An ampleness criterion for rank 2 vector bundles on surfaces

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Main result

S smooth projective complex surface.

$N^1(S) = \mathrm{Div}(S)/\sim_{num} = \mathrm{NS}(S)/\text{torsion} = \mathbb{Z}^\rho$.

Proposition

E globally generated rank 2 vector bundle on S.

Assume $h^0(E) \geqslant 4$, and $N^1(S) = \mathbb{Z}\cdot c_1(E)$. Then :

$$E \text{ is ample} \quad \text{or} \quad E = \mathcal{O}_S \oplus L\,.$$

- The two cases are distinguished by $c_2(E) > 0$ or $= 0$.
- Recall: E is ample iff $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample. For E globally generated, $\varphi_E : \mathbb{P}(E) \to \mathbb{P}(H^0(E))$ is a morphism (induced by $H^0(E) \otimes \mathcal{O}_S \twoheadrightarrow E$); then : E ample $\iff \varphi_E$ finite.
- In particular, $h^0(E) \geqslant 4$ necessary. But $N^1(S) = \mathbb{Z}\, c_1(E)$ strong. $\exists\, E$ on $\mathbb{P}^2$ globally generated not ample, $\det(E) = \mathcal{O}_{\mathbb{P}^2}(2)$.

Application I: Lazarsfeld-Mukai bundles

Recall: $C \subset S$, $L \in \mathrm{Pic}(C)$ globally generated by $V \subset H^0(C, L)$.

The Lazarsfeld-Mukai bundle $E_{C,V}$ is defined by the exact sequence

$$0 \to E^*_{C,V} \longrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_S \xrightarrow{\text{ev}} L \to 0\,, \qquad \text{or dually:}$$

$$0 \to V^* \otimes_{\mathbb{C}} \mathcal{O}_S \longrightarrow E_{C,V} \longrightarrow N_C \otimes L^{-1} \to 0\,, \text{ with } N_C := \mathcal{O}_S(C)_{|C}\,.$$

$E_{C,V}$ has rank $\dim V$, $c_1 = [C]$, $c_2 = \deg(L)$.

Proposition

Assume: $\dim V = 2$, $H^1(S, \mathcal{O}_S) = 0$, $N^1(S) = \mathbb{Z} \cdot [C]$,
L and $N_C \otimes L^{-1}$ globally generated and $\not\cong \mathcal{O}_C$. Then:

$E_{C,V}$ is globally generated and ample.

Proof :

$0 \to V^* \otimes_{\mathbb{C}} \mathcal{O}_S \to E_{C,V} \to N_C \otimes L^{-1} \to 0$ gives an exact sequence on H^0, hence a commutative diagram:

$$\begin{array}{ccccccc} 0 \to & V^* \otimes \mathcal{O}_S & \longrightarrow & H^0(E_{C,V}) \otimes \mathcal{O}_S & \longrightarrow & H^0(N_C \otimes L^{-1}) \otimes \mathcal{O}_S \to & 0 \\ & \| & & \downarrow \text{ev} & & \downarrow \text{ev} & \downarrow \\ 0 \to & V^* \otimes \mathcal{O}_S & \longrightarrow & E_{C,V} & \longrightarrow & N_C \otimes L^{-1} \longrightarrow & 0 \end{array}$$

$\implies$ E globally generated.

$c_1(E_{C,V}) = [C]$, $c_2 = \deg(L) > 0$, $h^0(E) \geqslant 4 \implies E$ ample. ■

Example : S K3 with $\mathrm{Pic}(S) = \mathbb{Z} \cdot [C]$, $|L|$ "primitive" linear series – i.e. $|L|$ and $|K_C \otimes L^{-1}|$ base point free.

Question : Does the result extend (say, for K3) for $\dim(V) \geqslant 3$?

Application II: congruences of lines

Let $\mathbb{G} := \mathbb{G}(2,4) \subset \mathbb{P}^5$. A surface $S \subset \mathbb{G}$ gives rise to a 2-dimensional family of lines in $\mathbb{P}^3$, called a **congruence**.

The **fundamental locus** $\mathcal{F}_S$ of the congruence is the set of points in $\mathbb{P}^3$ through which pass ∞ lines of the congruence.

Proposition

If $N^1(S) = \mathbb{Z} \cdot [\mathcal{O}_S(1)]$ and $\deg(S) \geqslant 2$, $\mathcal{F}_S = \varnothing$.

Proof : $E :=$ restriction to S of the universal quotient bundle on $\mathbb{G}$.

E globally generated, $h^0(E) \geqslant 4$ (otherwise S is a plane),
$\det(E) = \mathcal{O}_S(1)$, $c_2(E) > 0 \implies E$ ample.

But $\mathbb{P}(E) = \{(x,\ell) \in \mathbb{P}^3 \times S \mid x \in \ell\}$. The projection $p : \mathbb{P}(E) \to \mathbb{P}^3$ satisfies $p^*\mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_{\mathbb{P}(E)}(1)$, hence p finite $\iff \mathcal{F}_S = \varnothing$. ∎

Congruences of lines (continued)

Corollary

$S = \mathbb{G} \cap H_{d_1} \cap H_{d_2}$, H_{d_i} very general hypersurface of degree d_i, with $(d_1, d_2) \neq (1,1)$ or $(1,2) \implies \mathcal{F}_S = \varnothing$.

Proof : $\mathrm{Pic}(S) = \mathbb{Z} \cdot [\mathcal{O}_S(1)]$ by Noether-Lefschetz. ∎

Example.— Perhaps the simplest nontrivial example of congruence is the family of bisecants to a twisted cubic $T \subset \mathbb{P}^3$. Then $S = \mathrm{Sym}^2 T = \mathbb{P}^2$ embedded in $\mathbb{P}^5$ as a Veronese surface. The vector bundle E on $\mathbb{P}^2$ is globally generated but not ample since $\mathcal{F}_S = T$. It has $h^0(E) = 4$, $\det(E) = \mathcal{O}_{\mathbb{P}^2}(2)$, $c_2 = 3$.

Application III: surfaces with ample cotangent bundle (?)

My original motivation was to find new examples of surfaces with ample cotangent bundle. These surfaces have very interesting properties, but few concrete examples are known.

Applying the Proposition to Ω^1_S gives:

Corollary

Assume that Ω^1_S is globally generated, $q(S) \geqslant 4$ and $N^1(S) = \mathbb{Z} \cdot [K_S]$. Then Ω^1_S is ample.

Unfortunately I do not know any example of such a surface. The problem is that the condition $N^1(S) = \mathbb{Z} \cdot [K_S]$ is very difficult to check. Help appreciated!

Gieseker's lemma

The proof uses the method of Bogomolov to prove that the restriction of a stable bundle to a sufficiently ample curve is stable.

The starting point is the following easy observation:

Gieseker's lemma

E globally generated vector bundle on X projective irreducible.

$$E \text{ not ample} \iff \exists\, C \subset X \text{ irreducible and } u : E \twoheadrightarrow \mathcal{O}_C\,.$$

$\Longleftarrow$: E ample $\Rightarrow$ $E_{|C}$ is ample $\Rightarrow$ any quotient of $E_{|C}$ ample.

$\Longrightarrow$: $\exists\, C' \subset \mathbb{P}(E)$ with $\varphi_E(C') = \{\mathrm{pt}\}$, i.e. $\mathcal{O}_{\mathbb{P}(E)}(1)_{|C'} = \mathcal{O}_{C'}$.

Let $p : \mathbb{P}(E) \to S$. $\varphi_E : \mathbb{P}(E_s) \hookrightarrow \mathbb{P}(H^0(E))$ $\Rightarrow$ $p : C' \xrightarrow{\sim} C$.

$\rightsquigarrow$ section $s : C \xrightarrow{\sim} C' \subset \mathbb{P}(E)$; under s, $p^*E \twoheadrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ pulls back to $E_{|C} \twoheadrightarrow \mathcal{O}_C$. ∎

Proof of the Proposition :

E globally generated rank 2 vector bundle on S, $h^0(E) \geqslant 4$, $N^1(S) = \mathbb{Z} \cdot [c_1(E)]$. We assume that E is **not** ample.

Gieseker's lemma: $\exists\, C \subset S$ and $u : E \twoheadrightarrow \mathcal{O}_C$ · Define:

$$0 \to F \longrightarrow E \xrightarrow{u} \mathcal{O}_C \to 0 \quad (\star) \quad F \text{ rank 2 bundle on } S\,.$$

Strategy : We want to show that the **discriminant** $\Delta_F := 4c_2(F) - c_1^2(F)$ is < 0, so that F is **unstable** (Bogomolov); more precisely, F, hence also E, contains a positive line bundle. This will imply that E^* has a nonzero section, which splits a trivial factor out of E.

From $(\star)$ we get $\quad c(F) = c(E) \cdot c(\mathcal{O}_C)^{-1} = c(E) \cdot (1 - [C])$.

Put $c_i := c_i(E)$: $\;c_1(F) = c_1 - [C]$, $c_2(F) = c_2 - c_1 \cdot [C] - [C]^2$

$$\text{hence} \qquad \Delta_F = \Delta_E - 2c_1 \cdot [C] - [C]^2\,.$$

Proof (continued)

$$0 \to F \longrightarrow E \xrightarrow{u} \mathcal{O}_C \to 0 . \qquad (\star)$$

We have found: $c_1(F) = c_1 - [C]\,, \quad \Delta_F = \Delta_E - 2c_1 \cdot [C] - [C]^2$.

Now $N^1(S) = \mathbb{Z} \cdot c_1 \implies C \sim_{num} rc_1$ for some $r \geqslant 1$. Then

$$c_1(F) = (1-r)c_1\,,\ \Delta_F = \Delta_E - (r^2+2r)c_1^2 = 4(c_2 - c_1^2) - (r^2+2r-3)c_1^2\,,$$

therefore $\quad \Delta_F \leqslant 4(c_2 - c_1^2) - (r^2-1)c_1^2 < -(r^2-1)c_1^2 \quad$ by

Lemma

G globally generated rank 2 on a surface, $h^0(G) \geqslant 4$ and $H^1(\det(G)^{-1}) = 0 \implies c_1^2(G) > c_2(G)$.

End of the proof

$$\Delta_F < 0 \quad \overset{\text{(Bogomolov)}}{\Longrightarrow} \quad \exists \;\; 0 \to L \to F \to \mathcal{I}_Z M \to 0 \quad (\star\star),$$

$Z \subset S$ finite, $c_1(L) = ac_1$, $c_1(M) = bc_1$ in $N^1(S)$, with $a > b$.

$(\star\star)$ gives $c_1(F) = (a+b)c_1$, $c_2(F) = ab\, c_1^2 + \deg(Z)$, hence $\Delta_F = -(a-b)^2 c_1^2 + \deg(Z)$. Recall $\Delta_F < -(r^2-1)c_1^2$.

Therefore $(a-b)^2 c_1^2 \geqslant -\Delta_F > (r^2-1)c_1^2 \;\Rightarrow\; a-b \geqslant r$.

Comparing $c_1(F)$ gives $a+b = 1-r$, hence $a \geqslant 1$.

Thus : $E \supset L$ with $c_1(L) = ac_1$, $a \geqslant 1$.

$0 \neq H^0(E \otimes L^{-1}) = H^0(E^* \otimes \det(E) \otimes L^{-1})$ and $E^* \hookrightarrow H^0(E)^* \otimes \mathcal{O}_S$

$\Longrightarrow \;\; H^0(\det(E) \otimes L^{-1}) \neq 0 \;\Longrightarrow\; L = \det(E) \;\Longrightarrow\; H^0(E^*) \neq 0$

$\Longrightarrow \;\; E = \mathcal{O}_S \oplus \det(E)$. ■

Proof of the lemma

Lemma

G globally generated rank 2 on a surface, $h^0(G) \geqslant 4$ and $H^1(\det(G)^{-1}) = 0 \implies c_1^2(G) > c_2(G)$.

Proof : Choosing 4 general sections of G gives

$$0 \to N \longrightarrow \mathcal{O}_S^4 \longrightarrow G \to 0\,. \qquad (\star)$$

Then $c_1^2(G) - c_2(G) = c_2(N) = c_2(N^*)$.

Since N^* is globally generated, a general section $s \in H^0(N^*)$ vanishes at $c_2(N^*)$ points. Assume $c_2(N^*) = 0$; then

$$0 \to \mathcal{O}_S \xrightarrow{s} N^* \longrightarrow L \to 0 \quad \text{with } L = \det(N^*) = \det(G)\,.$$

Since $H^1(\det(G)^{-1}) = 0$, the extension splits, so that $N \cong \mathcal{O}_S \oplus \det(G)^{-1}$. Then $(\star)$ becomes

$$0 \to \det(G)^{-1} \to \mathcal{O}_S^3 \to G \to 0\,,$$

which implies $h^0(G) = 3$, a contradiction. ■

The end