Nodal surfaces and Gauss genus theory

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Gauss genus theory

Gauss genus theory deals with binary quadratic forms. I will only discuss one of its main consequences: the determination of $\mathrm{Cl}(\mathbb{Q}(\sqrt{d}))[2]$, the 2-torsion of the ideal class group.

Set-up: $d = \pm p_1 \ldots p_s$, $K := \mathbb{Q}(\sqrt{d})$, $\mathcal{O} :=$ ring of integers.

Ramification: $R = \{p_1, \ldots, p_s\} + \{2\}$ if $d \equiv 3 \pmod 4$. $\#R := r$.

$K_+^* := \{\alpha \in K \mid \sigma(\alpha) > 0 \ \forall\, \sigma : K \hookrightarrow \mathbb{R}\}$ ("totally positive").

$\mathrm{Cl}(K) := \mathrm{Pic}(\mathcal{O})$: $K^* \to \mathrm{Div}(\mathcal{O}) \to \mathrm{Cl}(K) \to 0$.

$\mathrm{Cl}^+(K)$: $K_+^* \to \mathrm{Div}(\mathcal{O}) \to \mathrm{Cl}^+(K) \to 0$ ("narrow class group").

$[\mathrm{Cl}^+(K) : \mathrm{Cl}(K)] = 1$ or 2.

$\big(1 \Leftrightarrow d < 0$ or $d > 0$, $\mathrm{Nm}(\mathcal{O}^*) = \{\pm 1\}.\big)$

Theorem (Gauss)

$$\mathrm{Cl}^+(K)[2] = (\mathbb{Z}/2)^{r-1}.$$

Comments

• The result is remarkable, because completely isolated: we know very little about p-torsion for $p > 2$, or 2-torsion of $\mathrm{Cl}(K)$ for $\deg(K) > 2$ (bounds by Bhargava, Venkatesh, ...).

Some consequences $\quad (h(d) := \#\,\mathrm{Cl}(\mathbb{Q}(\sqrt{d})) =$ class number):

• d prime $> 0 \;\Rightarrow\; h(d)$ odd.

• $h(d)$ odd (in particular $= 1$) $\Rightarrow\; d = p_1$ or $p_1 p_2$.

• Recall: it is still unknown whether $h(d) = 1$ for ∞ d.

Expected: $h(p) = 1$ for $\sim 3/4$ of primes p (Cohen-Lenstra).

Nodal surfaces

$\Sigma_d \subset \mathbb{P}^3$ degree d, $\mathrm{Sing}(\Sigma_d) = \mathscr{N} = \{\text{nodes}\}$.

Question: What is $\mu(d) := \max \#\mathscr{N}(\Sigma_d)$?

Classical: $\mu(3) = 4$, max realized by Cayley surface: $\sum \dfrac{1}{X_i} = 0$;

$\mu(4) = 16$, max realized by Kummer surfaces.

Severi 1946: claims $\mu(d) \leqslant \binom{d+2}{3} - 4 \;\Rightarrow\; \mu(5) \leqslant 31$.

B. Segre 1947: counter-examples.

Theorem

$\mu(5) = 31$ (AB 1979); $\mu(6) = 65$ (Jaffe-Ruberman 1986).

$=$ realized by the **Togliatti quintic** and the **Barth sextic.**

Wide open for $d \geqslant 7$; best bound $\mu(d) \leqslant \dfrac{4}{9} d(d-1)^2$ (Miyaoka).

How to prove $\mu(5) = 31$?

Resolution $b : S \to \Sigma_5$. For $n \in \mathscr{N}$, $E_n := b^{-1}(n)$ rational curve; $E_n^2 = -2$, $(E_n \cdot E_p) = 0$. Thus $\#\mathscr{N} \leqslant b_2(S) = 53$, not good...

Key observation: In $H^2(S, \mathbb{Z}/2)$, $\langle E_n \rangle$ **totally isotropic** subspace.

Suppose $\#\mathscr{N} = 32$. $\varphi : (\mathbb{Z}/2)^{32} \xrightarrow{[E_n]} H^2(S, \mathbb{Z}/2)$, $K := \operatorname{Ker}\varphi$.

Then $\dim \operatorname{Im}\varphi \leqslant \frac{1}{2} b_2(S) = 26.5 \implies \dim K \geqslant 6$.

For $A \subset \mathscr{N}$, $\sum_{i \in A} e_i \in K \iff \sum_{i \in A} E_i = 2D$ in $\operatorname{Pic}(S) \iff$ $\exists\, \pi : X \to S$ branched along $\bigcup E_i$. We say that $A \subset \mathscr{N}$ is **even**.

Proposition

A even $\Rightarrow$ $\#A = 16$ or 20.

To get a contradiction, we use easy linear algebra (coding theory):

For $x = \sum_{i \in A} e_i \in (\mathbb{Z}/2)^{32}$, $w(x) := \#A$ (*weight* of x).

$K \subset (\mathbb{Z}/2)^{32}$, $x \in K \Rightarrow w(x) = 0, 16$ or $20 \Rightarrow \dim K \leqslant 5$. ■

The key lemma

Proposition: $\pi : X \to S$, branch locus: $\bigcup_{n\in A} E_n \Rightarrow \# A = 16$ or 20.

Proof uses standard surface theory, plus:

Lemma

X, S smooth projective, $\pi : X \xrightarrow{2:1} S$, branch locus $E_1 \cup \ldots \cup E_r$,
$\mathrm{Pic}(S)[2] = 0$. *Put $\varphi : (\mathbb{Z}/2)^r \xrightarrow{(E_i)} H^2(S, \mathbb{Z}/2)$. Then*

$$\mathrm{Pic}(X)[2] \xrightarrow{\sim} \mathrm{Ker}\,\varphi / \left(\sum e_i\right) .$$

Sketch of proof of the Proposition:

(1) Riemann-Roch + Castelnuovo $\rightsquigarrow$ $4 \mid \#A$ and $\#A \geqslant 16$.

(2) $20 < \#A < 32$: R-R $\Longrightarrow q(X) \geqslant 1 \Longrightarrow \dim Ker\varphi \geqslant 1 \Longrightarrow$

$\exists\, B \subsetneq A$ even. Then B or $A \smallsetminus B$ even with $\# < 16$, contradicts (1).

(3) $\#A = 32$: analogous, + some coding theory. ∎

Proof of the key lemma

Lemma

X, S smooth projective, $\pi : X \xrightarrow{2:1} S$, branch locus $E_1 \cup \ldots \cup E_r$, $\mathrm{Pic}(S)[2] = 0$. Put $\varphi : (\mathbb{Z}/2)^r \xrightarrow{[E_i]} H^2(S, \mathbb{Z}/2)$. Then

$$\mathrm{Pic}(X)[2] \xrightarrow{\sim} \mathrm{Ker}\,\varphi/\left(\sum e_i\right).$$

We start from the exact sequence

$$1 \to \mathbb{C}^* \to K_S^* \xrightarrow{\mathrm{div}} \mathrm{Div}(S) \to \mathrm{Pic}(S) \to 0\,,$$

which we break as

$$1 \to \mathbb{C}^* \to K_S^* \to K_S^*/\mathbb{C}^* \to 1, \quad 1 \to K_S^*/\mathbb{C}^* \to \mathrm{Div}(S) \to \mathrm{Pic}(S) \to 0\,.$$

σ involution of X associated to π, $G := \langle\sigma\rangle \cong \mathbb{Z}/2$. **Recall:**

$$H^1(G, M) = \mathrm{Ker}(1+\sigma)/\mathrm{Im}(1-\sigma), \quad H^2(G, M) = \mathrm{Ker}(1-\sigma)/\mathrm{Im}(1+\sigma).$$

Proof of the key lemma

Compare 2nd exact sequences for S and X:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_S^*/\mathbb{C}^* & \longrightarrow & \mathrm{Div}(S) & \longrightarrow & \mathrm{Pic}(S) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 1 & \longrightarrow & (K_X^*/\mathbb{C}^*)^G & \longrightarrow & \mathrm{Div}(X)^G & \longrightarrow & \mathrm{Pic}(X)^G & \longrightarrow & H^1(G, K_X^*/\mathbb{C}^*)\,. \end{array}$$

Fact 1: $\mathbf{H^1(G, K_X^*/C^*) = 0}$: because $H^1(G, K_X^*) = 0$ (Hilbert 90) and $H^2(G, \mathbb{C}^*) = \mathbb{C}^*/\mathbb{C}^{*2} = 0$.

Fact 2: Coker α = $\mathbf{Z/2}$: follows from the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & K_S^* & \longrightarrow & K_S^*/\mathbb{C}^* & \longrightarrow & 1 \\ & & \| & & \| & & \downarrow{\scriptstyle\alpha} & & \\ 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & (K_X^*)^G & \longrightarrow & (K_X^*/\mathbb{C}^*)^G & \longrightarrow & H^1(G, \mathbb{C}^*) \longrightarrow 0 \end{array}$$

and $H^1(G, \mathbb{C}^*) = \mathrm{Ker}\big(\mathbb{C}^* \xrightarrow{\times 2} \mathbb{C}^*\big) = \mathbb{Z}/2$.

Proof of the key lemma

Apply snake lemma to

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K_S^*/\mathbb{C}^* & \longrightarrow & \mathrm{Div}(S) & \longrightarrow & \mathrm{Pic}(S) & \longrightarrow & 0 \\
 & & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\
1 & \longrightarrow & (K_X^*/\mathbb{C}^*)^G & \longrightarrow & \mathrm{Div}(X)^G & \longrightarrow & \mathrm{Pic}(X)^G & \longrightarrow & 0
\end{array}$$

$$\rightsquigarrow \quad 0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\sum e_i} (\mathbb{Z}/2)^r \longrightarrow \mathrm{Coker}\,\gamma \longrightarrow 0.$$

Hence exact sequence

$$0 \to \mathrm{Pic}(S) \to \mathrm{Pic}(X)^G \to (\mathbb{Z}/2)^r/(\sum e_i) \to 0.$$

Apply snake lemma to $\times 2 \rightsquigarrow$

$$0 \to \mathrm{Pic}(X)^G[2] \to (\mathbb{Z}/2)^r/(\sum e_i) \xrightarrow{[E_i]} \mathrm{Pic}(S) \otimes \mathbb{Z}/2.$$

$\mathrm{Pic}(X)^G[2] = \mathrm{Pic}(X)[2]$: $L \in \mathrm{Pic}(X)[2] \Rightarrow \mathrm{Nm}(L) \in \mathrm{Pic}(S)[2]$

$\Rightarrow \pi^* \mathrm{Nm}(L) = L \otimes \sigma^* L = \mathcal{O}_X \Rightarrow \sigma^* L = L^{-1} = L.$ ■

Proof of Gauss theorem

We apply the same proof with $S = \mathrm{Spec}(\mathbb{Z})$, $X = \mathrm{Spec}(\mathcal{O})$ $\rightsquigarrow$

$1 \to \mathcal{O}_+^* \to K_+^* \to \mathrm{Div}(\mathcal{O}) \to \mathrm{Cl}^+(K) \to 0$, and diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Q}_+^* & \longrightarrow & \mathrm{Div}(\mathbb{Z}) & \longrightarrow & \mathrm{Cl}(\mathbb{Q}) = 0 & & \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
1 & \longrightarrow & (K_+^*/\mathcal{O}_+^*)^G & \longrightarrow & \mathrm{Div}(\mathcal{O})^G & \longrightarrow & \mathrm{Cl}^+(K)^G & \longrightarrow & H^1(G, K_+^*/\mathcal{O}_+^*)
\end{array}$$

For simplicity, case $d > 0$: then $(\mathcal{O}_+^*, \sigma) \cong (\mathbb{Z}, -1)$.

Fact 1: $\mathbf{H^1(G, K_+^*/\mathcal{O}_+^*) = 0}$:

- $H^1(G, K_+^*) = 0$: $\quad 1 \to K_+^* \to K^* \xrightarrow{\mathrm{sgn}, \mathrm{sgn}\circ\sigma} \{\pm 1\} \times \{\pm 1\} \to 1$

  $\rightsquigarrow$ $\mathbb{Q}^* \to \{\pm 1\} \to H^1(G, K_+^*) \to 0$, hence $H^1 G, K_+^*) = 0$.

- $H^2(G, \mathcal{O}_+^*) = H^2(G, (\mathbb{Z}, -1)) = 0$.

Fact 2: $H^1(G, \mathcal{O}_+^*) = H^1(G, (\mathbb{Z}, -1)) = \mathbb{Z}/2$. ■

THE END

Happy retirement, Alex!