so on. A classical smooth ℓ-adic sheaf provides an example of a profinite sheaf.

Let $\mathscr{F} = \{\mathscr{F}_i\}_{i\in I}$ be a profinite sheaf on S. Then

$$\mathscr{F}_{\bar{s}} := \varprojlim \mathscr{F}_{i,\bar{s}}$$

is the stalk of the profinite sheaf $\mathscr{F}$ over $\bar{s}$. The group $\pi_1^{\text{ét}}(S,\bar{s})$ acts continuously on $\mathscr{F}_{\bar{s}}$. Hence we get two representations:

$$\rho_{\bar{s}} \ : \ \pi_1^{\text{ét}}(S_{\bar{K}},\bar{s}) \to \operatorname{Aut}(\mathscr{F}_{\bar{s}}),$$

called the monodromy representation in the stalk over $\bar{s}$ and

$$G_K \to \operatorname{Aut}(\mathscr{F}_{\bar{s}}),$$

the Galois representation in the stalk over $\bar{s}$. The relation between Galois representations in the stalks over $\bar{s}$ and $\bar{t}$, where t is another K-point of S, are given by the formula

$$\sigma_{\bar{\mathbf{t}}} \circ \mathbf{p}_* = \sigma(\mathbf{p})_* \circ \sigma_{\bar{\mathbf{s}}}, \tag{14.8}$$

where p_* (resp. $\sigma(p)_*$) is the parallel transport along the path p (resp. $\sigma(p)$) from $\bar{s}$ to $\bar{t}$, $\sigma_{\bar{s}}$ (resp. $\sigma_{\bar{t}}$) is the action of $\sigma \in G_K$ in the stalk over $\bar{s}$ (resp. over $\bar{t}$) and $\sigma(p)$ is the image of p by σ in the torsor of paths from $\bar{s}$ to $\bar{t}$.

It is clear from (14.8) that the Galois representation in the stalk over $\bar{s}$ together with the cocycle $\sigma \mapsto p^{-1}\cdot\sigma(p)$ determines uniquely the Galois representation in the stalk over $\bar{t}$. Let us write (14.8) in the form

$$p_*^{-1} \circ \sigma_{\bar{t}} \circ p_* = \big(p^{-1}\cdot\sigma(p)\big)_* \circ \sigma_{\bar{s}}.$$

Therefore it is crucial to calculate the element

$$f_p(\sigma) := p^{-1}\cdot\sigma(p) \in \pi_1^{\text{ét}}(S_{\bar{K}},\bar{s}).$$

We started to study $f_p(\sigma)$ in the series of papers on ℓ-adic iterated integrals.

For the next proposition we introduce the following notation

$$F_{S,\bar{s}}\big(G_K\big) := \{T^{-1}\cdot\sigma(T) \in \pi_1^{\text{ét}}(S_{\bar{K}},\bar{s}) \ \mid \ T \in \pi_1^{\text{ét}}(S_{\bar{K}},\bar{s}),\ \sigma \in G_K\}.$$

Proposition 19. *Let us assume that* $F_{S,\bar{s}}\big(G_K\big)$ *topologically and normally generates* $\pi_1^{\text{ét}}(S_{\bar{K}},\bar{s})$. *Let* $\mathscr{F}$ *be a profinite sheaf on* S *such that the monodromy representation*

$$\rho_{\bar{s}} \ : \ \pi_1^{\text{ét}}(S_{\bar{K}},\bar{s}) \to \operatorname{Aut}(\mathscr{F}_{\bar{s}})$$

is non-trivial. Then the Galois representation $G_K \to \operatorname{Aut}(\mathscr{F}_{\bar{s}})$ *in the stalk* $\mathscr{F}_{\bar{s}}$ *is also non-trivial.*

Proof. For $T \in \pi_1^{\text{ét}}(S_{\bar{K}}, \bar{s})$ and $\sigma \in G_K$ formula (14.8) leads to

$$T_*^{-1} \circ \sigma_{\bar{s}} \circ T_* \circ (\sigma_{\bar{s}})^{-1} = (T^{-1} \cdot \sigma(T))_*.$$

If $\sigma_{\bar{s}} = \mathrm{id}$ for all $\sigma \in G_K$ then the set $F_{S,\bar{s}}\big(G_K\big)$ lies in the kernel of $\rho_{\bar{s}}$. But the set $F_{S,\bar{s}}\big(G_K\big)$ topologically and normally generates $\pi_1^{\text{ét}}(S_{\bar{K}}, \bar{s})$. Hence $\sigma_{\bar{s}}$ cannot be the identity for all $\sigma \in G_K$. □

14.7 On profinite sheaves related to bundles of fundamental groups

In this section we shall study examples of profinite sheaves for which the monodromy representation determines Galois representations in the stalks. The notation and assumptions are as in Section 14.6. We recall only that $\pi_1(S_{\bar{K}}, \bar{s})$ is the pro-ℓ completion of $\pi_1^{\text{ét}}(S_{\bar{K}}, \bar{s})$.

For $\sigma \in G_K$ we denote by σ the induced automorphisms of $\pi_1^{\text{ét}}(S_{\bar{K}}, \bar{s})$ and of $\pi_1(S_{\bar{K}}, \bar{s})$. We have the surjective map $\pi_1^{\text{ét}}(S_{\bar{K}}, \bar{s}) \to \pi_1(S_{\bar{K}}, \bar{s})$. If $T \in \pi_1^{\text{ét}}(S_{\bar{K}}, \bar{s})$ we denote also by T its image in $\pi_1(S_{\bar{K}}, \bar{s})$.

Proposition 20. *Let S and s be as in Section 14.6. We assume that $\pi_1(S_{\bar{K}}, \bar{s})$ is a free noncommutative pro-ℓ group. Let $\mathscr{P}_{\bar{s}}$ be a profinite sheaf of ℓ-groups on S whose stalk over $\bar{s}$ is the group $(\mathscr{P}_{\bar{s}})_{\bar{s}} = \pi_1(S_{\bar{K}}, \bar{s})$ with the monodromy representation*

$$\rho_{\bar{s}} \; : \; \pi_1^{\text{ét}}(S_{\bar{K}}, \bar{s}) \to \mathrm{Aut}(\pi_1(S_{\bar{K}}, \bar{s}))$$

given by $\rho_{\bar{s}}(T)(w) = T \cdot w \cdot T^{-1}$. Then for any $\sigma \in G_K$ and any $w \in \pi_1(S_{\bar{K}}, \bar{s})$ we have

$$\sigma_{\bar{s}}(w) = \sigma(w).$$

Proof. Let $\sigma \in G_K$, $T \in \pi_1^{\text{ét}}(S_{\bar{K}}, \bar{s})$ and $w \in \pi_1(S_{\bar{K}}, \bar{s})$. The formula (14.8) implies

$$\sigma_{\bar{s}}(T \cdot w \cdot T^{-1}) = \sigma(T) \cdot \sigma_{\bar{s}}(w) \cdot \sigma(T)^{-1}.$$

Let us take T such that its image in $\pi_1(S_{\bar{K}}, \bar{s})$ is w. Then

$$\sigma_{\bar{s}}(w) = \sigma(w) \cdot \sigma_{\bar{s}}(w) \cdot \sigma(w)^{-1}.$$

The assumption that $\pi_1(S_{\bar{K}}, \bar{s})$ is a free pro-ℓ group implies that $\sigma_{\bar{s}}(w) = \sigma(w)^{\eta(\sigma,w)}$, where $\eta(\sigma, w) \in \mathbb{Z}_\ell$. Let $w_1, w_2 \in \pi_1(S_{\bar{K}}, \bar{s})$ be two arbitrary noncommuting elements. Then

$$\sigma_{\bar{s}}(w_1 \cdot w_2) = \sigma(w_1 \cdot w_2)^{\eta(\sigma, w_1 \cdot w_2)} = (\sigma(w_1) \cdot \sigma(w_2))^{\eta(\sigma, w_1 \cdot w_2)}$$

and

$$\sigma_{\bar{s}}(w_1) \cdot \sigma_{\bar{s}}(w_2) = \sigma(w_1)^{\eta(\sigma, w_1)} \cdot \sigma(w_2)^{\eta(\sigma, w_2)}.$$

Hence we get

$$(\sigma(w_1)\cdot\sigma(w_2))^{\eta(\sigma,w_1\cdot w_2)} = \sigma(w_1)^{\eta(\sigma,w_1)}\cdot\sigma(w_2)^{\eta(\sigma,w_2)}$$

for two noncommuting elements $\sigma(w_1)$, $\sigma(w_2)$ in the free pro-ℓ group $\pi_1(S_{\bar{K}},\bar{s})$ and for $\eta(\sigma,w_1\cdot w_2)\neq 0$, $\eta(\sigma,w_1)\neq 0$ and $\eta(\sigma,w_2)\neq 0$. This implies that $\eta(\sigma,w)=1$ for all σ and w. □

Proposition 21. *Let S and s be as above. Let $\mathscr{P}$ be a profinite sheaf on $S\times S$ whose stalk over $(\bar{s},\bar{s})$ is $\mathscr{P}_{(\bar{s},\bar{s})}=\pi_1(S_{\bar{K}},\bar{s})$ considered as a set. We assume that the monodromy representation*

$$\rho_{(\bar{s},\bar{s})}\ :\ \pi_1^{\text{ét}}(S_{\bar{K}},\bar{s})\times\pi_1^{\text{ét}}(S_{\bar{K}},\bar{s})\to Bijections(\pi_1(S_{\bar{K}},\bar{s}))$$

is given by $\rho_{(\bar{s},\bar{s})}(T_1,T_2)(w)=T_1\cdot w\cdot T_2^{-1}$. We assume also that the center of the group $\pi_1(S_{\bar{K}},\bar{s})$ is 1. Then for any $\sigma\in G_K$ and any $w\in\pi_1(S_{\bar{K}},\bar{s})$ we have

$$\sigma_{(\bar{s},\bar{s})}(w)=\sigma(w).$$

Proof. The formula (14.8) implies

$$\sigma(T_1)\cdot\sigma_{(\bar{s},\bar{s})}(w)\cdot\sigma(T_2)^{-1}=\sigma_{(\bar{s},\bar{s})}(T_1\cdot w\cdot T_2^{-1}). \tag{14.9}$$

Let us take $T_1=T_2=T$ and $w=1$. Then we get

$$\sigma(T)\cdot\sigma_{(\bar{s},\bar{s})}(1)\cdot\sigma(T)^{-1}=\sigma_{(\bar{s},\bar{s})}(1).$$

Hence $\sigma_{(\bar{s},\bar{s})}(1)$ lies in the center of $\pi_1(S_{\bar{K}},\bar{s})$ and $\sigma_{(\bar{s},\bar{s})}(1)=1$. Let us take $T_2=w=1$ in the formula (14.9). Then we get $\sigma(T_1)=\sigma_{(\bar{s},\bar{s})}(T_1)$ for any $T_1\in\pi_1(S_{\bar{K}},\bar{s})$. □

14.8 Polylogarithmic profinite sheaves and ℓ-adic polylogarithms

We shall show that if a profinite sheaf of $\mathbb{Z}_\ell$-modules on $\mathbb{P}^1_K\setminus\{0,1,\infty\}$ has the same monodromy representation as the classical complex polylogarithm then the Galois representation in the stalk over $z\in\big(\mathbb{P}^1_K\setminus\{0,1,\infty\}\big)(K)$ is given by the ℓ-adic polylogarithms evaluated at z.

We start by recalling a result about the monodromy of classical complex polylogarithms. The constant vector bundle with fibre $\mathrm{Pol}(X,Y)$

$$\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}\times\mathrm{Pol}(X,Y)\to\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$$

is endowed with the connection $\nabla=d-\omega$ defined by the 1-form

$$\omega=\frac{1}{2\pi i}\frac{dz}{z}\otimes X+\frac{1}{2\pi i}\frac{dz}{z-1}\otimes Y.$$

The space of horizontal sections $\Lambda : \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\} \to \mathrm{Pol}(X,Y)$ is the solution space of the differential equation

$$d\Lambda(z) - (\frac{1}{2\pi i}\frac{dz}{z} \otimes X + \frac{1}{2\pi i}\frac{dz}{z-1} \otimes Y) \cdot \Lambda(z) = 0.$$

One checks that

$$\Lambda_{\overrightarrow{01}}(z) := \exp\big(\frac{1}{2\pi i}\log(z)\big)X + \frac{1}{2\pi i}\log(1-z)Y + \sum_{k=2}^{\infty} \frac{-1}{(2\pi i)^k} Li_k(z) X^{k-1} Y$$

is locally a horizontal section. The functions $\log(z)$, $\log(1-z)$ and $Li_k(z)$ are calculated along a path α from $\overrightarrow{01}$ to z.

Let x and y be the standard generators of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, \overrightarrow{01})$. To calculate the monodromy of $\Lambda_{\overrightarrow{01}}(z)$ we integrate along the paths $\alpha \cdot x$ and $\alpha \cdot y$. The monodromy transformation of $\Lambda_{\overrightarrow{01}}(z)$ is given by

$$x \,:\, \Lambda_{\overrightarrow{01}}(z) \to \Lambda_{\overrightarrow{01}}(z) \cdot \exp(X)$$

$$y \,:\, \Lambda_{\overrightarrow{01}}(z) \to \Lambda_{\overrightarrow{01}}(z) \cdot (1+Y).$$

The group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, z)$ is freely generated by

$${}^{\alpha}x = \alpha \cdot x \cdot \alpha^{-1} \quad \text{and} \quad {}^{\alpha}y = \alpha \cdot y \cdot \alpha^{-1}.$$

The monodromy representation of $(\mathrm{Pol}(X,Y), \nabla)$ in z

$$\rho_z \,:\, \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, z) \to \mathrm{GL}(\mathrm{Pol}(X,Y))$$

is given by

$$\rho_z({}^{\alpha}x) = R_{\exp(X)}$$

$$\rho_z({}^{\alpha}y) = R_{1+Y}.$$

For a word $w({}^{\alpha}x, {}^{\alpha}y)$ in ${}^{\alpha}x$ and ${}^{\alpha}y$ we thus find

$$\rho_z\big(w({}^{\alpha}x, {}^{\alpha}y)\big) = R_{w(\exp(X),1+Y)} = R_{\bar{E}(w)}.$$

Now we shall study the ℓ-adic situation. Let z_0 be a K-point of $\mathbb{P}^1_K \setminus \{0,1,\infty\}$. We start with the description of the action of G_K on

$$\pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, z_0).$$

Let γ be a path from z_0 to $\overrightarrow{01}$ and let p be the standard path from $\overrightarrow{01}$ to $\overrightarrow{10}$. We recall that x and y are the standard generators of $\pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, \overrightarrow{01})$. Then

$$x_{z_0} = \gamma^{-1} \cdot x \cdot \gamma \quad \text{and} \quad y_{z_0} = \gamma^{-1} \cdot y \cdot \gamma$$

are free generators of the pro-ℓ group $\pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, z_0)$. The following lemma is a standard exercice.

Lemma 22. *The action of* G_K *on* $\pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, z_0)$ *is given by the formulae*

$$\begin{aligned}
\sigma(x_{z_0}) &= f_\gamma(\sigma)^{-1} \cdot {x_{z_0}}^{\chi(\sigma)} \cdot f_\gamma(\sigma) \\
\sigma(y_{z_0}) &= f_\gamma(\sigma)^{-1} \cdot (\gamma^{-1} \cdot f_p(\sigma)^{-1} \cdot \gamma) \cdot {y_{z_0}}^{\chi(\sigma)} \cdot (\gamma^{-1} \cdot f_p(\sigma) \cdot \gamma) \cdot f_\gamma(\sigma).
\end{aligned}$$

Let z be another K-point of $\mathbb{P}^1_K \setminus \{0,1,\infty\}$. Let δ be a path from z to z_0. Let us set

$$\gamma_z := \gamma \cdot \delta.$$

The following equalities can be found in [Wo04].

$$f_{\gamma\cdot\delta}(\sigma) = \delta^{-1} \cdot f_\gamma(\sigma) \cdot \delta \cdot f_\delta(\sigma) \text{ and } f_{\delta^{-1}}(\sigma)^{-1} = \delta \cdot f_\delta(\sigma) \cdot \delta^{-1}. \qquad (14.10)$$

Hence we get

$$\delta \cdot f_{\gamma\cdot\delta}(\sigma) \cdot \delta^{-1} = f_\gamma(\sigma) \cdot f_{\delta^{-1}}(\sigma)^{-1}. \qquad (14.11)$$

The group $\pi_1(P^1_{\bar{K}} \setminus \{0,1,\infty\}, z)$ is freely generated by the elements

$$x_z := \gamma_z^{-1} \cdot x \cdot \gamma_z \quad \text{and} \quad y_z := \gamma_z^{-1} \cdot x \cdot \gamma_z$$

as a pro-ℓ group. We use the following exponential embedings.

$$\begin{gathered}
E_{\overrightarrow{01}} \ : \ \pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, \overrightarrow{01}) \to \mathbb{Q}_\ell\{\{X,Y\}\} \\
E_{\overrightarrow{01}}(x) := \exp(X) \quad \text{and} \quad E_{\overrightarrow{01}}(y) := \exp(Y), \\
E_{z_0} \ : \ \pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, z_0) \to \mathbb{Q}_\ell\{\{X,Y\}\} \\
E_{z_0}(x_{z_0}) := \exp(X) \quad \text{and} \quad E_{z_0}(y_{z_0}) := \exp(Y), \\
E_z \ : \ \pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, z) \to \mathbb{Q}_\ell\{\{X,Y\}\} \\
E_z(x_z) := \exp(X) \quad \text{and} \quad E_z(y_z) := \exp(Y).
\end{gathered}$$

In other words we have trivialized the bundle of fundamental groups along the path γ_z. The action of G_K on $\mathbb{Q}_\ell\{\{X,Y\}\}$ considered over a K-point s is deduced from the action of G_K on $\pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}, s)$ so it depends on s

Using the embeddings E_a, for $a \in \{\overrightarrow{01}, z_0, z\}$, we can define the Λ-series as

$$\Lambda_\delta(\sigma) := E_z(f_\delta(\sigma)) \quad \text{and} \quad \Lambda_\gamma(\sigma) := E_{z_0}(f_\gamma(\sigma)).$$

The composition of an embedding E_a with the quotient map $\mathbb{Q}\{\{X,Y\}\} \to \mathrm{Pol}(X,Y)$ we denote by $\bar{E}_a$. We recall that the images of Λ-series by the quotient map are Ω-series. For example we have

$$\Omega_{\delta^{-1}}(\sigma) = \bar{E}_{z_0}(f_{\delta^{-1}}(\sigma)).$$

Because of the trivialization of the bundle of fundamental groups we can compare various Λ-series. It follows from (14.10) and (14.11) that

$$\Lambda_{\gamma\cdot\delta}(\sigma) = \Lambda_\gamma(\sigma)\cdot\Lambda_\delta(\sigma),$$

$$\big(\Lambda_{\delta^{-1}}(\sigma)\big)^{-1} = \Lambda_\delta(\sigma)$$

$$\Lambda_{\gamma\cdot\delta}(\sigma) = \Lambda_\gamma(\sigma)\cdot\big(\Lambda_{\delta^{-1}}(\sigma)\big)^{-1}. \tag{14.12}$$

These equalities imply the analogous equalities for Ω-series.

Theorem 23. *Let z_0 be a K-point of $\mathbb{P}^1_K\setminus\{0,1,\infty\}$, and let $\mathscr{P}$ be a profinite sheaf of $\mathbb{Z}_\ell$-algebras over $\mathbb{P}^1_K\setminus\{0,1,\infty\}$ such that*

(i) $\mathscr{P}_{z_0}\otimes\mathbb{Q}_\ell = \mathrm{Pol}(X,Y)$ *as a $\mathbb{Q}_\ell$-algebra, and*
(ii) *the monodromy representation of $\mathscr{P}$ on the stalk over z_0 tensored by $\mathbb{Q}_\ell$*

$$\rho_{z_0}\ :\ \pi_1^{\text{ét}}(\mathbb{P}^1_{\bar K}\setminus\{0,1,\infty\},z_0)\to \mathrm{GL}(\mathrm{Pol}(X,Y))$$

is given by the formula

$$\rho_{z_0}(w) = L_{\bar E_{z_0}(w)}.$$

Let z be another K-point of $\mathbb{P}^1_K\setminus\{0,1,\infty\}$. Let δ be a path from z to z_0 and let α be a path from $\overrightarrow{01}$ to z. Then

$$\delta_*\circ\sigma_z\circ(\delta_*)^{-1} = L_{\Omega_\alpha(\sigma)}\circ R_{B(\sigma)}\circ\tau(\chi(\sigma)),$$

where $B: G_K\to\mathrm{Pol}(X,Y)$ satisfies $B(\sigma\cdot\sigma_1) = \big(\tau(\chi(\sigma))(B(\sigma_1))\big)\cdot B(\sigma)$ and

$$\log\big(\Omega_\alpha(\sigma)\big) = l(z)_\alpha(\sigma)X + \sum_{i=1}^{\infty}(-1)^{i-1}l_i(z)_\alpha(\sigma)X^{i-1}Y.$$

Proof. Let us set $\gamma = (\delta\cdot\alpha)^{-1}$. Then γ is a path from z_0 to $\overrightarrow{01}$. Let $\sigma\in G_K$ and let $w\in\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar K}\setminus\{0,1,\infty\},z_0)$. It follows from Lemma 22 that

$$\bar E_{z_0}\big(\sigma(w)\big)) = (\Omega_\gamma(\sigma))^{-1}\cdot\big(\bar E_{z_0}(w)(\chi(\sigma)X,\chi(\sigma)Y)\big)\cdot\Omega_\gamma(\sigma). \tag{14.13}$$

It follows from (14.8) that

$$\sigma_{z_0}\big(\bar E_{z_0}(w)\big) = \bar E_{z_0}(\sigma(w))\cdot\sigma_{z_0}(1)$$

Hence it follows from (14.13) that

$$\sigma_{z_0}\big(\bar E_{z_0}(w)\big) = (\Omega_\gamma(\sigma))^{-1}\cdot\bar E_{z_0}(w)(\chi(\sigma)X,\chi(\sigma)Y)\cdot\Omega_\gamma(\sigma)\cdot\sigma_{z_0}(1). \tag{14.14}$$

Hence for any $W(X,Y)\in\mathrm{Pol}(X,Y)$ we have

$$\sigma_{z_0}(W(X,Y)) = (\Omega_\gamma(\sigma))^{-1}\cdot W(\chi(\sigma)X,\chi(\sigma)Y)\cdot\Omega_\gamma(\sigma)\cdot\sigma_{z_0}(1). \tag{14.15}$$

We shall calculate the representation of G_K in $\mathscr{P}_z \otimes \mathbb{Q}_\ell$. It follows from the fundamental formula (14.8) that

$$\delta_* \circ \sigma_z \circ \delta_*^{-1} = \delta_* \circ \sigma(\delta)_*^{-1} \circ \sigma_{z_0}.$$

Observe that

$$\delta_* \circ \sigma(\delta)_*^{-1} = (\delta \cdot \sigma(\delta^{-1}))_* = (f_{\delta^{-1}}(\sigma))_* = \rho_{z_0}(f_{\delta^{-1}}(\sigma)) = L_{\Omega_{\delta^{-1}}(\sigma)}.$$

Hence we get

$$\delta_* \circ \sigma_z \circ \delta_*^{-1} = L_{\Omega_{\delta^{-1}}(\sigma)} \circ \sigma_{z_0}.$$

The formula (14.15) implies that

$$\begin{aligned} L_{\Omega_{\delta^{-1}}(\sigma)} \circ \sigma_{z_0} &= L_{\Omega_{\delta^{-1}}(\sigma)} \circ L_{(\Omega_\gamma(\sigma))^{-1}} \circ R_{\Omega_\gamma(\sigma)\cdot\sigma_{z_0}(1)} \circ \tau(\chi(\sigma)) \\ &= L_{\Omega_{\delta^{-1}}(\sigma)\cdot(\Omega_\gamma(\sigma))^{-1}} \circ R_{\Omega_\gamma(\sigma)\cdot\sigma_{z_0}(1)} \circ \tau(\chi(\sigma)). \end{aligned}$$

By (14.12) we deduce that

$$\Omega_{\delta^{-1}}(\sigma) \cdot (\Omega_\gamma(\sigma))^{-1} = (\Omega_{\gamma\cdot\delta}(\sigma))^{-1} = (\Omega_{\alpha^{-1}}(\sigma))^{-1} = \Omega_\alpha(\sigma).$$

With $B(\sigma) = \Omega_\gamma(\sigma) \cdot \sigma_{z_0}(1)$ we therefore finally get

$$\delta_* \circ \sigma_z \circ \delta_*^{-1} = L_{\Omega_\alpha(\sigma)} \circ R_{B(\sigma)} \circ \tau(\chi(\sigma)).$$

The equality $(\tau \cdot \sigma)_z = \tau_z \circ \sigma_z$ implies that $B : G_K \to \mathrm{Pol}(X,Y)$ indeed satisfies the formula of the theorem.

Since the path α is from $\overrightarrow{01}$ to z, the formula for $\log\big(\Omega_\alpha(\sigma)\big)$ follows from the very definition of ℓ-adic polylogarithms. □

Remark 24. (1) We need one more condition to show that $B = 1$. We can require for example that over $\overrightarrow{01}$ the Galois group G_K acts on $\mathrm{Pol}(X,Y)$ through $\tau \circ \chi$.

(2) Let us set

$$\Omega_\alpha(\sigma) = 1 + \exp(l(z)(\sigma)X) + \sum_{n=1}^{\infty} -li_n(z)(\sigma)X^{n-1}Y.$$

Then the matrice of $L_{\Omega_\alpha(\sigma)}$ in the base $1, Y, XY, X^2Y, X^3Y, \dots$ is exactly as the matrice $L(z)$ expressing the monodromy of polylogarithms in [BD94].

(3) If ξ is a root of unity, then $l(\xi)_\alpha$ vanishes for a suitable choice of a path α and $l_n(\xi)_\alpha$ are cocycles, see [Wo05a, Corollary 11.0.12. and its proof]. Hence the representation of G_K on $\mathscr{P}_\xi \otimes \mathbb{Q}_\ell$ is an extension of $\mathbb{Q}_\ell(0)$ by $\prod_{n=1}^\infty \mathbb{Q}_\ell(n)$ if $B = 1$.

14.9 Cosimplicial spaces and Galois actions

In this last section we will work more generally over a field k, still with a fixed complex embedding $k \subset \mathbb{C}$, but not necessarily a number field. Let V be a smooth quasi-projective, geometrically connected algebraic variety over k and let v be a k-point of V. The étale fundamental group $\pi_1^{\text{ét}}(V_{\bar{k}}, v)$ and its maximal pro-ℓ quotient $\pi_1(V_{\bar{k}}, v)$ are equipped with the action of $G_k = \mathrm{Gal}(\bar{k}/k)$ induced by conjugation and the canonical section v_* as

$$G_K = \pi_1^{\text{ét}}(\mathrm{Spec}(k), \mathrm{Spec}(\bar{k})) \xrightarrow{v_*} \pi_1^{\text{ét}}(V, v) \xrightarrow{\gamma \mapsto \gamma(-)\gamma^{-1}} \mathrm{Aut}\big(\pi_1^{\text{ét}}(V_{\bar{k}}, v)\big). \tag{14.16}$$

On the other side, given an algebraic variety V and a k-point v there is a cosimplicial algebraic variety $V^\bullet$, which is a model in algebraic geometry for the loop space based at v (see [Wo93] and [Wo02]). Let $V(\mathbb{C})$ (resp. $V^\bullet(\mathbb{C})$) be the set of $\mathbb{C}$-points of V endowed with its natural structure as a (resp. cosimplicial) complex variety. The de Rham cohomology group of complex differential forms

$$\mathrm{H}^0_{DR}(V^\bullet(\mathbb{C}))$$

is the algebra of polynomial complex valued functions on the Malcev $\mathbb{Q}$-completion

$$\pi_1(V(\mathbb{C}), v) \otimes \mathbb{Q}.$$

The étale cohomology group

$$\mathrm{H}^0_{\text{ét}}(V^\bullet_{\bar{k}}, \mathbb{Q}_\ell)$$

can be interpreted as the algebra of polynomial $\mathbb{Q}_\ell$-valued functions on the $\mathbb{Q}$-completion, or better on the Malcev $\mathbb{Q}_\ell$-completion, for which we have the comparison isomorphism

$$\pi_1(V(\mathbb{C}), v) \otimes \mathbb{Q}_\ell = \pi_1(V_{\bar{k}}, v) \otimes \mathbb{Q}_\ell,$$

with the Malcev $\mathbb{Q}_\ell$-completion $\pi_1(V_{\bar{k}}, v) \otimes \mathbb{Q}_\ell$ of the pro-ℓ group $\pi_1(V_{\bar{k}}, v)$. The Galois group G_k acts on $\mathrm{H}^0_{\text{ét}}(V^\bullet_{\bar{k}}, \mathbb{Q}_\ell)$. We interpret $\mathrm{H}^0_{\text{ét}}(V^\bullet_{\bar{k}}, \mathbb{Q}_\ell)$ as the algebra of polynomial $\mathbb{Q}_\ell$-valued functions on $\pi_1(V_{\bar{k}}, v) \otimes \mathbb{Q}_\ell$. Therefore G_k acts also on

$$\pi_1(V_{\bar{k}}, v) \otimes \mathbb{Q}_\ell.$$

In this section we shall compare these two Galois actions. Our arguments will be very sketchy in some places because of a lot of technical material.

We first fix some notation. The sheaf $A_{X_{\text{ét}}}$ (resp. $A_{X(\mathbb{C})}$) is the constant sheaf with values in A on the étale site $X_{\text{ét}}$ (resp. $X(\mathbb{C})$) for an algebraic variety X. With $\Delta[1]$ we denote the standard simplicial model of the 1-simplex, while $\partial\Delta[1]$ is its boundary. The n-th truncation of a cosimplicial object $X^\bullet$ will be denoted by $X^\bullet_{[n]}$.

Let X be a smooth quasi-projective, geometrically connected algebraic variety over k. The inclusion of simplicial sets

$$\partial\Delta[1] \hookrightarrow \Delta[1]$$

induces the morphism of cosimplicial algebraic varieties

$$p^{\bullet} \ : \ X^{\Delta[1]} \to X^{\partial\Delta[1]}.$$

Therefore for each n we get the morphism between their n-th truncations

$$p^{\bullet}_{[n]} \ : \ X^{\Delta[1]}_{[n]} \longrightarrow X^{\partial\Delta[1]}_{[n]}.$$

For each i, the map

$$p^{i} \ : \ X^{\Delta[1]_i} = X \times X^{i} \times X \to X^{\partial\Delta[1]_i} = X \times X$$

is the projection map on the first and the last factor.

First we shall study the Gauss-Manin connection associated to the morphism $p^{\bullet} : X^{\Delta[1]} \to X^{\partial\Delta[1]}$. We review briefly the results from [Wo93] in a form suitable for our study here. We apply to the map between the n-th truncations

$$p^{\bullet}_{[n]} \ : \ X^{\Delta[1]}_{[n]} \to X^{\partial\Delta[1]}_{[n]}$$

the standard construction of the Gauss-Manin connection (see [Wo93]). For each $0 \leq i \leq n$ the complex of sheaves $\Omega^{\cdot}_{X^{\Delta[1]_i}}$ is equipped with a canonical filtration

$$F^{j}\Omega^{\cdot}_{X^{\Delta[1]_i}} := \mathit{Image}\big(\Omega^{\cdot - i}_{X^{\Delta[1]_i}/X^{\partial\Delta[1]_i}} \otimes_{\mathscr{O}_{X^{\Delta[1]_i}}} p^{i,*}\Omega^{j}_{X^{\partial\Delta[1]_i}} \to \Omega^{\cdot}_{X^{\Delta[1]_i}}\big).$$

Hence on $X^{\partial\Delta[1]_i} = X \times X$ we have a filtered complex $\mathrm{R}p^{i}_{*}\big(\Omega^{\cdot}_{X^{\Delta[1]_i}}\big)$. We form the total complex

$$\mathrm{Tot}\big(\mathrm{R}p^{\bullet}_{[n],*}(\Omega^{\cdot}_{X^{\Delta[1]}_{[n]}})\big) = \bigoplus_{i=0}^{n} \mathrm{R}p^{i}_{*}\big(\Omega^{\cdot}_{X^{\Delta[1]_i}}\big).$$

The filtration on each summand on the right hand side induces a filtration on the left hand side. Applying the spectral sequence of a finitely filtered object, we get a spectral sequence converging to the cohomology sheaves

$$\mathscr{H}^{j}\Big(\mathrm{Tot}\big(\mathrm{R}p^{\bullet}_{[n],*}(\Omega^{\cdot}_{X^{\Delta[1]}_{[n]}})\big)\Big)$$

on $X \times X$, the E_1-term of which reads

$$\mathrm{E}^{p,q}_{1} = \Omega^{p}_{X \times X} \otimes_{\mathscr{O}_{X \times X}} \mathscr{H}^{q}\Big(\mathrm{Tot}\big(\mathrm{R}p^{\bullet}_{[n],*}(\Omega^{\cdot}_{X^{\Delta[1]}_{[n]}/X^{\partial\Delta[1]}_{[n]}})\big)\Big),$$

where

$$\mathrm{Tot}\big(\mathrm{R}p^{\bullet}_{[n],*}(\Omega^{\cdot}_{X^{\Delta[1]}_{[n]}/X^{\partial\Delta[1]}_{[n]}})\big) = \bigoplus_{i=0}^{n} \mathrm{R}p^{i}_{*}\big(\Omega^{\cdot}_{X^{\Delta[1]_i}/X^{\partial\Delta[1]_i}}\big).$$

Farther we denote the relative de Rham complex $\Omega^{\boldsymbol{\cdot}}_{X^{\Delta[1]}_{[n]}/X^{\partial\Delta[1]}_{[n]}}$ on $X^{\Delta[1]}_{[n]}$ by $\Omega^{\boldsymbol{\cdot}}$ in the algebraic case, by $\Omega^{\boldsymbol{\cdot}}_{hol}$ in the holomorphic case and by $\Omega^{\boldsymbol{\cdot}}_{\mathscr{C}^\infty}$ in the smooth complex case. The differential

$$d_1^{0,q} \;:\; \mathrm{E}_1^{0,q} \to \mathrm{E}_1^{1,q}$$

is the integrable Gauss-Manin connection on the relative de Rham cohomology sheaves

$$\mathscr{H}^q\big(\mathrm{Tot}\big(\mathrm{R}p^\bullet_{[n],*}\Omega^{\boldsymbol{\cdot}}\big)\big).$$

Let x and y be two k-points of X. The fiber of $\mathscr{H}^q\big(\mathrm{Tot}\big(\mathrm{R}p^\bullet_{[n],*}\Omega^{\boldsymbol{\cdot}}\big)\big)$ over a point $(x,y) \in X \times X$ is

$$\mathrm{H}^q_{DR}\big((p^\bullet_{[n]})^{-1}(x,y)\big).$$

Note that if $x = y$ then $(p^\bullet_{[n]})^{-1}(x,x)$ is the n-th truncation of the cosimplicial algebraic variety, which is a model in algebraic geometry for the loop space based at x from the very beginning of the section.

Recall that we fixed an embedding $k \subset \mathbb{C}$. Then we get the morphism of cosimplicial complex varieties

$$p(\mathbb{C})^\bullet \;:\; X(\mathbb{C})^{\Delta[1]} \longrightarrow X(\mathbb{C})^{\partial\Delta[1]}$$

and the maps between the n-th truncations

$$p(\mathbb{C})^\bullet_{[n]} \;:\; X(\mathbb{C})^{\Delta[1]}_{[n]} \longrightarrow X(\mathbb{C})^{\partial\Delta[1]}_{[n]}.$$

We do the same construction for holomorphic differentials. The holomorphic de Rham sheaf $\Omega^{\boldsymbol{\cdot}}_{X(\mathbb{C})^{\Delta[1]}_{[n]}}$ is the resolution of the constant sheaf $\mathbb{C}_{X(\mathbb{C})^{\Delta[1]}_{[n]}}$ on $X(\mathbb{C})^{\Delta[1]}_{[n]}$. Let us set

$$\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^\bullet_{[n],*}(\mathbb{C}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big) = \bigoplus_{i=0}^{n} \mathrm{R}p^i_*(\mathbb{C}_{X(\mathbb{C})^{\Delta[1]_i}}).$$

Hence we get that

$$\mathscr{H}^q\Big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^\bullet_{[n],*}(\mathbb{C}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\Big)$$

is the sheaf of the flat sections of the holomorphic Gauss-Manin connection

$$(d_1^{0,q})_{hol} \,:\, \mathscr{H}^q\big(\mathrm{Tot}(\mathrm{R}p(\mathbb{C})^\bullet_{[n],*}\Omega^{\boldsymbol{\cdot}}_{hol})\big) \to \Omega^1_{X(\mathbb{C})^2} \otimes_{\mathscr{O}_{X(\mathbb{C})^2}} \mathscr{H}^q\big(\mathrm{Tot}(\mathrm{R}p(\mathbb{C})^\bullet_{[n],*}\Omega^{\boldsymbol{\cdot}}_{hol})\big).$$

We shall calculate the monodromy representation of the locally constant sheaf

$$\mathscr{H}^0\Big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^\bullet_{[n],*}(\mathbb{C}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\Big).$$

The de Rham complexes of smooth differentials are acyclic for direct image functors. Hence there is a quasi-isomorphism

$$\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^{\bullet}_{[n],*}\Omega^{\cdot}_{hol}\big) \simeq \mathrm{Tot}\big(p(\mathbb{C})^{\bullet}_{[n],*}\Omega^{\cdot}_{\mathscr{C}^\infty}\big).$$

Let $\omega_1,\dots,\omega_n \in \Omega^1_{\mathscr{C}^\infty}(X(\mathbb{C}))$ be closed one-forms on $X(\mathbb{C})$. Let us assume that $\omega_i \wedge \omega_{i+1} = 0$ for all i. Then $1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$ defines an element of $\mathscr{O}_{X(\mathbb{C})}\otimes \Omega^{\cdot}_{\mathscr{C}^\infty}(X(\mathbb{C})^n)\otimes\mathscr{O}_{X(\mathbb{C})}$. Hence

$$1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$$

defines a global section of $\mathscr{H}^0\big(\mathrm{Tot}\big(p(\mathbb{C})^{\bullet}_{[n],*}\Omega^{\cdot}_{\mathscr{C}^\infty}\big)\big)$. We shall calculate the action of $d^0 := (d_1^{0,0})_{\mathscr{C}^\infty}$ on the section $1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$. The connection d^0 is the boundary homomorphism of the long exact sequence associated to the short exact sequence

$$0\to F^1/F^2\to F^0/F^2\to F^0/F^1\to 0\,.$$

We recall that the coface maps

$$\delta^i\ :\ X\times X^{n-1}\times X\to X\times X^n\times X$$

are given by

$$\delta^i(x_0,x_1,\dots,x_n) = (x_0,\dots,x_{i-1},x_i,x_i,\dots,x_n)$$

for $0\le i\le n$. We set

$$\delta_n := \sum_{i=0}^{n}(-1)^{n-i}(\delta^i)^*.$$

The boundary operator of the total complex is given by $D=\delta_n+(-1)^n d$, where d is the exterior differential of the de Rham complex.

We denote by $\int_a \omega_1,\dots,\omega_i$ a function defined on a contractible subset of $X(\mathbb{C})$ containing a and sending z to the iterated integral $\int_a^z\omega_1,\dots,\omega_i$ along any path contained in this contractible subset. After calculations we get the following result.

Lemma 25. *Let* $(a,b)\in X(\mathbb{C})\times X(\mathbb{C})$. *We have*

$$D\Big(\sum_{0\le i\le j\le n}\int_a\omega_1,\dots,\omega_i\otimes\omega_{i+1}\otimes\ldots\otimes\omega_j\otimes(-1)^{n-j}\int_b\omega_n,\dots,\omega_{j+1}\Big)=0\,.$$

We denote by $\pi_1(X(\mathbb{C});b,a)$ the right $\pi_1(X(\mathbb{C}),a)$-torsor of paths from a to b on $X(\mathbb{C})$ and by

$$\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q}$$

the induced right $\pi_1(X(\mathbb{C}),a)\otimes\mathbb{Q}$-torsor. We denote by

$$\mathrm{Alg}_{\mathbb{C}}(\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q})$$

the algebra of complex valued polynomial functions on $\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q}$. The fiber of the sheaf $\mathscr{H}^0\big(\mathrm{Tot}\big(p(\mathbb{C})^{\bullet}_{*}\Omega^{\cdot}_{\mathscr{C}^\infty}\big)\big)$ over (a,b) is $\mathrm{H}^0_{DR}((p(\mathbb{C})^{\bullet})^{-1}(a,b))$. The shuffle product defines a multiplication on $\mathrm{H}^0_{DR}((p(\mathbb{C})^{\bullet})^{-1}(a,b))$, hence the 0-th cohomology group is a $\mathbb{C}$-algebra and if $a=b$ it is a Hopf algebra.

The element $1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$ in $\mathrm{H}^0_{DR}((p(\mathbb{C})^\bullet)^{-1}(a,b))$ determines a polynomial complex valued function on the torsor of rational paths $\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q}$, which to a path γ from a to b associates the iterated integral $\int_\gamma \omega_1\ldots,\omega_n$. Hence we get an isomorphism of $\mathbb{C}$-algebras

$$\mathrm{H}^0_{DR}\big((p(\mathbb{C})^\bullet)^{-1}(a,b)\big)\cong Alg_{\mathbb{C}}(\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q})$$

and if $a=b$ we get an isomorphism of Hopf algebras by the work of Chen.

Observe that

$$\varinjlim_n \mathrm{H}^0_{DR}((p(\mathbb{C})^\bullet_{[n]})^{-1}(a,b))=\mathrm{H}^0_{DR}((p(\mathbb{C})^\bullet)^{-1}(a,b)).$$

The same holds also for cohomology sheaves, considered by us and for the connections.

Farther we shall need the following lemma.

Lemma 26. *Let X be a smooth quasi-projective, geometrically connected algebraic variety over a field $k\subset\mathbb{C}$. Then there is an affine smooth algebraic curve S over k and an algebraic map $f:S\to X$ over k such that the induced map*

$$f_*\;:\;\mathrm{H}_1(S(\mathbb{C}),\mathbb{Q})\to\mathrm{H}_1(X(\mathbb{C}),\mathbb{Q})$$

is surjective.

Proof. The lemma follows from the successive applications of the Lefschetz hyperplane theorem for quasi-projective varieties (see [GM88, pages 22 and 23]). To assure that S and $f:S\to X$ are over k one takes successive hyperplanes over k. □

Proposition 27. *Let X be a smooth quasi-projective, geometrically connected algebraic variety over a field $k\subset\mathbb{C}$. The monodromy representation of the bundle of flat sections of the Gauss-Manin connection*

$$d^0=(d_1^{0,0})_{hol}\;:\;\mathscr{H}^0\big(\mathrm{Tot}(\mathrm{R}p(\mathbb{C})^\bullet_*\Omega^\cdot_{hol})\big)\to\Omega^1_{X(\mathbb{C})^2}\otimes_{\mathscr{O}_{X(\mathbb{C})^2}}\mathscr{H}^0\big(\mathrm{Tot}(\mathrm{R}p(\mathbb{C})^\bullet_*\Omega^\cdot_{hol})\big)$$

at a point $(a,b)\in X(\mathbb{C})\times X(\mathbb{C})$

$$\rho_{a,b}\;:\;\pi_1(X(\mathbb{C}),a)\times\pi_1(X(\mathbb{C}),b)\to\mathrm{Aut}(\mathrm{Alg}_{\mathbb{C}}(\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q}))$$

is given by the formula

$$((\rho_{a,b}(\alpha,\beta))(f))(\gamma)=f(\beta^{-1}\cdot\gamma\cdot\alpha),\tag{14.17}$$

where $(\alpha,\beta)\in\pi_1(X(\mathbb{C}),a)\times\pi_1(X(\mathbb{C}),b)$ acts on $f\in\mathrm{Alg}_{\mathbb{C}}(\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q})$. and where $\gamma\in\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q}$.

Proof. First we suppose that X is an affine smooth algebraic curve over a field $k\subset\mathbb{C}$. We can find smooth closed one-forms

$$\eta_1,\dots,\eta_r \in \Omega^1_{\mathscr{C}^\infty}(X(\mathbb{C}))$$

such that their classes form a base of $\mathrm{H}^1_{DR}(X(\mathbb{C}))$ and $\eta_i \wedge \eta_j = 0$ for $1 \le i,j \le r$. Then all possible tensor products $1\otimes \eta_{i_1}\otimes\ldots\otimes\eta_{i_k}\otimes 1$ form a base of

$$\mathrm{H}^0_{DR}((p(\mathbb{C})^\bullet)^{-1}(a,b)).$$

Let $1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$ be one of such products. The stalk of the locally constant sheaf

$$\mathscr{H}^0\big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^\bullet_{[n],*}(\mathbb{C}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\big)$$

over the point (a,b) is equal $\mathrm{H}^0((p(\mathbb{C})^\bullet_{[n]})^{-1}(a,b),\mathbb{C})$, which in turn we calculate using complexes of smooth differential forms. The element $1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$ is considered in the stalk of the sheaf $\mathscr{H}^0\big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^\bullet_{[n],*}(\mathbb{C}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\big)$ over the point (a,b). We prolongate $1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$ to a continuous section s of the locally constant sheaf $\mathscr{H}^0\big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^\bullet_{[n],*}(\mathbb{C}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\big)$ along

$$(\alpha,\beta)\in\pi_1(X(\mathbb{C}),a)\times\pi_1(X(\mathbb{C}),b).$$

We have $s(0) = 1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1$. It follows from Lemma 25 that

$$s(1) = \sum_{0\le i\le j\le n}\Big(\int_\alpha \omega_1,\dots,\omega_i\Big)\otimes\omega_{i+1}\otimes\ldots\otimes\omega_j\otimes(-1)^{n-j}\Big(\int_\beta \omega_n,\dots,\omega_{j+1}\Big)$$

as an element of

$$\mathrm{H}^0((p(\mathbb{C})^\bullet)^{-1}(a,b),\mathbb{C}) = \mathrm{H}^0_{DR}((p(\mathbb{C})^\bullet)^{-1}(a,b)) = \mathrm{Alg}_{\mathbb{C}}(\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q}).$$

Then, for any path γ from a to b, we have

$$s(1)(\gamma) = \sum_{0\le i\le j\le n}\Big(\int_\alpha \omega_1,\dots,\omega_i\Big)\cdot\Big(\int_\gamma \omega_{i+1},\dots,\omega_j\Big)\cdot(-1)^{n-j}\Big(\int_\beta \omega_n,\dots,\omega_{j+1}\Big)$$

which by Chen's formulae, see [Ch75], equals

$$s(1)(\gamma) = \int_{\beta^{-1}\cdot\gamma\cdot\alpha}\omega_1,\dots,\omega_n.$$

Hence the monodromy transformation along (α,β) maps the function

$$f(-) := s(0) \in \mathrm{Alg}_{\mathbb{C}}(\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q})$$

into the function $f(\beta^{-1}\cdot - \cdot\alpha)\in\mathrm{Alg}_{\mathbb{C}}(\pi_1(X(\mathbb{C});b,a)\otimes\mathbb{Q})$. Therefore the proposition is proved for affine smooth algebraic curves over k.

Now we assume that X is a smooth quasi-projective, geometrically connected algebraic variety over a field $k\subset\mathbb{C}$. It follows from Lemma 26 that there is an affine smooth algebraic curve S over k and an algebraic map $f : S \to X$ over k such that the induced map

$$f_* \;:\; \mathrm{H}_1(S(\mathbb{C}),\mathbb{Q}) \to \mathrm{H}_1(X(\mathbb{C}),\mathbb{Q})$$

is surjective.

The morphism f induces a morphism of locally constant sheaves

$$f^*\mathscr{H}^0\Big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{C}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\Big) \longrightarrow \mathscr{H}^0\Big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{C}_{S(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\Big).$$

Consider first $(a,b) \in X(\mathbb{C}) \times X(\mathbb{C})$ which is the image of $(s,t) \in S(\mathbb{C}) \times S(\mathbb{C})$. Then $\mathrm{H}^0((p(\mathbb{C})^{\bullet}_{[n]})^{-1}(a,b))$ is a subalgebra of $\mathrm{H}^0((p(\mathbb{C})^{\bullet}_{[n]})^{-1}(s,t))$. Hence it follows from what we have already proved for smooth affine curves that the monodromy representation of the sheaf

$$\mathscr{H}^0\Big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{C}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\Big)$$

at the point (a,b) is given by the formula (14.17). But then it is given by the formula (14.17) at any point of $X(\mathbb{C}) \times X(\mathbb{C})$. □

We recall that X is a smooth quasi-projective, geometrically connected algebraic variety over the field k with a fixed complex embedding $k \subset \mathbb{C}$. Let us set

$$\mathrm{Tot}\big(\mathrm{R}p^{\bullet}_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]}_{[n],\text{ét}}})\big) = \bigoplus_{i=0}^{n} \mathrm{R}p^{i}_{*}\big(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]_i}_{\text{ét}}}\big),$$

where Tot is the total complex of a bicomplex. Let us define

$$\mathrm{R}^i p^{\bullet}_{[n],*}\big(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]}_{[n],\text{ét}}}\big) = \mathscr{H}^i\Big(\mathrm{Tot}\big(\mathrm{R}p^{\bullet}_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]}_{[n],\text{ét}}})\big)\Big). \tag{14.18}$$

Observe that the stalks of the sheaves in (14.18) are equipped with Galois actions.

From now on we assume that the field k is algebraically closed with a fixed complex embedding $k \subset \mathbb{C}$.

Lemma 28. *The cohomology sheaves* $\mathrm{R}^i p^{\bullet}_{[n],*}\big(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]}_{[n],\text{ét}}}\big)$ *are sheaves of finitely generated* $\mathbb{Z}/\ell^m\mathbb{Z}$*-modules on* $(X \times X)_{\text{ét}}$.

Proof. The spectral sequence of the bicomplex

$$\bigoplus_{i=0}^{n} \mathrm{R}p^{i}_{*}\big(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]_i}_{\text{ét}}}\big),$$

converges to the cohomology sheaves in question. The E_1-term reads

$$\mathrm{E}_1^{i,j} = \mathrm{R}^i p^{j}_{*}\big(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]_j}_{\text{ét}}}\big)$$

and is a constant sheaf on $(X \times X)_{\text{ét}}$ with finitely generated $\mathbb{Z}/\ell^m\mathbb{Z}$-modules as stalks. As only finitely many E_1-terms are nonzero, the lemma follows. □

We need to know if the sheaves

$$\mathrm{R}^i p^{\bullet}_{[n],*}\big(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]}_{[n],\text{ét}}}\big)$$

are locally constant and we need to calculate their monodromy representations.

Let Y be a topological space. We denote by Y_{lh} the site of local homeomorphisms on Y. The functors R^i for the sites $X(\mathbb{C})_{\mathrm{lh}}$ and $X(\mathbb{C})$ are defined as in (14.18) for $X_{\text{ét}}$.

The morphisms of sites

$$\varepsilon : (X \times X)(\mathbb{C})_{\mathrm{lh}} \to (X \times X)_{\text{ét}} \text{ and } \alpha : (X \times X)(\mathbb{C})_{\mathrm{lh}} \to (X \times X)(\mathbb{C})$$

induce the comparison isomorphisms

$$\mathrm{R}^i p^{\bullet}_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]}_{[n],\text{ét}}}) \cong \varepsilon_*\big(\mathrm{R}^i p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z}_{X(\mathbb{C})^{\Delta[1]}_{[n],\mathrm{lh}}})\big) \tag{14.19}$$

and

$$\mathrm{R}^i p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z}_{X(\mathbb{C})^{\Delta[1]}_{[n]}}) \cong \alpha_*\big(\mathrm{R}^i p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z}_{X(\mathbb{C})^{\Delta[1]}_{[n],\mathrm{lh}}})\big). \tag{14.20}$$

We do not know whether the sheaves in (14.19) and (14.20) are locally constant. However we have

$$\big(\varprojlim_{m} \mathrm{R}^i p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big) \otimes \mathbb{Q}_\ell \cong \mathscr{H}^i\Big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{Z}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\Big) \otimes \mathbb{Q}_\ell.$$

The sheaf $\mathscr{H}^i\Big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{C}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\Big)$ is locally constant as the sheaf of flat sections of the integrable connection d^0. Hence the sheaf

$$\mathscr{H}^i\Big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{Z}_{X(\mathbb{C})^{\Delta[1]}_{[n]}})\big)\Big) \otimes \mathbb{Q}$$

is locally constant on $(X \times X)(\mathbb{C})$. This implies that the sheaf

$$\mathscr{H}^i\Big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{Z}_{X(\mathbb{C})^{\Delta[1]}_{[n],\mathrm{lh}}})\big)\Big) \otimes \mathbb{Q}$$

is locally constant on $(X \times X)(\mathbb{C})_{\mathrm{lh}}$. Therefore the sheaf

$$\Big(\mathscr{H}^i\Big(\mathrm{Tot}\big(\mathrm{R}p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{Z}_{X(\mathbb{C})^{\Delta[1]}_{[n],\mathrm{lh}}})\big)\Big)\Big)/\mathrm{Torsion}$$

is also locally constant on $(X \times X)(\mathbb{C}))_{\mathrm{lh}}$.

If $\mathscr{A}$ is locally constant sheaf of finite sets on T_{lh}, where T is a topological space, then the stalk of $\mathscr{A}$ in $t \in T$ can be naturally identified with $\mathscr{A}(\bar{T})$ for some finite covering $\bar{T} \to T$ of T that trivializes $\mathscr{A}$. Moreover if $\bar{T} \to T$ is a Galois covering then the finite quotient $\pi_1(T,t)/\pi_1(\bar{T},t)$ acts on $\mathscr{A}(\bar{T})$. We apply this to calculate the stalk

$$\Big(\varprojlim_m \mathrm{R}^i p(\mathbb{C})^{\bullet}_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z})_{X(\mathbb{C})^{\Delta[1]}_{[n],\mathrm{lh}}} \otimes \mathbb{Q}_\ell\Big)_{(a,b)}$$

in $(a,b) \in (X \times X)(\mathbb{C})$. By the comparison isomorphism (14.19) the same is true for the stalk in the correspinding geometric point (a,b) and the projective system of sheaves

$$\{\mathrm{R}^i p^{\bullet}_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]}_{[n],\text{ét}}})\}_{m\in N}.$$

Moreover we get an action of $\pi_1^{\text{ét}}(X \times X, (a,b))$ on

$$\varprojlim_m \Big(\mathrm{R}^i p^{\bullet}_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z}_{X^{\Delta[1]}_{[n],\text{ét}}})\Big)_{(a,b)} \otimes \mathbb{Q}_\ell \cong \mathrm{H}^i_{\text{ét}}((p^{\bullet}_{[n]})^{-1}(a,b), \mathbb{Q}_\ell). \tag{14.21}$$

It follows from the work of Chen that

$$\mathrm{H}^0_{DR}\big((p(\mathbb{C})^{\bullet})^{-1}(a,b)\big) \cong \mathrm{Alg}_{\mathbb{C}}(\pi_1(X(\mathbb{C});b,a) \otimes \mathbb{Q})\ .$$

We shall use Sullivan's polynomial differential forms with $\mathbb{Q}$-coefficients, see [Su77] page 297. The subscript *SDR* denotes the corresponding cohomology groups. We get the corresponding isomorphism of $\mathbb{Q}$-algebras

$$\mathrm{H}^0_{SDR}\big((p(\mathbb{C})^{\bullet})^{-1}(a,b)\big) \cong \mathrm{Alg}_{\mathbb{Q}}(\pi_1(X(\mathbb{C});b,a) \otimes \mathbb{Q})$$

where $\mathrm{Alg}_{\mathbb{Q}}$ denotes the algebra of polynomial functions on $\pi_1(X(\mathbb{C});b,a) \otimes \mathbb{Q}$ with values in $\mathbb{Q}$. If $a = b$ then we get even an isomorphism of Hopf algebras. The comparison isomorphism leads to isomorphisms

$$\mathrm{H}^0_{\text{ét}}\big((p^{\bullet})^{-1}(a,b), \mathbb{Q}_\ell\big) \cong \mathrm{H}^0\big((p(\mathbb{C})^{\bullet})^{-1}(a,b), \mathbb{Q}\big) \otimes \mathbb{Q}_\ell$$

$$\cong \mathrm{H}^0_{SDR}\big((p(\mathbb{C})^{\bullet})^{-1}(a,b)\big) \otimes \mathbb{Q}_\ell$$

from étale via singular to de Rham cohomology, the last one calculated using Sullivan polynomial differential forms. We conclude a natural isomorphism

$$\mathrm{H}^0_{\text{ét}}\big((p^{\bullet})^{-1}(a,b), \mathbb{Q}_\ell\big) \cong \mathrm{Alg}_{\mathbb{Q}_\ell}(\pi_1(X(\mathbb{C});b,a) \otimes \mathbb{Q})$$

where $\mathrm{Alg}_{\mathbb{Q}_\ell}$ denotes the algebra of polynomials with values in $\mathbb{Q}_\ell$. On the other side we have an isomorphisms of torsors

$$\pi_1(X(\mathbb{C});b,a) \otimes \mathbb{Q}_\ell \cong \pi_1(X;b,a) \otimes \mathbb{Q}_\ell,$$

where $\pi_1(X;b,a)$ is the right $\pi_1(X,a)$-torsor of ℓ-adic paths from a to b on X and $\pi_1(X;b,a) \otimes \mathbb{Q}_\ell$ is the induced right $\pi_1(X,a) \otimes \mathbb{Q}_\ell$-torsor. Therefore we get an isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$\mathrm{H}^0_{\text{ét}}\big((p^\bullet)^{-1}(a,b),\mathbb{Q}_\ell\big) \cong \mathrm{Alg}_{\mathbb{Q}_\ell}(\pi_1(X;b,a)\otimes\mathbb{Q}_\ell)\,. \tag{14.22}$$

The shuffle product on H^0_{DR} is defined using codegeneracies, and can thus be defined also on $\mathrm{H}^0_{\text{ét}}$. The Hopf algebra structure on $\mathrm{H}^0_{DR}((p(\mathbb{C})^\bullet)^{-1}(a,a))$ is defined by the maps

$$1\otimes\omega_1\otimes\ldots\otimes\omega_n\otimes 1 \to \sum_{i=0}^{n}(1\otimes\omega_1\otimes\ldots\otimes\omega_i\otimes 1)\otimes(1\otimes\omega_{i+1}\otimes\ldots\otimes\omega_n\otimes 1),$$

hence one can use maps $X^n \to X^i\times X^{n-i}$ to define it. Therefore the isomorphism (14.22) is an isomorphism of $\mathbb{Q}_\ell$-algebras and if $a=b$ it is an isomorphism of Hopf algebras. We get that the monodromy representation on the projective limit of stalks (14.21) for $i=0$ reads

$$\rho_{(a,b)}\;:\;\pi_1^{\text{ét}}(X,a)\times\pi_1^{\text{ét}}(X,b)\longrightarrow \mathrm{Aut}\big(\mathrm{Alg}_{\mathbb{Q}_\ell}(\pi_1(X;b,a)\otimes\mathbb{Q}_\ell)\big)$$

and is given by the formula

$$((\rho_{(a,b)}(\alpha,\beta))(f))(\gamma) = f(\beta^{-1}\cdot\gamma\cdot\alpha). \tag{14.23}$$

Now we suppose that X is defined over a number field K with a complex embedding $K\subset\mathbb{C}$ and that a and b are two K-points of X. Then the Galois group G_K acts on $\mathrm{H}^0_{\text{ét}}\big((p^\bullet)^{-1}(a,b),\mathbb{Q}_\ell\big)$, where

$$p^\bullet\;:\;X_{\bar{K}}^{\Delta[1]}\to X_{\bar{K}}^{\partial\Delta[1]}.$$

On the other side G_K acts on $\pi_1(X_{\bar{K}};b,a)$, the set of isomorphisms of fiber functors (see [SGA1] Exp V), which in this paper we call the set of ℓ-adic paths, by

$$\sigma(\alpha) = \sigma_{\bar{b}}\circ\alpha\circ\sigma_{\bar{a}}^{-1}.$$

If $a=b$ this is the action described in (14.16).

Proposition 29. *Let X be a smooth quasi-projective, geometrically connected algebraic variety defined over a number field K with a complex embedding $K\subset\mathbb{C}$. Let a and b be two K-points of X. Then the isomorphism of $\mathbb{Q}_\ell$-algebras*

$$\mathrm{H}^0_{\text{ét}}((p^\bullet)^{-1}(a,b),\mathbb{Q}_\ell) \cong \mathrm{Alg}_{\mathbb{Q}_\ell}(\pi_1(X_{\bar{K}};b,a)\otimes\mathbb{Q}_\ell)$$

is an isomorphism of G_K-modules, where on the left hand side G_K acts on the étale cohomology group and on the right hand side the action of G_K is deduced from the action on $\pi_1(X_{\bar{K}};b,a)$.

Proof. The projective system of sheaves

$$\{\mathscr{H}^0\Big(\mathrm{Tot}\big(\mathrm{R}p^\bullet_{[n],*}(\mathbb{Z}/\ell^m\mathbb{Z}_{(X_{\bar{K}})^{\Delta[1]}_{[n],\text{ét}}})\big)\Big)\}_{m\in N}$$

over $X_{\bar{K}} \times X_{\bar{K}}$ is equipped with Galois action in each stalk. Moreover the projective limit tensored with $\mathbb{Q}_\ell$ is locally constant. The Galois action and parallel transport satisfy the formula (14.8). To see this one need to do all constructions over $\mathrm{Spec}(K)$ instead of $\mathrm{Spec}(\bar{K})$.

More naively, if $f : X_{\bar{K}} \to X_{\bar{K}}$ is a morphism of algebraic varieties over $\bar{K}$, then for any $\sigma \in G_K$ we have $\sigma \circ f = \sigma(f) \circ \sigma$ on $X_{\bar{K}}$. Let $Z \to X_{\bar{K}} \times X_{\bar{K}}$ be an étale Galois covering and let $f : Z \to Z$ be a covering transformation. Let $\mathscr{A}$ be a sheaf on $(X_{\bar{K}} \times X_{\bar{K}})_{\text{ét}}$, like the sheaves considered in this paper. Then f induces $f_* : \mathscr{A}(Z) \to \mathscr{A}(Z)$ and once more one have $\sigma \circ f_* = \sigma(f)_* \circ \sigma$ on $\mathscr{A}(Z)$. The map f_* is the monodromy along an element of $\pi_1^{\text{ét}}(X_{\bar{K}} \times X_{\bar{K}}, (a,b))$, which induces f. The equality $\sigma \circ f_* = \sigma(f)_* \circ \sigma$ is the formula (14.8).

For $(\alpha, \beta) \in \pi_1^{\text{ét}}(X_{\bar{K}}, a) \times \pi_1^{\text{ét}}(X_{\bar{K}}, a)$, for $\sigma \in G_K$ and for

$$f \in \mathrm{H}^0_{\text{ét}}((p^\bullet)^{-1}(a,a), \mathbb{Q}_\ell)$$

we have a formula for the Galois action that reads

$$\begin{aligned} \sigma_{(a,a)}\big((\alpha,\beta)_*(f)\big) &= (\sigma(\alpha), \sigma(\beta))_*(\sigma_{(a,a)}(f)) \\ &= \Big(\gamma \mapsto (\sigma_{(a,a)}(f))(\sigma(\beta)^{-1} \cdot \gamma \cdot \sigma(\alpha))\Big) \end{aligned} \tag{14.24}$$

by (14.8) and (14.23).

The function $\gamma \to f(\beta^{-1} \cdot \gamma \cdot \alpha)$ is calculated using the Hopf algebra structure on $\mathrm{H}^0_{\text{ét}}((p^\bullet)^{-1}(a,a), \mathbb{Q}_\ell)$. The Galois group G_K acts on $\mathrm{H}^0_{\text{ét}}((p^\bullet)^{-1}(a,a), \mathbb{Q}_\ell)$ through $\mathbb{Q}_\ell$-isomorphisms. Therefore after applying $\sigma_{(a,a)}$ and setting $\beta = 1$ and $\gamma = 1$ we get that the left hand side of (14.24) is equal $f(\alpha)$.

If we plug $\beta = 1$ and $\gamma = 1$ into the right hand side we get $(\sigma_{(a,a)}(f))(\sigma(\alpha))$. Hence for any $\sigma \in G_K$ and any $\alpha \in \pi_1(X_{\bar{K}}, a)$ we have

$$(\sigma_{(a,a)}(f))(\alpha) = f(\sigma^{-1}(\alpha)).$$

Therefore the G_K-modules $\mathrm{H}^0_{\text{ét}}((p^\bullet)^{-1}(a,a), \mathbb{Q}_\ell)$ and $\mathrm{Alg}_{\mathbb{Q}_\ell}(\pi_1(X_{\bar{K}}, a) \otimes \mathbb{Q})$, where the action of G_K on the second one comes from the action of G_K on $\pi_1(X_{\bar{K}}, a)$, are isomorphic. We deduce also a G_K isomorphism

$$\mathrm{H}^0_{\text{ét}}((p^\bullet)^{-1}(a,b), \mathbb{Q}_\ell) \cong \mathrm{Alg}_{\mathbb{Q}_\ell}(\pi_1(X_{\bar{K}}; b, a) \otimes \mathbb{Q}_\ell)$$

for any pair (a,b). ☐

Acknowledgements

We would like to thank very much the Max Planck Institute for Mathematics in Bonn for its hospitality where during two summer visits this paper was written.

Very warm thanks are also due to Jakob Stix for enormous help in the editorial preparation of the paper and for a lot of very helpful mathematical suggestions.

References

[BD94] A.A. Beilinson and P. Deligne. Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs. In U. Jannsen, S.L. Kleiman, J.-P. Serre, *Motives, Proc. of Sym. in Pure Math.*, pages 97–121, **55**, Part II AMS 1994

[BL94] A.A. Beilinson and A. Levin. The Elliptic Polylogarithm. In U. Jannsen, S.L. Kleiman, J.-P. Serre, *Motives, Proc. of Sym. in Pure Math.*, pages 123–190, **55**, Part II AMS 1994

[Ch75] K.T. Chen Iterated integrals, fundamental groups and covering spaces. *Trans. of the Amer. Math. Soc.*, 206:83–98, 1975.

[DW04] J.-C. Douai and Z. Wojtkowiak. On the Galois Actions on the Fundamental Group of $\mathbb{P}^1_{\mathbb{Q}(\mu_n)} \setminus \{0,1,\infty\}$. *Tokyo Journal of Math.*, 27(1):21–34, 2004.

[GM88] M. Goresky and R. MacPherson. Stratified Morse Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. **14**, Springer 1988.

[NW02] H. Nakamura and Z. Wojtkowiak. On the explicit formulae for ℓ-adic polylogarithms. In *Arithmetic Fundamental Groups and Noncommutative Algebra, Proc. Symp. Pure Math.*, (AMS) 70:285–294, 2002.

[Su77] D. Sullivan. Infinitesimal computations in topology. *Publications Mathématiques, Institut des Hautes Études Scientifiques*, 47:269–332, 1977.

[Wo91] Z. Wojtkowiak. The Basic Structure of Polylogarithmic Functional Equations. In L. Lewin, *Structural Properties of Polylogarithms, Mathematical Surveys and Monographs*, pages 205–231, Vol 37, 1991.

[Wo93] Z. Wojtkowiak. Cosimplicial objects in algebraic geometry. In *Algebraic K-Theory and Algebraic Topology*, Kluwer Academic Publishers, 407:287–327, 1993.

[Wo02] Z. Wojtkowiak. Mixed Hodge Structures and Iterated Integrals,I. In F. Bogomolov and L. Katzarkov, *Motives, Polylogarithms and Hodge Theory. Part I: Motives and Polylogarithms, International Press Lectures Series* , Vol.3:121–208, 2002.

[Wo04] Z. Wojtkowiak. On ℓ-adic iterated integrals,I Analog of Zagier Conjecture. *Nagoya Math. Journals*, 176:113–158, 2004.

[Wo05a] Z. Wojtkowiak. On ℓ-adic iterated integrals,II Functional equations and l-adic polylogarithms. *Nagoya Math. Journals*, 177:117–153, 2005.

[Wo05b] Z. Wojtkowiak. On ℓ-adic iterated integrals,III Galois actions on fundamental groups. *Nagoya Math. Journals*, 178:1–36, 2005.

[Wo07] Z. Wojtkowiak. On the Galois Actions on Torsors of Paths I, Descent of Galois Representations. *J. Math. Sci. Univ. Tokyo*, 14:177–259, 2007.

[Wo09] Z. Wojtkowiak. On ℓ-adic Galois Periods, relations between coefficients of Galois representations on fundamental groups of a projective line minus a finite number of points. *Publ. Math. de Besançon, Algèbra et Théorie des Nombres*, 155–174, 2007–2009, *Février* 2009.

[SGA1] Revêtements Étales et Groupes fondamental, Un séminaire dirigé par A. Grothendieck. *Documents Mathématiques 3, Societé Mathématique de France 2003.*