Recent progress on rationality problems

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Lüroth 1875

Theorem (Lüroth, 1875)

C plane curve, defined by polynomial $f(x,y)=0$, which can be parametrized by rational functions :

$$t \mapsto (x(t),y(t)) \ : \ f(x(t),y(t)) = 0 \ .$$

Then there exists another parametrization $u \mapsto (x(u),y(u))$ such that $u \in \mathbb{C} \xleftrightarrow{1:1} (x,y) \in C$, with finitely many exceptions.

In geometric terms : "map" $\mathbb{C} \dashrightarrow C$, $t \mapsto (x(t),y(t))$

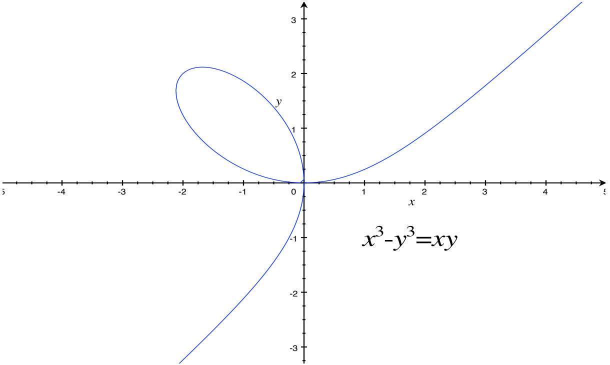
$=$ **rational** map (well-defined outside finite subset of $\mathbb{C}$),

dominant (surjective except for finite subset of C)

$\Longrightarrow\ \exists\ \mathbb{C} \overset{\sim}{\dashrightarrow} C$ **birational** (1-to-1 except for finite subsets;

$\Longleftrightarrow\ \exists$ inverse rational map $C \overset{\sim}{\dashrightarrow} \mathbb{C}$).

An example



Strophoid : parametrized by $x(t) = \dfrac{t}{1-t^3}$, $y(t) = \dfrac{t^2}{1-t^3}$.

$\big($Here $t \mapsto (x(t), y(t))$ is birational: inverse $(x,y) \mapsto \dfrac{y}{x}\big)$.

Lüroth gives a clever, but somewhat mysterious algebraic proof.

(Modern) proof : $C \smallsetminus \mathrm{Sing}(C) \subset \overline{C}$ compact Riemann surface.

$$\begin{array}{ccc} \mathbb{C} & \dashrightarrow & C \smallsetminus \mathrm{Sing}(C) \\ \cap & & \cap \\ \mathbb{P}^1 := \mathbb{C} \cup \infty & \xrightarrow{f} & \bar{C} \end{array}$$

Riemann: $\bar{C} \cong \mathbb{P}^1 \iff$ any holomorphic form ω on $\bar{C}$ is zero.

Here: $f^*\omega = 0 \ \Rightarrow \ \omega = 0 \ \Rightarrow \ \bar{C} \cong \mathbb{P}^1$. ■

Castelnuovo-Enriques

In the years 1890-1900, Castelnuovo and Enriques develop the theory of algebraic surfaces.

Starting from a rather primitive stage, they obtain in a few years a rich harvest of results, culminating with an elaborate classification – called nowadays the Enriques classification.

"An entirely new and beautiful chapter of geometry was opened" (Lefschetz, 1968).

Castelnuovo's theorem

One of the first questions Castelnuovo considers is the analogue of the Lüroth theorem for surfaces :

Theorem (Castelnuovo, 1893)

S algebraic surface, $\exists\ \mathbb{C}^2 \dashrightarrow\!\!\!\!\to S \implies \exists\ \mathbb{C}^2 \overset{\sim}{\dashrightarrow} S$.

or : "*S unirational*" $\implies$ "*S rational*".

($\dashrightarrow\!\!\!\!\to$: replace "finite subset of S" by "strict subvariety of S")

As before, we can assume S smooth and projective ($S \subset \mathbb{P}^N_{\mathbb{C}}$).

Castelnuovo observes that, by refining the previous argument:

S unirational $\Rightarrow$ S has no holomorphic 1-form or 2-form

(locally, 1-form $= p(x,y)dx + q(x,y)dy$, 2-form $= r(x,y)dx \wedge dy$).

[*Footnote :* This is the modern formulation. Castelnuovo used the (equivalent) vanishing of the "geometric" and "numerical" genera.]

The Enriques surface

At first Castelnuovo tried to prove that the vanishing of holomorphic 1- and 2-forms characterizes rational surfaces, but he could not eliminate one particular type of surfaces. He asked Enriques, who found a non-rational surface with no holomorphic form, now called the Enriques surface :

" *Guarda un po' se fosse tale una superficie del* 6^{o} *ordine avente como doppi i* 6 *spigoli d'un tetraedro (se esiste)?* "

These surfaces play an important role in the Enriques classification.

An Enriques surface

Castelnuovo's rationality criterion

Then Castelnuovo found the correct characterization (again, in modern terms) :

Theorem

S rational $\iff$ no (holomorphic) 1-form and quadratic 2-form.

Quadratic 2-form = in local coordinates, $f(x, y)\,(dx \wedge dy)^2$.

A unirational surface has no such forms, hence is rational.

This is a major step in the classification of surfaces; even today, with our powerful modern methods, it is still a highly nontrivial theorem.

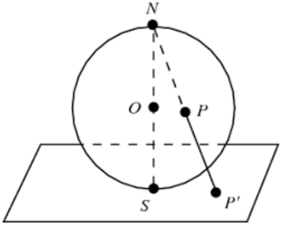
Definitions

Definition

X complex algebraic variety

- X **rational** if $\exists\ \mathbb{C}^n \overset{\sim}{\dashrightarrow} X$;
- X **unirational** if $\exists\ \mathbb{C}^n \dashrightarrow\!\!\!\!\rightarrow X$.

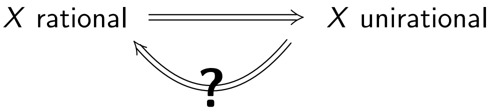
Examples : • $V_2 \subset \mathbb{P}^n$ is rational :



$P \mapsto P'$, $V_2 \overset{\sim}{\dashrightarrow} \mathbb{P}^{n-1}$

• $V_3 \subset \mathbb{P}^n$ is unirational for $n \geqslant 3$, rational for $n = 3$.

The Lüroth problem



Lüroth problem

A parenthesis for algebraists :

The rational functions $X \dashrightarrow \mathbb{C}$ form a field $\mathbb{C}(X)$.

$$X \text{ rational} \iff \exists\ \mathbb{C}^n \overset{\sim}{\dashrightarrow} X \iff \mathbb{C}(X) \xrightarrow{\sim} \mathbb{C}(t_1,\ldots,t_n)\,;$$

$$X \text{ unirational} \iff \exists\ \mathbb{C}^n \dashrightarrow\!\!\!\!\rightarrow X \iff \mathbb{C}(X) \hookrightarrow \mathbb{C}(t_1,\ldots,t_n).$$

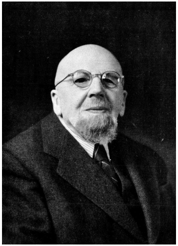
Lüroth problem: $\mathbb{C} \subset K \subset \mathbb{C}(t_1,\ldots,t_n) \implies K \cong \mathbb{C}(u_1,\ldots,u_p)\,?$

But in dimension $\geqslant 2$, this formulation does not help (no known algebraic proof of Castelnuovo's theorem).

Higher dimension

Does "unirational $\Longrightarrow$ rational" hold in dimension $\geqslant 3$?

In 1912, Enriques claims to give a counter-example: a smooth complete intersection of a quadric and a cubic $V_{2,3} \subset \mathbb{P}^5$. Actually he proves that it is unirational, and relies on an earlier paper of Fano (1908) for the non-rationality.



But Fano's analysis is incomplete. The geometry in dimension $\geqslant 3$ is much more complicated than for surfaces; the intuitive methods of the Italian geometers were insufficient.

Doubts

Fano made various other attempts (1915, 1947); in the last one he claims that a smooth $V_3 \subset \mathbb{P}^4$ is not rational, a longstanding conjecture.

But none of these attempts are acceptable by modern standards.

A detailed criticism of Fano's attempts appears in the 1955 book *Algebraic threefolds, with special regard to problems of rationality* by the British mathematician Leonard Roth, who concludes that none of these can be considered as correct.

More doubts

Roth goes on giving a counter-example of his own, by mimicking in dimension 3 the construction of Enriques' surface.

He shows that his example is unirational, and not simply-connected – hence not rational, because a rational (smooth, projective) variety is simply-connected.

Alas, 4 years later Serre showed that a *unirational* variety is simply-connected, so Roth also was wrong...

The modern era

In 1971 appeared almost simultaneously 3 indisputable examples of unirational, non rational varieties, using modern technology:

Authors	**Example**	**Method**
Clemens-Griffiths	$V_3 \subset \mathbb{P}^4$	Hodge theory (JV)
Iskovskikh-Manin	some $V_4 \subset \mathbb{P}^4$	Fano's idea $(\mathrm{Bir}(V))$
Artin-Mumford	specific	Tors $H^3(V,\mathbb{Z})$

A brief overview of the methods

• Clemens and Griffiths associate to a 3-fold V with no holomorphic 3-form a complex torus [$= \mathbb{C}^g/lattice$], the **intermediate Jacobian** JV, with a distinguished hypersurface $\Theta \subset JV$ – generalizing the classical Jacobian of a curve. They prove:

$$V \text{ rational} \Rightarrow (JV, \Theta) \text{ is the Jacobian of a curve.}$$

This is not the case for $V = V_3 \subset \mathbb{P}^4$: one can show $\mathrm{Sing}(\Theta) = \{\mathrm{pt}\}$ and $\dim JV = 5$, while $\dim \mathrm{Sing}(\Theta) \geqslant g - 4$ for the Jacobian of a curve of genus g (Riemann).

• Iskovskikh and Manin, using one of Fano's ideas, prove that any birational map $V_4 \overset{\sim}{\dashrightarrow} V_4$ is actually an automorphism, hence $\mathrm{Bir}(V_4)$ is finite. Since $\mathrm{Bir}(\mathbb{P}^3)$ is enormous, V_4 is not rational.

• The Artin-Mumford method (discussed later) is the only one to give examples (quite particular) in dimension > 3. In contrast, the first two methods give many examples **in dimension 3**.

Examples: complete intersections

e.g. for $V_{d_1\ldots d_{n-3}} \subset \mathbb{P}^n$ with no holomorphic 3-forms ($\Leftrightarrow \sum d_i \leqslant n$):

Variety	Unirational	Rational	Method
$V_3 \subset \mathbb{P}^4$	yes	no	JV
$V_4 \subset \mathbb{P}^4$	some	no	$\mathrm{Bir}(V)$
$V_{2,2} \subset \mathbb{P}^5$	yes	yes	
$V_{2,3} \subset \mathbb{P}^5$	yes	no (generic)	$JV, \mathrm{Bir}(V)$
$V_{2,2,2} \subset \mathbb{P}^6$	yes	no	JV

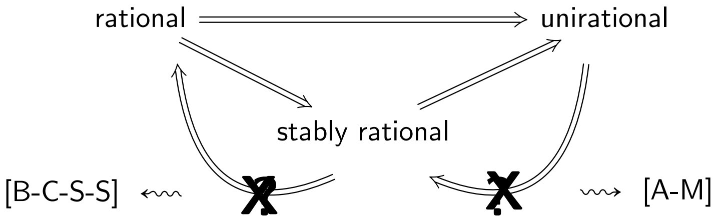
So most of these are unirational, not rational.

The stable Lüroth problem

More generally, Fano studied a class of threefolds, now called *Fano threefolds*, which are good candidates for being unirational. Most of them are unirational, not rational.

This leads to search for an intermediate notion :

X **stably rational** if $X \times \mathbb{P}^m$ rational for some m (Zariski, 1949).



The Artin-Mumford example

Artin and Mumford gave an example of a unirational, non stably rational variety V (hence same for $V \times \mathbb{P}^n$). They prove:

$$V \text{ stably rational } \implies \operatorname{Tors} H^3(V,\mathbb{Z}) = 0\,.$$

Start from: $L = (L_{ij})$ symmetric 4×4 matrix of linear forms in $\mathbb{P}^3$.

$\det(L) = 0$: surface $\Delta \subset \mathbb{P}^3$ with 10 nodes (*quartic symmetroid*).

(node = ordinary double point $\underset{loc}{\cong}$ $x^2 + y^2 + z^2 = 0$ in $\mathbb{C}^3$.)

X defined by $w^2 = \det(L)$: $X \xrightarrow{2:1} \mathbb{P}^3$ branched along Δ.

X has 10 nodes; the desingularization $\tilde{X}$ has $\operatorname{Tors} H^3(\tilde{X},\mathbb{Z}) = \mathbb{Z}/2$.

Till 3 years ago, very few examples of unirational varieties V with $\operatorname{Tors} H^3(V,\mathbb{Z}) \neq 0$.

New results

The situation changed dramatically 3 years ago with a new idea of Claire Voisin:

Theorem 1 (Voisin, 2015)

A double covering of $\mathbb{P}^3$ branched along a **general** quartic surface is not stably rational.

- general := outside a countable union of strict subvarieties of the moduli space
- Known to be unirational, not rational (AB 77, Voisin 86)

Other results

Elaborations of Voisin's idea give the non-stable rationality of the general (*in chronological order*):

1. $V_4 \subset \mathbb{P}^4$ (Colliot-Thélène-Pirutka).
2. $V \xrightarrow{2:1} \mathbb{P}^3$ branched along $S_6 \subset \mathbb{P}^3$ (AB).
3. $V_d \subset \mathbb{P}^{n+1}$, $d \geqslant 2\left\lceil \frac{n+2}{3} \right\rceil$ (Totaro); improved as:
4. $V_d \subset \mathbb{P}^{n+1}$, $d \geqslant \log_2 n + 2$ (Schreieder, 01/2018).
5. The remaining complete intersection threefolds $V_{2,3}$, $V_{2,2,2}$ (Hassett-Tschinkel; more generally, all the non-rational *Fano threefolds*) **except** the cubic threefold $V_3 \subset \mathbb{P}^4$.

Deformations

The most spectacular consequence :

Theorem 2 (Hassett-Pirutka-Tschinkel, 2016)

Let $(V_b)_{b\in B}$ be the family of smooth fourfolds $V_{2,2,2} \subset \mathbb{P}^7$.

1. For general b, V_b is not rational (not even stably);
2. There exists a dense subset $B_{rat} \subset B$ such that V_b is rational for $b \in B_{rat}$.

The existence of a family containing both rational and non rational smooth varieties was unknown, and is still unknown in dimension 3.

However, Hassett-Kresch-Tschinkel have constructed a family of smooth 3-folds containing both **stably rational** and non stably rational varieties (02/2018).

The degeneration argument

Idea : degenerate general quartic into symmetroid:

$B = \{b(X,Y,Z,T) \mid \deg(b) = 4\}$. $b \leftrightsquigarrow$ surface $b = 0$ in $\mathbb{P}^3$.

$X_b := \{w^2 = b\}$ = double covering of $\mathbb{P}^3$ branched along $\{b = 0\}$.

For $o \in B \rightsquigarrow$ quartic symmetroid, X_o has 10 ordinary double points, desingularization $\tilde{X}_o$ satisfies Tors $H^3(\tilde{X}_o, \mathbb{Z}) \neq 0$.

Theorem 3 (Voisin)

$(X_b)_{b\in B}$ family of projective varieties, B smooth, X_b smooth for b general, $o \in B$. Assume:

(i) X_o has only ordinary double points;

(ii) A desingularization $\tilde{X}_o$ of X_o satisfies Tors $H^3(\tilde{X}_o, \mathbb{Z}) \neq 0$.

Then X_b is not stably rational for general b.

$\Rightarrow$ Theorem 1. ■

Comments

- For $(X_b)_{b\in B}$ family of projective threefolds, $J\tilde{X}_o$ not Jacobian $\Longrightarrow$ for general b, JX_b not Jacobian $\Longrightarrow$ X_b not rational.
This is how one proves the "generic" non-rationality results.

- Here Tors $H^3(X_b,\mathbb{Z}) = 0$ for general b, so need a more subtle argument, using **decomposition of the diagonal** in $CH(X_b)$.

- Stronger results last year by Nicaise-Shinder, then Kontsevich-Tschinkel :

Theorem 4

$(X_b)_{b\in B}$ family as above.

- X_o not rational $\Longrightarrow$ X_b not rational for general b.
- X_o not stably rational $\Longrightarrow$ X_b not stably rational for general b.

The proof uses ideas from **motivic integration**.

The mistery of the cubic hypersurface

Why such an interest for cubic hypersurfaces?

- They are very simple to define;
- In dimension 2, 3 and 4, they have a beautiful geometry.

Conjecture (folklore)

The general cubic n-fold is not rational for $n \geqslant 3$.

- Some known rational smooth cubic n-folds for n even; no known example for n odd.
- Smooth cubic 3-folds are not rational (Clemens-Griffiths); but for $n \geqslant 4$, no example of a non-rational cubic known.
- Much studied for cubic 4-folds $V_3 \subset \mathbb{P}^5$ (discussed below).
- **Stable rationality**: nothing known, even for cubic 3-folds. (Nodal cubics are rational, so the above methods do not apply).

The cubic fourfold

Moduli space $\mathcal{C} := \{\text{smooth } V_3 \subset \mathbb{P}^5\}/\mathrm{PGL}(6)$, of dimension **20**.

For some V_3's, $H^4(V_3,\mathbb{Z}) \sim H^2(S,\mathbb{Z})$ for a certain *K3 surface S*:
we say that V is *associated* to S (Hassett).

But K3s depend on **19** parameters.

Fact : In $\mathcal{C}$, the cubics associated to a K3 form a countable union of hypersurfaces $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots$ (defined in terms of $H^4(V_3,\mathbb{Z})$).
For instance, $\mathcal{C}_1 = \{\textit{pfaffian cubics}\}$ defined by $\mathrm{Pf}(L) = 0$, with L a 6×6 skew-symmetric matrix of linear forms.

Conjecture (Kuznetsov + Hassett, ...)

$V_3 \subset \mathbb{P}^5$ rational $\iff$ V_3 has an associated K3 surface.
(equivalently, $[V_3] \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots$)

$\Rightarrow$: nothing known.

$\Leftarrow$: known for $\mathcal{C}_1$ (Fano), $\mathcal{C}_2$ and $\mathcal{C}_3$ (Russo-Staglianò, July 2017).

Conclusion : We know only the tip of the iceberg. Many beautiful open problems!